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GENERALIZED ANALYTIC SPACES, COMPLETENESS AND FRAGMENTABILITY

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Dedicated to Ivo Vrkoč on the occasion of his 70th birthday

Abstract. Classical analytic spaces can be characterized as projections of Polish spaces. We prove analogous results for three classes of generalized analytic spaces that were introduced by Z. Frolík, D. Fremlin and R. Hansell. We use the technique of complete sequences of covers. We explain also some relations of analyticity to certain fragmentability properties of topological spaces endowed with an additional metric.

Keywords: scattered-K-analytic space, isolated-K-analytic space, Čech analytic space, σ-fragmented space, complete sequence of covers

MSC 2000: 54H05, 54D15, 54F65, 54C35

Introduction

We consider especially the classes of scattered-K-analytic, Čech-analytic and isolated-K-analytic spaces. We prove several characterizations of them in Section 1, mainly those which are formulated in terms of projections along the Polish space $\mathbb{N}^{\mathbb{N}}$ of some "complete" space, and some internal ones employing complete sequences of covers. Using these characterizations, we explain some relationships of different classes of "analytic spaces" in Section 2. The last section is devoted to the notion of σ-fragmentability which was introduced in [11] and which we extend to show connections to various classes of generalized analytic spaces. Proposition 5 and Theorem 6, together with Theorem 1 and Theorem 4, improve our [8, Theorems 5

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and 6] and also [11, Theorem 4.1, (c) implies (a) for Čech-analytic spaces]. The proof of Proposition 5 here is quite straightforward and it needs essentially no reference to results which are not proved here (except for Lemma 3 ([11], Lemma 4.4]) concluding the proof).

We should remark that due to our Theorem 1 (the equivalence of (a) and (c)) and to [15, Lemma 2.3], the result [15, Theorem 5.2] is stronger than our Proposition 5 within completely regular spaces if $\mathcal{D}$ stands for scattered families. There is also a new paper [16] which studies more σ-fragmented subsets of $C_p(K)$ (the space of continuous functions on a compact space K endowed with the topology of pointwise convergence and the supremum metric) than our Corollary below. Many results on generalized analytic spaces can be found in, or are related to, the outstanding paper [6] of R. W. Hansell, where the relations to nonseparable Banach spaces play a central role. Our Section 1 can be understood as a supplement to the results which originate in [6], or to those which can be found in [5], [8] and [10]. Our aim is to add a few new characterizations and to present missing proofs of some announced results.

All topological spaces are supposed to be Hausdorff in what follows, mostly they are regular, but this assumption will be pointed out explicitly wherever it is needed.

1. Generalized analytic and complete spaces

A collection $\mathcal{N}$ of subsets of X is called a *network for a collection* $\mathcal{C}$ of subsets of X if each $C \in \mathcal{C}$ equals $\bigcup\{N \in \mathcal{N};\ N \subset C\}$. Let us recall that a topological space X has a network $\mathcal{N}$ if $\mathcal{N}$ is a network for the collection of all open subsets of X.

A compact-valued map f of a topological space X to compact subsets of a topological space Y is *upper semi-continuous* if $\{x \in X;\ f(x) \subset G\}$ is open whenever G is an open subset of Y. We write *usc-K* instead of upper semi-continuous and compact-valued in what follows.

An indexed family $(C_a;\ a \in A)$ of subsets of X is *point-countable* if $|\{a;\ x \in C_a\}| \leqslant \aleph_0$ for every $x \in X$.

We introduce first a notion of generalized analytic spaces which generalizes [5, Definition 6.7] and [9, Definition 9]. Later on we are primarily interested in special cases which coincide with the notions introduced first by Z. Frolík in [1], and by R. Hansell in [6]. Frolík's "WT-analytic spaces" coincide with Hansell's "K-descriptive spaces". They are called here, in the same way as in [5, Definition 6.7], isolated-K-analytic. Hansell's "almost-K-descriptive spaces" are called here, also in accordance with [5, Definition 6.7], scattered-K-analytic spaces.

In what follows $\mathcal{D}$ stands exclusively for particular collections of families from $\mathcal{P}(\mathcal{P}(X))$ for all topological spaces X under consideration.

Definition 1. Let X be a topological space and $\mathcal{D}$ be a collection of families of subsets of X. The space X is called *$\mathcal{D}$-K-analytic* if there is an usc-K map f of a complete metric space M onto X such that the indexed family $(f(C);\ C \in \mathcal{C})$ is point-countable and $\{f(C);\ C \in \mathcal{C}\}$ has a network $\mathcal{N}$ from $\mathcal{D}_\sigma$, i.e. $\mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathcal{N}_n$ with $\mathcal{N}_n \in \mathcal{D}$, for every discrete family $\mathcal{C}$ of subsets of M.

A topological space X is *$\mathcal{D}$-analytic* if there is a continuous map f of a complete metric space onto X with the same properties as the compact-valued map f above. (We identify continuous $f\colon\ M \to X$ with the usc-K map defined by $\tilde{f}(x) = \{f(x)\}$ in what follows.

Remark 1. Here and in what follows a family $\mathcal{C}$ of subsets of a topological space X is *discrete* in X if every point of X has a neighbourhood intersecting at most one element of $\mathcal{C}$. It is known that the notion of generalized analytic spaces would not change if we used the discreteness in the metric of M instead of the topological discreteness of $\mathcal{C}$ in the above definition.

We have in mind mainly the following examples of collections $\mathcal{D}$.

Definition 2. Let X denote a topological space. We say that a family $\mathcal{C}$ of pairwise disjoint subsets of X is *scattered* if every nonempty subfamily $\mathcal{C}_0$ of $\mathcal{C}$ contains an element that is relatively open in $\bigcup \mathcal{C}_0$. If $\mathcal{S}$ denotes the collection of all scattered families, we say that X is *scattered-K-analytic* if it is $\mathcal{S}$-K-analytic and that X is *scattered-analytic* if it is $\mathcal{S}$-analytic.

We denote by $\mathcal{I}$ the collection of all *isolated* families (or, equivalently, *relatively discrete* families), i.e. of families $\mathcal{E}$ of subsets of a topological space which are discrete in $\bigcup \mathcal{E}$. The $\mathcal{I}$-K-analytic spaces are called *isolated-K-analytic* and $\mathcal{I}$-analytic spaces are called *isolated-analytic*.

Another significant class, say $\mathcal{O}$, is formed by *relatively open* families, i.e. families $\mathcal{R}$ such that every $R \in \mathcal{R}$ is open in $\bigcup \mathcal{R}$.

Let us remark that the assumption of point-countability in Definition 1 is not necessary for the case of scattered-K-analytic spaces (see [10, Theorem 1]). However, there is a space X which is not isolated-K-analytic and such that there is an usc-K map f of a complete metric space onto X with $\{f(C);\ C \in \mathcal{C}\}$ having a σ-isolated network if $\mathcal{C}$ is discrete in M. We may consider the space $X = \bigcup X_\alpha \cup \{p_\infty\}$ of H. Junnila and J. Pelant described in [5, Example 6.22] to show that the implication (b) $\Rightarrow$ (a) of Theorem 2 below does not hold without the additional assumption that X is hereditarily weakly θ-refinable. We may notice that $f(\alpha) = X_\alpha \cup \{p_\infty\}$, in the notation from [5], gives an usc-K map of $[0, \omega_1)$ with the discrete metric onto X and every union of $X_\alpha \cup \{p_\infty\}$ has a σ-isolated network consisting of $\{p_\infty\}$ and some X_α's.

We recall the definition of D. Fremlin of another class of "analytic spaces" which turn out to be closely related with the notion of relatively open families. More information can be found in [5, Section 5].

Definition 3. A completely regular topological space X is called *Čech-analytic* if it is the projection of some Čech-complete subspace of $X \times \mathbb{N}^{\mathbb{N}}$.

Let us notice that $\mathcal{I} = \mathcal{O} \cap \mathcal{S}$ and that every relatively open family has a scattered refinement. Clearly, isolated families are exactly those which are relatively open and pairwise disjoint.

We recall now two useful auxiliary constructions related to isolated, relatively open, and scattered families of sets (cf. e.g. [5, Definition 6.1]).

Associated open sets. If $\mathcal{C}$ is an arbitrary scattered family of sets, then we may choose some well ordering $<$ of $\mathcal{C}$ and open sets $U(C),\ C \in \mathcal{C}$, such that $C \subset U(C)$ and $U(C) \cap B = \emptyset$ for $C, B \in \mathcal{C}$ if $C < B$. The existence is ensured for example by Lemma 1 of [10] and follows easily from Definition 2. The family $\{U(C);\ C \in \mathcal{C}\}$ is called the associated family of open sets in this case.

If $\mathcal{C}$ is an arbitrary relatively open family of sets, then we can find open sets $U(C)$ such that $U(C) \cap \bigcup \mathcal{C} = C$. Since isolated family is relatively open (and disjoint), we may and shall use the same $U(C)$ for the associated family of open sets for isolated families.

Associated Borel sets. If $\mathcal{C}$ is a scattered family and $U(C),\ C \in \mathcal{C}$, form the associated family of open sets as above, the sets $B(C) = \overline{C} \cap (U(C) \setminus \bigcup\{U(E);\ E < C\})$ are useful. They form a scattered family of sets of the form $B(C) = F \cap G$, where F is closed and G is open, and we call them the associated Borel sets.

It is also useful to consider the sets $B(C) = \overline{C} \cap U(C)$ for elements C of an isolated family $\mathcal{C}$ and $U(C)$ the associated open sets. The "associated family of Borel sets" $\{B(C);\ C \in \mathcal{C}\}$ is isolated in this case.

Now we point out several properties of collections $\mathcal{D}$ of families of subsets of a topological space which are significant for our further investigation.

Heredity. We say that the collection $\mathcal{D}$ is hereditary if for any family $\mathcal{E} = \{E_a;\ a \in A\}$ from $\mathcal{D}$ and any family $\{F_a;\ a \in A\}$ such that $F_a \subset E_a,\ a \in A$, the family $\{F_a;\ a \in A\}$ is also in $\mathcal{D}$.

We may notice that the collections $\mathcal{I}$, $\mathcal{S}$, $\mathcal{I}_\sigma$ or $\mathcal{S}_\sigma$ are hereditary. Clearly, the same is not true for $\mathcal{O}$.

Property of unions. The collection $\mathcal{D}$ has the property of unions if, given $\mathcal{E}_a \in \mathcal{D}$, $a \in A$, such that the family $\{\bigcup \mathcal{E}_a = E_a;\ a \in A\}$ is also in $\mathcal{D}$, then the union $\bigcup_{a \in A} \mathcal{E}_a$ is in $\mathcal{D}$.

It can be easily checked that the collections $\mathcal{S}$, $\mathcal{S}_\sigma$, $\mathcal{I}$, $\mathcal{I}_\sigma$ and $\mathcal{O}$ have the property of unions.

However, $\mathcal{O}_\sigma$ does not have the property of unions. Consider e.g. the intervals $[0,\alpha+1)$ of ordinals as open subsets of the space $X=[0,\omega_1)$ endowed with the order topology. Put $\mathcal{E}_\alpha=\{\{\beta\};\ \beta\in[0,\alpha+1)\}$. The families $\mathcal{E}_\alpha$ are countable and thus in $\mathcal{O}_\sigma$ but $\{\{\beta\};\ \beta\in X\}$ is not in $\mathcal{O}_\sigma$.

Cross-section property. We say that the collection $\mathcal{D}$ has the cross-section property if, whenever $\mathcal{E}_1,\mathcal{E}_2$ are from $\mathcal{D}$, then the family $\mathcal{E}_1\wedge\mathcal{E}_2=\{E_1\cap E_2;\ E_1\in\mathcal{E}_1, E_2\in\mathcal{E}_2\}$ is in $\mathcal{D}$, too.

The collections $\mathcal{S}$, $\mathcal{O}$ and $\mathcal{I}$ have the cross-section property. It follows immediately that the same is true concerning $\mathcal{S}_\sigma$, $\mathcal{O}_\sigma$, or $\mathcal{I}_\sigma$.

Trace property. We say that $\mathcal{D}$ has the trace property if given any $\mathcal{E}\in\mathcal{D}$ and $F\subset X$ the family $\{E\cap F;\ E\in\mathcal{E}\}$ is in $\mathcal{D}$.

The key role in our investigation is played by the following notion.

Definition 4. We say that $\mathcal{C}_n$ is a *complete sequence of covers* of a topological space X if every filter $\mathcal{U}$, with $\mathcal{U}\cap\mathcal{C}_n\neq\emptyset$ for every $n\in\mathbb{N}$, has a cluster point, i.e. $\bigcap\{\overline{U};\ U\in\mathcal{U}\}\neq\emptyset$.

Remark 2. Let us remark that the notion of the complete sequence of covers was used by Z. Frolík ([2], Theorem 2.8] and [3, Theorem 9.3]) to characterize Čech-complete spaces and regular K-analytic spaces.

Notice that, if $\mathcal{C}_n$ is a complete sequence of covers of X, then any sequence $\mathcal{E}_n$ of covers of X such that $\mathcal{E}_n$ refines $\mathcal{C}_n$ is also complete.

If X is a regular space, then the completeness of the sequence $\mathcal{C}_n$ of covers of X is equivalent to the following property.

For every (respectively, for every centered) sequence of elements $C_n\in\mathcal{C}_n$, the sequence $\overline{C_1\cap\ldots\cap C_n}$ converges to a (respectively nonempty) compact set $K=\bigcap\limits_{n=1}^{\infty}\overline{C_1\cap\ldots\cap C_n}$, i.e. for every open $G\supset K$ there is an n_0 such that $K\subset\overline{C_1\cap\ldots\cap C_n}\subset G$ for $n>n_0$ (see e.g. Proposition 1 from [10]).

It follows from the above property that the map $f\colon\prod\limits_{n=1}^{\infty}\mathcal{C}_n\to X$ defined by the equality $f((C_n)_{n=1}^{\infty})=\bigcap\limits_{n=1}^{\infty}\overline{\bigcap\limits_{k=1}^{n}C_k}$ is an usc-K map of the complete metric space $\prod\limits_{n=1}^{\infty}\mathcal{C}_n$, defined as the countable product of $\mathcal{C}_n$'s endowed with the discrete metric, onto X.

We want to make clear the relations of our "generalized analytic" spaces to the projections of spaces which are "complete" in an appropriate sense.

Definition 5.

(a) A topological space X is *$\mathcal{D}$-K-complete* if there is a complete sequence of covers from $\mathcal{D}$.

(b) The space X is *$\mathcal{D}$-complete* if there is a complete sequence of covers $\mathcal{C}_n$ from $\mathcal{D}$ such that $\bigcap_{n\in\mathbb{N}} \overline{C_1 \cap \ldots \cap C_n}$ is a singleton for an arbitrary centered sequence of sets C_n with $C_n \in \mathcal{C}_n$.

Remark 3. We should mention that scattered-K-complete spaces, i.e. $\mathcal{S}$-K-complete spaces, in our present terminology are called "cover-complete" e.g. in [7] or "scattered-complete" in [5, 6.17]. Their projections along $\mathbb{N}^{\mathbb{N}}$ are called "cover-analytic" in [7]. Also isolated-K-complete spaces are called "hypercomplete" or "partition complete" (see [5, 6.17]).

The $\mathcal{O}$-K-complete completely regular spaces coincide with the Čech-complete ones ([2, Theorem 2.8]). We have introduced our terminology to cover several related situations simultaneously.

We are going to summarize some characterizations of scattered-analytic, scattered-K-analytic, Čech-analytic, isolated-analytic and isolated-K-analytic spaces. We begin with an implication proved implicitly in [10, Lemma 2], where only scattered-K-analytic spaces were investigated.

Proposition 1. *Let X be a regular $\mathcal{D}$-K-analytic, or even $\mathcal{D}$-analytic, space and let $\mathcal{D}_\sigma$ have the property of unions, the trace property, and the heredity property. Then X is $\mathcal{D}_\sigma$-K-complete, or even $\mathcal{D}_\sigma$-complete, respectively.*

P r o o f. Let $f\colon M \to X$ be a parameterization from the definition of $\mathcal{D}$-K-analytic spaces, or $\mathcal{D}$-analytic spaces, respectively. The statement can be checked by following word by word the proof of Lemma 2 in [10] considering any complete sequence $\mathcal{U}_n$ of σ-discrete covers of M by sets of diameter less than $1/n$ and substituting $\mathcal{D}$ for scattered. In fact, Lemma 2 of [10] gives additional information on the possibility to consider other than (σ-)discrete families in M in the definition of the parameterization f. □

Now we are interested in special continuous maps, the projections along $\mathbb{N}^{\mathbb{N}}$. We investigate the preservation of the relevant properties by such projections in the following two assertions.

Lemma 1. *Let X be a topological space and S be a topological space with a countable network $\mathcal{N}$. Let $\mathcal{C}$ be a family of subsets of $X \times S$ from $\mathcal{D}$, where $\mathcal{D}$ stands for $\mathcal{I}$, $\mathcal{O}$, or $\mathcal{S}$. Then $\{\pi(C);\ C \in \mathcal{C}\}$ has a network from $\mathcal{D}_\sigma$, where π is the projection of $X \times S$ onto X. Moreover, if $\mathcal{D} = \mathcal{I}$ or $\mathcal{D} = \mathcal{S}$, then $(\pi(C);\ C \in \mathcal{C})$ is point-countable.*

P r o o f. Let $U(C)$, $C \in \mathcal{C}$, be the associated family of open sets for $\mathcal{C}$ in any of the three cases. For $N \in \mathcal{N}$ and $C \in \mathcal{C}$ there is the maximal open set $G_C(N) \subset X$

such that $G_C(N) \times N \subset U(C)$. We define $D_C(N) = \pi(C \cap (X \times N)) \cap G_C(N)$. Notice that $\pi(C) = \bigcup\limits_{N \in \mathcal{N}} D_C(N)$ (the inclusion "$\subset$" follows because $U(C)$ is open, it contains C, and N is a network; the inclusion "$\supset$" follows since $D_C(N) \subset \pi(C)$ by the definition) and it is not difficult to check that the family $\{D_C(N);\ C \in \mathcal{C}\}$, for every $N \in \mathcal{N}$, has the property $\mathcal{I}$, $\mathcal{O}$, or $\mathcal{S}$, respectively, with the associated family of open sets $\{G_C(N);\ C \in \mathcal{C}\}$.

Let us explain the last claim for each of the three cases in more details.

Firstly, let $\mathcal{D} = \mathcal{O}$ or $\mathcal{D} = \mathcal{I}$. We will prove that $D_C(N) = \bigcup\limits_{C' \in \mathcal{C}} D_{C'}(N) \cap G_C(N)$. The inclusion "$\subset$" follows from the fact that $D_C(N) \subset G_C(N)$. Let $x \in D_{C'}(N) \cap G_C(N)$ for some $C' \in \mathcal{C}$. Then there is a $y \in N$ such that $(x, y) \in C'$. As $y \in N$ and $x \in G_C(N)$, we have that $(x, y) \in U(C)$. So $C' \cap U(C) \neq \emptyset$ and we conclude that $(x, y) \in C$. If $\mathcal{C}$ was disjoint, then so is $\{D_C(N);\ C \in \mathcal{C}\}$ for every $N \in \mathcal{N}$.

Secondly, let $\mathcal{D} = \mathcal{S}$. Let $<$ be the well ordering and the $U(C)$ above be the corresponding associated open sets for $\mathcal{C}$. Let $C < C'$ be two elements of $\mathcal{C}$. We want to show that $G_C(N) \cap D_{C'}(N) = \emptyset$ and so the sets $G_C(N)$ form an associated family for $\{D_C(N);\ C \in \mathcal{C}\}$. Let $x \in G_C(N) \cap D_{C'}(N)$. Then there is a $y \in N$ such that $(x, y) \in C'$. However, as $G_C(N) \times N \subset U(C)$, we have $(x, y) \in U(C) \cap C'$ which is a contradiction. Notice that $\{D_C(N);\ C \in \mathcal{C}\}$ is disjoint for every $N \in \mathcal{N}$ in this case.

So the claim follows. Also the assertion on point-countability follows using the supplements on disjointness of the families $\{D_C(N);\ C \in \mathcal{C}\}$. $\square$

Remark 4. Compare with [8, Lemma 5], [6, Lemma 7.1]. The proof of Lemma 6.9 in [5] seems to be not completely correct. It is not clear at all why the families $\mathcal{N}_n(a)$ and $\mathcal{N}_n(b)$, $a \neq b$, from that proof of part (b) should be disjoint. It seems however that this shortage could be circumvented by refining the proof.

Proposition 2. *Let $P \subset X \times \mathbb{N}^{\mathbb{N}}$ and $X = \pi(P)$ be a regular space, where π is the projection of $X \times \mathbb{N}^{\mathbb{N}}$ onto X.*

(a) *If P is $\mathcal{O}$-K-complete, or even $\mathcal{O}$-complete, then X is $\mathcal{O}_\sigma$-K-complete, or even $\mathcal{O}_\sigma$-complete, respectively.*
(b) *If P is $\mathcal{I}$-K-complete, or even $\mathcal{I}$-complete, then X is $\mathcal{I}_\sigma$-K-complete, or even $\mathcal{I}_\sigma$-complete, respectively.*

Remark 5. Analogical result holds also for scattered-K-analytic spaces but we derive it, using other arguments, in Theorem 1 below.

P r o o f. Let $\mathcal{E}_n$, $n \in \mathbb{N}$, form the complete sequence of covers of P with the corresponding property from (a) or (b).

We use $N(k_1,\dots,k_n)=\{\nu\in\mathbb{N}^{\mathbb{N}};\ (\nu_1,\dots,\nu_n)=(k_1,\dots,k_n)\}$, $(k_1,\dots,k_n)\in\mathbb{N}^n$, to denote the Baire intervals of order n. We shall construct families $\mathcal{C}_n(s_1,\dots,s_n)$ of subsets of X, for $n\in\mathbb{N}$, $s_1,\dots,s_n\in\mathbb{S}=\bigcup_{k\in\mathbb{N}}\mathbb{N}^k$, such that

(i) $\mathcal{C}_n:=\bigcup\{\mathcal{C}_n(s_1,\dots,s_n);\ s_1,\dots,s_n\in\mathbb{S}\}$ is a cover of X;
(ii) each $\mathcal{C}_n(s_1,\dots,s_n)$ is relatively open (and disjoint if $\mathcal{E}_n$ is disjoint);
(iii) each $\{\bigcup\mathcal{C}_n(s_1,\dots,s_n);\ s_n\in\mathbb{S}\}$ is disjoint;
(iv) $\bigcup\mathcal{C}_{n+1}(s_1,\dots,s_n,s_{n+1})\subset\bigcup\mathcal{C}_n(s_1,\dots,s_n)$; and
(v) if $C\in\mathcal{C}_n(s_1,\dots,s_n)$, then

$$C\subset\pi([X\times N(s_1,\dots,s_n)]\cap P)$$

and

$$\text{there is an } E\in\mathcal{E}_n \text{ such that } [C\times N(s_1,\dots,s_n)]\cap P\subset E.$$

Here $N(s_1,\dots,s_n)=N(s_1{}^\wedge\dots{}^\wedge s_n)$, where $s_1{}^\wedge\dots{}^\wedge s_n$ is the concatenation of the sequences $s_1,\dots,s_n$. □

We claim that the $\mathcal{C}_n$'s (from (i)) form the required sequence of covers of X if (i) to (v) hold. Using (i) we get that they are covers. Due to (ii) and the fact that $\mathbb{S}$ is countable, they are σ-relatively open (or σ-isolated). Let $C_n\in\mathcal{C}_n$ be centered. By (iii) and (iv) there is a unique sequence $s_n\in\mathbb{S}$ such that $C_n\in\mathcal{C}_n(s_1,\dots,s_n)$. From the first part of (v) we get that $[C_n\times N(s_1,\dots,s_n)]\cap P$, $n\in\mathcal{N}$, is centered and from the other part of (v) that each $[C_n\times N(s_1,\dots,s_n)]\cap P$ is contained in some $E_n\in\mathcal{E}_n$. Thus $\overline{E_1\cap\dots\cap E_n}$ converge to a compact set $K\times\{s\}\subset P\subset X\times\mathbb{N}^{\mathbb{N}}$. It follows that $\overline{C_1\cap\dots\cap C_n}$ converge to a compact subset of K in X and $\mathcal{C}_n$ is complete. Moreover, in cases of $\mathcal{O}$-complete and $\mathcal{I}$-complete spaces, $K\times\{s\}$ is a singleton and so K is also a singleton in X.

It remains to construct $\mathcal{C}_n(s_1,\dots,s_n)$ for $n\in\mathbb{N}$ and $s_1,\dots,s_n\in\mathbb{S}$ such that (i) to (v) hold. Let us first describe $\mathcal{C}_1(s)$ for $s\in\mathbb{S}$. For an $E\in\mathcal{E}_1$ and $s\in\mathbb{S}=\bigcup_{k=1}^{\infty}\mathbb{N}^k$ we may find the maximal open set $G_E(s)$ such that $(G_E(s)\times N(s))\cap P\subset E$. Further, we define $D_E(s)=\pi(E\cap(X\times N(s)))\cap G_E(s)$. We choose some well ordering $<$ of the set $\mathbb{S}$ and we define, by induction over $s\in\mathbb{S}$, $\mathcal{C}_1(s)=\{C_E(s)=D_E(s)\setminus\bigcup_{t<s}\bigcup\mathcal{C}_1(t);\ E\in\mathcal{E}_1\}$.

Now $\mathcal{C}_1=\bigcup\{\mathcal{C}_1(s);\ s\in\mathbb{S}\}$ is a cover of X because $\mathcal{E}_1$ is relatively open and so, for every $x\in E\in\mathcal{E}_1$, there are an $s\in\mathbb{S}$ and an open U such that $x\in U\times N(s)$ and $(U\times N(s))\cap P\subset E$.

The families $\mathcal{C}_1(s)$ are relatively open due to the fact that they are traces, to the complement of $\bigcup_{t<s}\mathcal{C}_1(t)$, of the families $\{D_E(s);\ E\in\mathcal{E}_1\}$ that are relatively

open since $G_E(s) \cap \bigcup_{E' \in \mathcal{E}_1} D_{E'}(s) = D_E(s)$. Indeed, the inclusion "$\supset$" is obvious and we check the validity of the other one. Let $x \in G_E(s) \cap D_{E'}(s)$. Then there is a $y \in N(s)$ such that $(x, y) \in E'$ since $x \in D_{E'}(s)$. Then $(x, y) \in P$ and, as $x \in G_E(s)$, $(x, y) \in E$ and so $x \in D_E(s)$.

As $\mathbb{S}$ is countable, the family $\mathcal{C}_1$ is σ-relatively open, and if $\mathcal{E}_1$ was disjoint, then $\mathcal{C}_1(s)$ are also disjoint and $\mathcal{C}_1$ is σ-isolated.

The sets $\bigcup \mathcal{C}_1(s)$, $s \in \mathbb{S}$, are pairwise disjoint since they were defined inductively by subtracting the previously constructed sets $\bigcup \mathcal{C}_1(t)$ with $t < s$.

Finally, for $E \in \mathcal{E}_1$, we get $C_E(s) \subset D_E(s) \subset \pi[E \cap (X \times N(s))] \subset \pi[P \cap (X \times N(s))]$ and $(C_E(s) \times N(s)) \cap P \subset (G_E(s) \times N(s)) \cap P \subset E$.

So the points (i), (ii), (iii), (v) are satisfied and (iv) says nothing for $n = 1$.

Put moreover $X_1(s) = \bigcup \mathcal{C}_1(s)$ and $P_1(s) = P \cap (X(s) \times N(s))$.

Let us suppose that we have constructed

$$\mathcal{C}_n(s_1, \dots, s_n) = \{C_E(s_1, \dots, s_n);\ E \in \mathcal{E}_n\},$$
$$X_n(s_1, \dots, s_n) = \bigcup \mathcal{C}_n(s_1, \dots, s_n),$$

and

$$P_n(s_1, \dots, s_n) = P_{n-1}(s_1, \dots, s_{n-1}) \cap (X_n(s_1, \dots, s_n) \times N(s_1, \dots, s_n)),$$

where $P_0(s_1, \dots, s_0) = P$. We define the same objects for $n + 1$ so that (i) to (v) are satisfied similarly as we did for $n = 1$ above.

Let $G_E(s_1, \dots, s_{n+1})$ be the maximal open set such that

$$[G_E(s_1, \dots, s_{n+1}) \times N(s_1, \dots, s_{n+1})] \cap P_n(s_1, \dots, s_n) \subset E \cap P_n(s_1, \dots, s_n)$$

for each $E \in \mathcal{E}_{n+1}$. Further, put

$$D_E(s_1, \dots, s_{n+1}) = \pi[(E \cap P_n(s_1, \dots, s_n)) \cap X(s_1, \dots, s_n)] \cap G_E(s_1, \dots, s_{n+1}).$$

Finally, we define

$$\begin{aligned}\mathcal{C}_{n+1}(s_1, \dots, s_{n+1}) = \{&C_E(s_1, \dots, s_{n+1}) \\ &= D_E(s_1, \dots, s_{n+1}) \setminus \bigcup_{t < s_{n+1}} \bigcup \mathcal{C}_{n+1}(s_1, \dots, s_n, t);\ E \in \mathcal{E}_{n+1}\},\end{aligned}$$

and for the next construction needed

$$X_{n+1}(s_1, \dots, s_{n+1}) = \bigcup \mathcal{C}_{n+1}(s_1, \dots, s_{n+1})$$

and

$$P_{n+1}(s_1, \dots, s_{n+1}) = P_n(s_1, \dots, s_n) \cap (X(s_1, \dots, s_{n+1}) \times N(s_1, \dots, s_{n+1})).$$

Now (i) to (v) follow by the facts that firstly $\bigcup_{s\in\mathbb{S}} \mathcal{C}_{n+1}(s_1,\dots,s_n,s)$ has the properties analogical to (i), (ii), (iii), (v) with respect to the sets $X_n(s_1,\dots,s_n)$, $P_n(s_1,\dots,s_n)$ which can be verified similarly as the corresponding properties of $\bigcup_{s\in\mathbb{S}} \mathcal{C}_1(s)$ with respect to X, P above, and secondly, the families

$$\{X_n(s_1,\dots,s_{n-1},s);\ s\in\mathbb{S}\} \quad \text{and} \quad \{P_n(s_1,\dots,s_{n-1},s);\ s\in\mathbb{S}\}$$

are disjoint covers of the sets $X_{n-1}(s_1,\dots,s_{n-1})$ and $P_{n-1}(s_1,\dots,s_{n-1})$, respectively. □

We need an extra covering property to study the $\mathcal{I}_\sigma$-K-complete spaces in Propositions 3 and 4. We formulate it in a more general setting because we are going to use it in Proposition 5 below in that form. Notice that the respective property is fulfilled automatically if $\mathcal{D}$ from the definition coincides with $\mathcal{S}$ or with $\mathcal{O}$, and it means "hereditarily weakly θ-refinable" if $\mathcal{D}$ coincides with $\mathcal{I}$.

Definition 6. Let X be a topological space and $\mathcal{D}$ be a collection of families of its subsets. We say that X has the *property of $\mathcal{D}_\sigma$ refinements* if every family $\mathcal{U}$ of open subsets of X has a refinement $\mathcal{C}$ from $\mathcal{D}_\sigma$ such that $\bigcup\mathcal{C}=\bigcup\mathcal{U}$.

In what follows we want to show the respective analyticity from the existence of corresponding complete sequence of covers. The first part (a) of the following Proposition 3 concerns the cases of $\mathcal{D}_\sigma$-K-complete spaces for $\mathcal{D}=\mathcal{S}$, and $\mathcal{D}=\mathcal{I}$ under the assumption that X has the property of $\mathcal{D}_\sigma$ refinements.

The part (b) is a partial result concerning the case of $\mathcal{O}$-K-complete spaces which we don't apply later and we introduce it just for completeness. The statement (c) shows that, in a very special case of $\mathcal{I}$-K-complete spaces, we can easily deduce the $\mathcal{I}$-K-analyticity even if no additional covering property of X is supposed.

Proposition 3.

(a) *Let X be a $\mathcal{D}_\sigma$-K-complete, or even a $\mathcal{D}_\sigma$-complete, regular space. Let $\mathcal{D}=\mathcal{S}$ or $\mathcal{D}=\mathcal{I}$, and X have the property of $\mathcal{I}_\sigma$ refinements in the latter case. Then X is $\mathcal{D}$-K-analytic, or even $\mathcal{D}$-analytic, respectively.*

(b) *Let X be an $\mathcal{O}$-K-complete, or even an $\mathcal{O}$-complete, regular space. Then there is a discrete metric space Γ and an usc-K map f of $\Gamma^{\mathbb{N}}$ onto X, or even a single-valued map of a closed subset $F\subset\Gamma^{\mathbb{N}}$ onto X respectively, such that $\{f(B);\ B\in\mathfrak{B}\}\in\mathcal{O}$, with*

$$\mathfrak{B}=\{\{\gamma\in\Gamma^{\mathbb{N}};\ (\gamma_1,\dots,\gamma_n)=(\delta_1,\dots,\delta_n)\};\ n\in\mathbb{N},\ (\delta_1,\dots,\delta_n)\in\Gamma^{\mathbb{N}}\}$$

the canonical base for $\Gamma^{\mathbb{N}}$.

(c) *Let X be $\mathcal{I}$-K-complete, or even $\mathcal{I}$-complete. Then X is $\mathcal{I}$-K-analytic, or even $\mathcal{I}$-analytic, respectively.*

Proof. (a) In fact, we use the following properties of the collections $\mathcal{D}$ here.

- $\mathcal{D}$ has the cross-section property;
- $\mathcal{D}$ has the heredity property;
- $\mathcal{D}$ has the following "property of regular refinements":

Let $\mathcal{C} \in \mathcal{D}$. There are $\mathcal{H}_C(\mathcal{C})$ for every $C \in \mathcal{C}$ such that

$$\mathcal{H}(\mathcal{C}) := \bigcup_{C \in \mathcal{C}} \mathcal{H}_C(\mathcal{C}) \in \mathcal{D}_\sigma, \bigcup \mathcal{H}_C(\mathcal{C}) = C, \Big\{ \bigcup \{\overline{H};\ H \in \mathcal{H}_C(\mathcal{C})\};\ C \in \mathcal{C} \Big\} \in \mathcal{D}_\sigma.$$

We verify first that the preceding properties of $\mathcal{D}$ are fulfilled in both cases considered.

The first two properties are satisfied by $\mathcal{S}$ and $\mathcal{I}$ as we already mentioned.

We shall verify the latter property. Let $\mathcal{C}$ be scattered or isolated in X and $B(C)$, $C \in \mathcal{C}$, form the family of "associated Borel sets". Since they are differences of open sets and X is regular, it is easy to realize that for every $C \in \mathcal{C}$ there is some relatively open cover $\mathcal{H}'_C(\mathcal{C})$ of C with $\overline{H'} \subset B(C)$ for every $H' \in \mathcal{H}'_C(\mathcal{C})$. Now we choose any $\mathcal{D}_\sigma$ refinement $\mathcal{H}_C(\mathcal{C})$ of $\mathcal{H}'_C(\mathcal{C})$ which exists because the property of $\mathcal{D}_\sigma$ refinements is obviously hereditary to subspaces for $\mathcal{D} = \mathcal{I}$ and it holds even in any space for $\mathcal{D} = \mathcal{S}$. Since $\bigcup\{\overline{H};\ H \in \mathcal{H}_C(\mathcal{C})\} \subset B(C)$ for every $C \in \mathcal{C}$ and the collections $\mathcal{S}$ and $\mathcal{I}$ have the heredity property, the family $\big\{\bigcup\{\overline{H};\ H \in \mathcal{H}_C(\mathcal{C}); C \in \mathcal{C}\big\}$ is in $\mathcal{D}$. Using this fact, the property of unions and the fact that each $\mathcal{H}_C(\mathcal{C})$ is in $\mathcal{D}_\sigma$, we conclude that $\mathcal{H}(\mathcal{C})$ is in $\mathcal{D}_\sigma$. So the properties hold for $\mathcal{D} = \mathcal{S}$ and $\mathcal{D} = \mathcal{I}$.

Now let $\mathcal{C}_n \in \mathcal{D}_\sigma$ form a complete sequence of covers of X. Let us define a new sequence of covers $\mathcal{E}_n \in \mathcal{D}_\sigma$ so that $\mathcal{E}_1 = \mathcal{H}(\mathcal{C}_1)$, and having $\mathcal{E}_n$ already defined, we put $\mathcal{E}_{n+1} = \mathcal{H}(\mathcal{C}_{n+1} \wedge \mathcal{E}_n)$.

We consider now the sets $\mathcal{E}_n$ endowed with the discrete metric and we define the compact-valued map $f\colon \prod_{n=1}^{\infty} \mathcal{E}_n \to X$ by $f((E_n)_{n=1}^{\infty}) = \bigcap\{\overline{E_n};\ n \in \mathbb{N}\}$ if $(E_n)_{n=1}^{\infty} \in F \subset \prod_{n=1}^{\infty} \mathcal{E}_n$, where $(E_n)_{n=1}^{\infty} \in F$ if there are $H_n \in \mathcal{H}_{E_n}(\mathcal{E}_n)$ such that $E_{n+1} \subset H_n (\subset E_n)$ for every $n \in \mathbb{N}$. Otherwise, we put $f((E_n)_{n=1}^{\infty}) = \emptyset$.

The map f is usc-K since $\mathcal{E}_n$ form a complete sequence of covers in the regular space X (see Remark 2) and F is a closed subset of $\prod_{n=1}^{\infty} \mathcal{E}_n$. The map f maps the base $\mathfrak{B}$ of $\prod_{n=1}^{\infty} \mathcal{E}_n$ formed by sets of the form $\Big\{(E'_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathcal{E}_n;\ (E'_1, \ldots, E'_n) = (E_1, \ldots, E_n)\Big\}$ with $(E_1, \ldots, E_n) \in \prod_{k=1}^{n} \mathcal{E}_k$ to families from $\mathcal{D}_\sigma$ because

$$f(\{(E'_k)_{k=1}^{\infty};\ (E'_1, \ldots, E'_n) = (E_1, \ldots, E_n)\}) \subset \bigcup\{\overline{H};\ H \in \mathcal{H}_{E_n}(\mathcal{E}_n)\}.$$

It suffices to use the fact that $\big\{\bigcup\{\overline{H};\ H\in\mathcal{H}_{E_n}(\mathcal{E}_n)\};\ E_n\in\mathcal{E}_n\big\}$ is in $\mathcal{D}_\sigma$ by our assumptions and that $\mathcal{D}_\sigma$ has the heredity property.

Let $\mathcal{E}$ be a discrete family of subsets of $F\subset\prod\limits_{n=1}^{\infty}\mathcal{E}_n$ in the product metric. Then there is an $n\in\mathbb{N}$ such that, for $E\in\mathcal{E}$,

$$E\subset\{B\in\mathfrak{B}_n;\ B\cap E\neq\emptyset\}=B_E,$$

where $\mathfrak{B}_n=\{\{(E'_k)_{k=1}^{\infty};\ E'_1=E_1,\dots,E'_n=E_n\}$ for some $E_1\in\mathcal{E}_1,\dots,\ E_n\in\mathcal{E}_n\}$. We know that $\{f(B);\ B\in\mathfrak{B}\}$ is in $\mathcal{D}_\sigma$, and thus it is also point-countable. Since $\{B_E;\ E\in\mathcal{E}\}$ is disjoint, the family $\{f(B_E);\ E\in\mathcal{E}\}$ is point-countable. By the heredity property of $\mathcal{D}$, we finally get that $\{f(B)\cap f(E);\ B\in\mathfrak{B}_n, E\in\mathcal{E}B\cap E\neq\emptyset\}$ forms a $\mathcal{D}_\sigma$ network for $\{f(E);\ E\in\mathcal{E}\}$. The family $\{f(E);\ E\in\mathcal{E}\}$ is point-countable because $\{f(B_E);\ E\in\mathcal{E}\}$ is point-countable and $f(E)\subset f(B_E)$.

The case of $\mathcal{D}_\sigma$-complete spaces can be treated in the same way and we arrive at a continuous map f of F onto X.

(b) Let $\mathcal{C}_n$ form a complete sequence of open covers of X by open sets. Similarly as in (a), we define a complete sequence of covers $\mathcal{E}_n$. Put $\mathcal{E}_1=\mathcal{C}_1$ and let $\mathcal{E}_n$ be already chosen. We choose for every $C\in\mathcal{E}_n\wedge\mathcal{C}_{n+1}$ an open family $\mathcal{H}_C(n+1)$ such that $\bigcup\{\overline{H};\ H\in\mathcal{H}_C(n+1)\}=C$. We define $\mathcal{E}_{n+1}=\bigcup\{\mathcal{H}_C(n+1);\ C\in\mathcal{E}_n\wedge\mathcal{C}_{n+1}\}$. Let $F\subset\prod\limits_{n=1}^{\infty}\mathcal{E}_n$ be the closed set of sequences $(E_n)_{n=1}^{\infty}$ such that $\overline{E_{n+1}}\subset E_n$. Clearly F is closed and the map f defined similarly as in (a) is usc-K and $f(B)$ is an open set for every $B\in\mathfrak{B}_n$ defined as above.

Again, if $\mathcal{E}_n$ is a complete sequence of open covers from the definition of $\mathcal{O}$-complete spaces, then f is a continuous map on F. Obviously, $\prod\limits_{n=1}^{\infty}\mathcal{E}_n$ is a closed subspace of some complete metric space of the form $\Gamma^{\mathbb{N}}$ desired in (b).

(c) Let $\mathcal{C}_n$ form the complete sequence of isolated covers. Define $\mathcal{E}_1=\mathcal{C}_1$ and $\mathcal{E}_{n+1}=\mathcal{E}_n\wedge\mathcal{C}_{n+1}$. Now put $f((C_n)_{n=1}^{\infty})=\bigcap\{\overline{E_n};\ n\in\mathbb{N}\}$. This gives the desired correspondence because the sets $E_n\in\mathcal{E}_n$ are clopen in this case.

If $\mathcal{C}_n$ comes from the definition of $\mathcal{I}$-complete spaces for X $\mathcal{I}$-complete, then f is a continuous map. □

The crucial step to obtain a description of analyticity by projections along $\mathbb{N}^{\mathbb{N}}$ seems to be the following Proposition 4.

Proposition 4.

(a) *Let X be a regular $\mathcal{S}_\sigma$-K-complete space. Then there is an $\mathcal{S}$-K-complete space $P\subset X\times\mathbb{N}^{\mathbb{N}}$ such that $\pi(P)=X$.*

(a′) *Let X be a regular $\mathcal{S}_\sigma$-complete space. Then there is an $\mathcal{S}$-complete space $P\subset X\times\mathbb{N}^{\mathbb{N}}$ such that $\pi(P)=X$.*

(b) *Let X be a regular $\mathcal{I}_\sigma$-K-complete space having the property of σ-isolated refinements. Then there is an $\mathcal{I}$-K-complete space $P \subset X \times \mathbb{N}^{\mathbb{N}}$ such that $\pi(P) = X$.*

(b′) *Let X be a regular $\mathcal{I}_\sigma$-complete space. Then there is an $\mathcal{I}$-complete space $P \subset X \times \mathbb{N}^{\mathbb{N}}$ such that $\pi(P) = X$.*

(c) *Let X be a regular $\mathcal{O}_\sigma$-K-complete space. Then there is an $\mathcal{O}$-K-complete space $P \subset X \times \mathbb{N}^{\mathbb{N}}$ such that $\pi(P) = X$.*

(c′) *Let X be a regular $\mathcal{O}_\sigma$-complete space. Then there is an $\mathcal{O}$-complete space $P \subset X \times \mathbb{N}^{\mathbb{N}}$ such that $\pi(P) = X$.*

Proof. Cases (a), (a′), (b), and (b′). Let us suppose that $\mathcal{C}_n = \bigcup\limits_{k\in\mathbb{N}} \mathcal{C}_n(k)$, $n \in \mathbb{N}$, form a complete sequence of covers of X and $\mathcal{C}_n(k)$ are scattered (or isolated, respectively). Due to the heredity and to the cross-section properties of $\mathcal{S}_\sigma$ and $\mathcal{I}_\sigma$, we may suppose without loss on the generality that $\mathcal{C}_n$'s are partitions and that $\mathcal{C}_{n+1}$ refines $\mathcal{C}_n$.

Let $U_{n,k}(C)$, $C \in \mathcal{C}_n(k)$, be the associated open sets. Notice that if X is an $\mathcal{I}_\sigma$-complete space, then the elements of the covers of the complete sequence of covers from the definition of $\mathcal{I}_\sigma$-complete spaces form a network and so X has the property of $\mathcal{I}_\sigma$ refinements. Due to the regularity (and moreover the property of σ-isolated refinements in cases (b) and (b′), respectively), by another refinement made inductively in n, we may achieve that, if $D \in \mathcal{C}_{n+1}$, there is some $C \in \mathcal{C}_n(k)$ such that $D \subset C$ and $\overline{D} \subset U_{n,k}(C)$. We put

$$\mathcal{C}_n(k_1, \ldots, k_n) = \{C \in \mathcal{C}_n(k_n);\ \exists (C_i)_{i=1}^{n-1}\ C \subset C_i,\ C_i \in \mathcal{C}_i(k_i),\ i = 1, \ldots, n-1\},$$

where $k_1, \ldots, k_n$ are arbitrary natural numbers. The sets $U_{n,k_n}(C)$ for $C \in \mathcal{C}_n(k_n)$ described above will be denoted by $U_{k_1,\ldots,k_n}(C)$ if $C \in \mathcal{C}_n(k_1, \ldots, k_n)$. Thus we have $\overline{D} \subset U_{k_1,\ldots,k_n}(C)$ whenever $C \in \mathcal{C}_n(k_1, \ldots, k_n)$, $D \in \mathcal{C}_{n+1}(k_1, \ldots, k_n, k_{n+1})$.

Now we define the set $P \subset X \times \mathbb{N}^{\mathbb{N}}$. We use the notation $N(k_1, \ldots, k_n)$ for the set of all (infinite) sequences of natural numbers such that $k_1, \ldots, k_n$ are the first n of them. Let us put

$$P = \bigcap_{n\in\mathbb{N}} \bigcup_{(k_1,\ldots,k_n)\in\mathbb{N}^n} \Big[\bigcup\{\overline{C} \cap U_{k_1,\ldots,k_n}(C);\ C \in \mathcal{C}_n(k_1, \ldots, k_n)\}\Big] \times N(k_1, \ldots, k_n).$$

Obviously, every $x \in X$ belongs to the projection of P along $\mathbb{N}^{\mathbb{N}}$ because $x \in \bigcap\limits_{n=1}^{\infty} C_n$ for some $C_n \in \mathcal{C}_n(k_1, \ldots, k_n)$, $n \in \mathcal{N}$, and we have $\overline{C_n} \cap U_{k_1,\ldots,k_n}(C_n) \supset C_n$. So $\pi(P) = X$.

We shall show that the families

$$\mathcal{E}_n = \{P \cap [(\overline{C} \cap U_{k_1,\dots,k_n}(C)) \times N(k_1,\dots,k_n)];\ C \in \mathcal{C}_n(k_1,\dots,k_n), (k_1,\dots,k_n) \in \mathbb{N}^n\}$$

form a complete sequence of scattered (or isolated) covers of P. It is obvious from the definition of P that all $\mathcal{E}_n$'s are covers. Every $\mathcal{E}_n$ is scattered (isolated) because $\{X \times N(k_1,\dots,k_n);\ (k_1,\dots,k_n) \in \mathbb{N}^n\}$ is discrete and thus isolated and scattered, and for a fixed $(k_1,\dots,k_n)$, the system

$$\{(\overline{C} \cap U_{k_1,\dots,k_n}(C)) \times N(k_1,\dots,k_n);\ C \in \mathcal{C}_n(k_1,\dots,k_n)\}$$

is clearly scattered (or isolated) due to the fact that the open sets $U_{k_1,\dots,k_n}(C)$ are the associated open sets for $\mathcal{C}_n(k_1,\dots,k_n)$.

It remains to show that the sequence $\{\mathcal{E}_n\}$ is complete. Let

$$E_n = P \cap \big\{[\overline{C_n} \cap U_{k_1(n),\dots,k_n(n)}(C_n)] \times N(k_1(n),\dots,k_n(n))\big\}$$

be a centered sequence of sets and $C_n \in \mathcal{C}_n(k_1(n),\dots,k_n(n))$. It is enough to show that this sequence converges to some compact set (see Remark 2 above). Since $(E_n)_{n=1}^{\infty}$ is centered, it is obvious that $k_1(n),\dots,k_n(n)$ are initial sequences of one and only one infinite sequence $(k_1,k_2\dots) \in \mathbb{N}^{\mathbb{N}}$. The sets $\overline{C_n} \cap U_{k_1,\dots,k_n}(C_n)$, $C_n \in \mathcal{C}_n(k_1,\dots,k_n)$, are pairwise disjoint and we claim that $C_n \subset C_{n-1}$. Namely, for every $C_n \in \mathcal{C}_n$ there is a $C'_{n-1} \in \mathcal{C}_{n-1}(k_1,\dots,k_{n-1})$ such that $C_n \subset C'_{n-1}$. If C_n is not a subset of C_{n-1}, then $[\overline{C'_{n-1}} \cap U(C'_{n-1})] \cap [\overline{C_{n-1}} \cap U(C_{n-1})] = \emptyset$. However, $E_n \subset \overline{C_n} \times \mathbb{N}^{\mathbb{N}} \subset [\overline{C'_{n-1}} \cap U(C'_{n-1})] \times \mathbb{N}^{\mathbb{N}}$ and $E_{n-1} \subset [\overline{C_{n-1}} \cap U(C_{n-1})] \times \mathbb{N}^{\mathbb{N}}$ which is a contradiction with the fact that E_n is centered. Thus the sequence $(C_n)_{n=1}^{\infty}$ is also centered in X. Thus the sequence of sets $\overline{C_n}$ converges to a compact set P_∞ in X and also the sequence of sets $\overline{C_n} \cap U_{k_1,\dots,k_n}(C_n)$ converges to P_∞ because $\overline{C_{n+1}} \subset U_{k_1,\dots,k_n}(C_n)$ due to our choice of the systems $\mathcal{C}_n(k_1,\dots,k_n)$. Since obviously

$$P_\infty \subset \bigcap_{n\in\mathbb{N}} \bigcup \{\overline{C} \cap U_{k_1,\dots,k_n}(C);\ C \in \mathcal{C}_n(k_1,\dots,k_n)\},$$

the sequence $(\overline{E_1 \cap \dots \cap E_n})_{n=1}^{\infty}$ converges to $P_\infty \times \{(k_1,k_2,\dots)\}$ in P. (Let us notice that $\overline{C_n} \supset \overline{\overline{C_n} \cap U(C_n)} \supset \overline{C_{n+1}} \supset P_\infty$ and so $P_\infty \times \{k\} \subset \overline{E_1 \cap \dots \cap E_n} \subset \overline{C_n} \times N(k_1,\dots,k_n)$.)

Cases (c) and (c'). Let $\mathcal{C}_n = \bigcup\limits_{k\in\mathbb{N}} \mathcal{C}_n(k)$ form a complete sequence of covers of X and $\mathcal{C}_n(k)$, $k \in \mathbb{N}$, be relatively open. We define $\mathcal{D}_n(k_1,\dots,k_n) = \mathcal{C}_1(k_1) \wedge \dots \wedge \mathcal{C}_n(k_n)$. Further, we put $\mathcal{D}_1^*(k) = \mathcal{D}_1(k)$ for $k \in \mathbb{N}$, and when $\mathcal{D}_n^*(k_1,\dots,k_n)$ are already defined for some $n \in \mathbb{N}$, using the regularity of X we find relatively open refinements $\mathcal{D}^*_{n+1}(k_1,\dots,k_{n+1})$ of $\mathcal{D}_{n+1}(k_1,\dots,k_{n+1})$ such that for every $D\in\mathcal{D}^*_{n+1}(k_1,\dots,k_{n+1})$ there is a set $C\in\mathcal{D}^*_n(k_1,\dots,k_n)$ such that $\overline{D}\subset U_{k_1,\dots,k_n}(C)$ and $D\subset C$. Here $U_{k_1,\dots,k_n}(C)$ is the corresponding associated open set for $C\in\mathcal{D}^*_n(k_1,\dots,k_n)$. We define

$$P=\bigcap_{n\in\mathbb{N}}\ \bigcup_{(k_1,\dots,k_n)\in\mathbb{N}^n}\Big[\bigcup\{\overline{C}\cap U_{k_1,\dots,k_n}(C);\ C\in\mathcal{D}^*_n(k_1,\dots,k_n)\}\Big]\times N(k_1,\dots,k_n)$$

similarly as in (a) above.

Now, the projection $\pi(P)$ of P along $\mathbb{N}^{\mathbb{N}}$ is X because $\overline{C}\cap U_{k_1,\dots,k_n}(C)\supset C$ and $\bigcup_{k_n\in\mathbb{N}}\mathcal{D}^*_n(k_1,\dots,k_n)$ is a cover of $\bigcup\mathcal{D}^*_{n-1}(k_1,\dots,k_{n-1})\bigcup\mathcal{D}_{n-1}(k_1,\dots,k_{n-1})$, and $\mathcal{D}^*_0(\emptyset)=X$.

We define collections

$$\mathcal{E}_n=\big\{P\cap\big([\overline{C}\cap U_{k_1,\dots,k_n}(C)]\times N(k_1,\dots,k_n)\big);\ (k_1,\dots,k_n)\in\mathbb{N}^{\mathbb{N}},\ C\in\mathcal{D}^*_n(k_1,\dots,k_n)\big\}.$$

Obviously, each $\mathcal{E}_n$ is a cover of P.

Each $\mathcal{E}_n$ consists of open subsets of P. Let $E_n\in\mathcal{E}_n$, i.e. $E_n=P\cap([\overline{C}\cap U_{k_1,\dots,k_n}(C)]\times N(k_1,\dots,k_n))$ for some $C\in\mathcal{D}^*_n(k_1,\dots,k_n)$, $k_1,\dots,k_n\in\mathcal{N}$. Namely, the inclusion $C'\cap U_{k_1,\dots,k_n}(C)\subset C$ for every $C,C'\in\mathcal{D}^*_n(k_1,\dots,k_n)$ holds and implies that

$$\overline{C}\cap U_{k_1,\dots,k_n}(C)=U_{k_1,\dots,k_n}(C)\cap\bigcup\{\overline{C'}\cap U_{k_1,\dots,k_n}(C');\ C'\in\mathcal{D}^*_n(k_1,\dots,k_n)\}$$

for every $C\in\mathcal{D}^*_n(k_1,\dots,k_n)$. Therefore

$$\bigcup\mathcal{E}_n\cap[U_{k_1,\dots,k_n}(C)\times N(k_1,\dots,k_n)]=P\cap([\overline{C}\cap U_{k_1,\dots,k_n}(C)]\times N(k_1,\dots,k_n)).$$

It remains to show that $\mathcal{E}_n$ is a complete sequence. Let

$$E_n=P\cap([\overline{C_n}\cap U_{k_1(n),\dots,k_n(n)}(C_n)]\times N(k_1(n),\dots,k_n(n)))\in\mathcal{E}_n$$

form a centered sequence. Then, by the definition of the Baire intervals $N(k_1,\dots,k_n)$, there is a unique sequence $k\in\mathbb{N}^{\mathbb{N}}$ such that $(k_1(n),\dots,k_n(n))=(k_1,\dots,k_n)$ for every $n\in\mathbb{N}$. Obviously, it suffices to study the centered sequence $P_k\cap[\overline{C_n}\cap U_{k_1,\dots,k_n}(C_n)]$, where $P_k=\{x\in X;\ (x,k)\in P\}$, $C_n\in\mathcal{D}^*_n(k_1,\dots,k_n)$. We realize first that the following inclusions hold for every $n\in\mathcal{N}$:

$$\begin{aligned}&P_k\cap U_{k_1}(C_1)\cap\ldots\cap U_{k_1,\dots,k_n}(C_n)\\&\subset\overline{\bigcup\mathcal{D}^*_n(k_1,\dots,k_n)}\cap U_{k_1}(C_1)\cap\ldots\cap U_{k_1,\dots,k_n}(C_n)\\&\subset\overline{\bigcup\mathcal{D}^*_n(k_1,\dots,k_n)\cap U_{k_1}(C_1)\cap\ldots\cap U_{k_1,\dots,k_n}(C_n)}\subset\overline{C_1\cap\ldots\cap C_n}.\end{aligned}$$

We used that $U_{k_1,\ldots,k_i}(C_i)$ are open for the second inclusion.

It follows that the sequence $(C_n)_{n=1}^{\infty}$ is centered in X and from the completeness of $(\mathcal{C}_n)_{n=1}^{\infty}$ and the definition of $\mathcal{D}_n^*$ we obtain that $\overline{C_1 \cap \ldots \cap C_n}$ converge to some compact set $K \subset X$. Moreover, we see from the above inclusions that

$$P_k \cap \overline{C_1} \cap U_{k_1}(C_1) \cap \ldots \cap \overline{C_n} \cap U_{k_1,\ldots,k_n}(C_n) \subset \overline{C_1 \cap \ldots \cap C_n}$$

and so we get that

$$\overline{P_k \cap \overline{C_1} \cap U_{k_1}(C_1) \cap \ldots \cap \overline{C_n} \cap U_{k_1,\ldots,k_n}(C_n)} \subset \overline{C_1 \cap \ldots \cap C_n}$$

converge to some compact $L \subset K \subset X$. Finally, we have that

$$L = \bigcap_{n\in\mathbb{N}} \overline{P_k \cap \overline{C_1} \cap U_{k_1}(C_1) \cap \ldots \cap \overline{C_n} \cap U_{k_1,\ldots,k_n}(C_n)} \subset \bigcap_{n\in\mathbb{N}} \overline{C_1 \cap \ldots \cap C_n}.$$

However, by our assumptions there are $D_{n-1} \in \mathcal{D}^*(k_1, \ldots, k_{n-1})$ such that

$$L \subset \bigcap_{n\in\mathbb{N}} \overline{C_1 \cap \ldots \cap C_n} \subset \bigcap_{n\in\mathbb{N}} \overline{D_n} \cap U_{k_1,\ldots,k_n}(D_n) \subset P_k.$$

So we have that the sequence $\mathcal{E}_n$ is complete in P. □

Now we formulate theorems containing characterizations of scattered-analytic, scattered-K-analytic, isolated-analytic, isolated-K-analytic and Čech-analytic spaces which follow from Propositions 1 to 4 above. Notice that the classes of scattered-analytic and scattered-K-analytic spaces allow characterizations by both the projections and the complete sequences of covers without any restriction.

Theorem 1. *Let X be a regular Hausdorff space. Then the following assertions* (a), (b), *and* (c) *are equivalent.*

(a) *X is scattered-K-analytic;*

(b) *X is $\mathcal{S}_\sigma$-K-complete;*

(c) *X is the projection of an $\mathcal{S}$-K-complete subspace of $X \times \mathbb{N}^{\mathbb{N}}$.*

Also the assertions (a′), (b′), and (c′) are equivalent, where

(a′) *X is scattered-analytic;*

(b′) *X is $\mathcal{S}_\sigma$-complete;*

(c′) *X is the projection of an $\mathcal{S}$-complete subspace of $X \times \mathbb{N}^{\mathbb{N}}$.*

Proof of Theorem 1.

The implications (a) implies (b) and (a′) implies (b′) follow from Proposition 1 above because σ-scattered families satisfy the assumptions on $\mathcal{D}_\sigma$ of Proposition 1.

Due to Proposition 3 (a), (b) implies (a) and (b′) implies (a′).

Now, let the space X be the projection $\pi(C)$ of a scattered-K-complete or scattered-complete space $C \subset X \times \mathbb{N}^{\mathbb{N}}$. By definition, we know that C admits a complete sequence of scattered covers and, by the implication (b) implies (a), or (b′) implies (a′), we get that C is scattered-K-analytic, or scattered-analytic, if it is regular. Really, it is regular as a subspace of $X \times \mathbb{N}^{\mathbb{N}}$. Finally, the projection of a scattered-K-analytic, or scattered-analytic, space C along $\mathbb{N}^{\mathbb{N}}$, i.e. X, is scattered-K-analytic, or scattered-analytic, by Lemma 1. Namely, we may compose π with the parameterization f of X from the definition of the respective analyticity.

To finish the proof of Theorem 1 it is enough to prove the implications (b) implies (c) and (b′) implies (c′). However, this follows from the statements (a) and (a′) of Proposition 4. □

Remark 6. The equivalence of (a) and (b) is proved in [10, Theorem 1], (a) ⇔ (d). Another proof in [5, Theorem 6.18] uses [5, Lemma 6.9] the proof of which needs some correction as indicated in Remark 4 above.

Instead of Lemma 1, we might use e.g. [8, Lemma 5 or its corollary] in this case.

Let us notice that there are other characterizations. In [5, Theorem 6.18] it is stated that scattered-K-analytic spaces are images of Čech-complete spaces by continuous maps taking scattered families to families that are countable unions of families having a scattered refinement. Another characterization says that a completely regular space X is scattered-K-analytic if and only if it is the result of the Souslin operation applied to sets which are elements of the smallest σ-algebra containing Borel sets and closed to the unions of scattered families in some (or every) compactification of X (see [10, Theorem 2]).

The statement [1, Theorem 1] claims that X is $\mathcal{I}_\sigma$-K-complete if and only if X is isolated-K-analytic. This is not correct as was shown by H. Junnila and J. Pelant (their example can be found in [5, Example 6.22]). A modification of Frolík's statement under a supplementary restriction is stated in [5, Theorem 6.19] and it is covered by Theorem 2 below. We use Definition 6 with $\mathcal{D}$ standing for isolated families now.

Theorem 2. *Let X be a regular space. Then the following assertions* (a), (b), *and* (c) *are equivalent if X has the property of σ-isolated refinements.*

(a) *The space X is isolated-K-analytic;*

(b) *the space X is $\mathcal{I}_\sigma$-K-complete;*

(c) *the space X is the projection of some $\mathcal{I}$-K-complete subspace of $X \times \mathbb{N}^{\mathbb{N}}$.*

The implications (a) *implies* (b) *and* (c) *implies* (a) *do not need the property of σ-isolated refinements for X to hold.*

Also the assertions (a′), (b′), *and* (c′) *are equivalent, where* (a′) *The space X is isolated-analytic;*

(b′) *the space X is $\mathcal{I}_\sigma$-complete;*

(c′) *the space X is the projection of some $\mathcal{I}$-complete subspace of $X \times \mathbb{N}^{\mathbb{N}}$.*

P r o o f. The implications (a) implies (b) and (a′) implies (b′) follow from Proposition 1 because $\mathcal{I}_\sigma$ has the property of unions, the trace property and the heredity property.

By Proposition 4 (b) and (b′), we get that (b) implies (c) and (b′) implies (c′).

Finally, using Lemma 1 and Proposition 3 (a), we get (c) implies (a) and (c′) implies (a′). □

The following characterization of Čech-analytic spaces was announced by Z. Frolík in [4, Theorem 1]. The proof can be found in [5, Theorem 5.7, the proof of (a) is equivalent to (c)]. (Notice that the statement (b) of Theorem 5.7 in [5] is formulated a little stronger than the one which is really proved, as R. W. Hansell remarked later.) We prove the analogous equivalence for a slightly more general class of not necessarily completely regular spaces. Since we do not use the embedding of the space admitting a complete sequence of σ-relatively open covers into a compactification K and we do not use the characterization of Čech-analytic spaces as spaces that are the results of the Souslin operation applied to Borel subsets of K, our proof is rather more straightforward. Since we have no analogue of the characterization (a) of Theorem 1 by a parameterization for Čech-analytic spaces, we have to prove the existence of the complete sequence of covers directly from the description by a projection as described by Proposition 2.

Theorem 3. *Let X be a regular space. Then the following are equivalent.*

(a) *The space X is the projection of some $\mathcal{O}$-K-complete subspace of $X \times \mathbb{N}^{\mathbb{N}}$.*

(b) *X is $\mathcal{O}_\sigma$-K-complete.*

Also (a′) and (b′) are equivalent, where

(a′) *The space X is the projection of some $\mathcal{O}$-complete subspace of $X \times \mathbb{N}^{\mathbb{N}}$.*

(b′) *X is $\mathcal{O}_\sigma$-complete.*

P r o o f. It follows from Propositions 2 (a) and 4 (c) and (c′). □

Let us remark that we don't know if there is some natural characterization of Čech-analytic spaces X in terms of an usc-K map $f\colon M \to X$ of a complete metric space M onto X. E.g. the assumption that $(f(C);\ C \in \mathcal{C})$ is point-countable and has a σ-relatively open network for $\mathcal{C}$ discrete cannot be fulfilled for every Čech-analytic space as the following example shows.

Example. The interval $[0, \omega_1)$ of ordinals less than ω_1 endowed with the order topology is Čech-analytic but there is no usc-K map of a metric space M onto

$[0,\omega_1)$ such that $\{f(C);\ C\in\mathcal{C}\}$ has a σ-relatively open refinement for every discrete family $\mathcal{C}$ of subsets of M and $\{f(C);\ C\in\mathcal{C}\}$ is point-countable. In particular, $[0,\omega_1)$ is not $\mathcal{O}$-K-analytic.

P r o o f. Let there be such a parameterization $f\colon M\to[0,\omega_1)$. Since every compact set in $[0,\omega_1)$ is contained in a countable open set and f is usc-K, there is a subfamily $\{I_a;\ a\in A\}$ of a σ-discrete base for M such that $f(I_a)$ is countable for every $a\in A$ and $f(I_a),\ a\in A$, form a cover of $[0,\omega_1)$.

The family $\{f(I_a);\ a\in A\}$ of countable sets is point-countable because $\{I_a;\ a\in A\}$ is σ-discrete. So we may find a partition B_b, $b\in C$, of A to countable subsets such that the family $\{f(J_b);\ b\in C\}$ is disjoint, where $J_b=\bigcup\limits_{a\in B_b} I_a$.

Moreover, the family $\{f(J_b);\ b\in C\}$ has a σ-relatively open refinement and being pairwise disjoint it has a σ-isolated refinement. We thus obtain a σ-isolated cover of $[0,\omega_1)$ by countable sets.

We realize that the complement to the union of any isolated family $\{S_a;\ a\in A\}$ of countable sets in $[0,\omega_1)$ contains a closed uncountable subset. It is immediate if the family is countable. If not, choose $s_a\in S_a$, $a\in A$, and consider the set $F=\overline{\{s_a;\ a\in A\}}\setminus\bigcup\limits_{a\in A}\{s_a\}$. The set F is closed and uncountable.

Let $\mathcal{C}_n$ be the isolated families of countable sets such that $\bigcup\limits_{n\in\mathbb{N}}\bigcup\mathcal{C}_n=[0,\omega_1)$. Let F_n be some closed unbounded subset of $[0,\omega_1)\setminus\bigcup\mathcal{C}_n$. Then the intersection $\bigcap\limits_{n=1}^{\infty}F_n$ is nonempty which is a contradiction to the assumption that $\bigcup\limits_{n\in\mathbb{N}}\mathcal{C}_n$ is a cover of $[0,\omega_1)$. □

2. Some subclasses of scattered-K-analytic spaces

We begin with a result containing our [8, Theorem 1] that can be now derived easily from our knowledge of complete sequences of covers.

Theorem 4. *Every completely regular isolated-K-analytic space is Čech-analytic and every Čech-analytic space X is scattered-K-analytic.*

P r o o f. The first part follows by the implication (a) implies (b) of Theorem 2, using the fact that the property of σ-isolated refinements is not needed for it, and by the implication (b) implies (a) of Theorem 3.

The other part follows from the implication (a) implies (b) of Theorem 3 and the implication (b) implies (a) of Theorem 1. □

Now we shall characterize those scattered-K-analytic spaces which are scattered-analytic or even *analytic*, i.e. continuous images of some complete separable metric space.

Theorem 5.

(a) *A regular Hausdorff space X is scattered-analytic if and only if it is scattered-K-analytic and has a σ-scattered network.*

(b) *A regular Hausdorff space X is analytic if and only if it is scattered-K-analytic and has a countable network.*

(c) *A regular Hausdorff space X is isolated-analytic if and only if it is scattered-K-analytic and has a σ-isolated network.*

Proof. (a) If X is scattered-analytic, then there is a complete sequence of σ-scattered covers $\mathcal{C}_n$ of X with the property (b) of Definition 5. For simplicity we shall suppose that $\mathcal{C}_{n+1}$ refines $\mathcal{C}_n$. This is possible due to the heredity property and the cross-section property of scattered families. If $x \in X$, then there is a sequence of sets $C_n \in \mathcal{C}_n$ such that $x \in \bigcap\limits_{n\in\mathbb{N}} C_n$. So the sequence C_n is centered and $\bigcap\limits_{k=1}^{n} \overline{C_k}$ converges to $\{x\}$. In particular, for every open neighbourhood G of x, there is an $n \in \mathbb{N}$ such that $x \in C_n \subset \overline{C_n} \subset G$ and thus $\bigcup\limits_{n\in\mathbb{N}} \mathcal{C}_n$ is a σ-scattered network for open sets in X.

Let X be scattered-K-analytic. Due to Theorem 1 there is a complete sequence of σ-scattered covers $\mathcal{C}_n$ of X. As above we may suppose that $\mathcal{C}_n$ are disjoint and that $\mathcal{C}_{n+1}$ refines $\mathcal{C}_n$. Let $\mathcal{N} = \bigcup\limits_{n\in\mathbb{N}} \mathcal{N}_n$ be a network for the open sets of X and $\mathcal{N}_n$ be scattered.

We may put $\mathcal{N}_n^* = \mathcal{N}_n \cup \{X \setminus \bigcup \mathcal{N}_n\}$ and then $\mathcal{E}_n = \mathcal{N}_1^* \wedge \ldots \wedge \mathcal{N}_n^* \wedge \mathcal{C}_n$. As $\mathcal{C}_n$ form a complete sequence of covers of X, also $\mathcal{E}_n$ form a complete sequence. As $\mathcal{N}$ is a network for X, the families $\mathcal{N}_n$ are disjoint, and X is regular, the sequence of covers $\mathcal{E}_n$ of X is a complete sequence, which proves that X is scattered-analytic.

(b) If X is analytic, then there is a continuous map f of a separable complete metric space M onto X. The metric space M has a countable basis and the images of elements of such a basis form a network for the open sets in X. Since every discrete family in M is countable, the map f fulfils the assumptions on f of the definition of the scattered-analytic space and X is scattered-analytic and also scattered-K-analytic.

If X is scattered-K-analytic and if it has a σ-scattered network for open sets, then according to (a) above X is scattered-analytic. Hence there is a complete sequence of σ-scattered covers $\mathcal{C}_n$ of X which has the property from Definition 5 (b). Every scattered system of sets in a space that has a countable network is at most countable and so $\mathcal{C}_n$ is a complete sequence of countable covers which fulfils $\left|\bigcap_{n\in\mathbb{N}} \overline{C_1 \cap \ldots \cap C_n}\right| = 1$ for every centered sequence $C_n \in \mathcal{C}_n$. This is sufficient for the analyticity of a regular space (see e.g. the proof of [3, Theorem 9.3] or Remark 2 above and notice that the images of the corresponding usc-K map f are singletons if we construct it using our complete sequence of covers).

(c) If X is isolated-analytic, then it is clearly isolated-K-analytic, and hence also scattered-K-analytic. It can be easily verified that X has a σ-isolated network formed by the elements of the covers from the complete sequence of covers the existence of which is ensured by Theorem 2, (a′) implies (b′).

Let X be scattered-K-analytic and let X have a σ-isolated network. By (a) above, X is scattered-analytic. By Theorem 1, (a′) implies (b′), it is $\mathcal{S}_\sigma$-complete. The covers of the corresponding complete sequence of covers can be refined by σ-isolated covers due to Lemma 3.5 of [6]. In this way we obtain that X is $\mathcal{I}_\sigma$-complete and using Theorem 2, (b′) implies (a′), we have that X is isolated-analytic. □

Remark 7. We used, in (a), only the existence of a σ-scattered system $\mathcal{N}$ separating points in the sense that, for $x, y \in X$, $x \neq y$, there is an $N \in \mathcal{N}$ which contains exactly one element of x, y.

If X is regular and Hausdorff, then X is analytic if and only if it is K-analytic and has a countable network and also if and only if X is Čech-analytic and has a countable network. This follows from the fact that both K-analytic spaces are scattered-K-analytic by the definitions and Čech-analytic spaces are scattered-K-analytic by Theorem 4, and from Theorem 5 (b).

For K-analytic spaces this assertion is contained in [13, Theorem 5.5.1]. Our proof could be also done with the use of usc-K maps. The procedure which decomposes our proof to the parts (a) and (b) of Theorem 5 causes that the proof of Theorem 5, which is similar to the proof of [13, Theorem 5.5.1, (h) $\Rightarrow$ (a)], is in a sense "more straightforward". This is, roughly speaking, due to the fact that open subsets of a K-analytic space need not be K-analytic, but they are scattered-K-analytic.

For Čech-analytic spaces this assertion was shown to me by C. Stegall using essentially different arguments.

It should be noted that the assumption that X has a countable network is equivalent to the assumption that X is a continuous image of some separable metric space.

3. Fragmentability by a metric

In what follows we have in mind mainly the following examples. The space $X = C(K)$ of continuous functions on a compact space K or the space $X = C_b(T)$ of bounded continuous functions on a topological space T. In these cases we consider the uniform metric and the topology of pointwise convergence.

For most of the following results it is essential that one of the topologies is (completely) metrizable by a metric ϱ that is lower semi-continuous with respect to a weaker topology τ.

Our [8, Theorem 5 and Theorem 6] were based on the use of results that were not published yet or which were published in a different form ([12] and [6]). We give a different, perhaps more straightforward proof, and we improve the statements to include more general situations. Another characterizations of σ-fragmented spaces and sets are contained in [15] and [16]. Using Theorem 1, our Proposition 5 restricted to scattered-K-analytic completely regular spaces follows from [15, Theorem 5.2].

We formulate our propositions below for general collections $\mathcal{D}$ although the main examples are $\mathcal{S}$ and $\mathcal{I}$. We hope that the reasons of the validity of Theorem 6 become more transparent.

Definition 7. Let $\mathcal{D}$ be a collection of families of subsets of a topological space X and ϱ be a metric on X. Then X is called *$\mathcal{D}$-fragmented* by the metric ϱ if, for any $\varepsilon > 0$, X has a cover from $\mathcal{D}$ by sets of ϱ-diameter less than ε.

We say that X is *fragmented* by the metric ϱ if it is scattered-fragmented, i.e. $\mathcal{S}$-fragmented, by ϱ.

Remark 8. The notion *σ-fragmented* means σ-scattered fragmented, i.e. $\mathcal{S}_\sigma$-fragmented, and was introduced in [11].

We prove the following crucial proposition first.

Proposition 5. *Let $\mathcal{D}$ be a collection of families of subsets of a regular topological space X such that $\mathcal{D}$ has the property of unions, the property of cross-sections, the trace property, and the heredity property.*

Let X have the property of $\mathcal{D}_\sigma$ refinements, let X be $\mathcal{D}_\sigma$-K-complete, and ϱ be any lower semi-continuous metric on X.

If every compact subset of X is fragmented by ϱ, then X is $\mathcal{D}_\sigma$-fragmented by ϱ.

Remark 9. It is proved in [11, Corollary 3.1.1] that every compact (or even every hereditarily Baire) space K with the topology τ is σ-fragmented by a lower semi-continuous (with respect to τ) metric ϱ if and only if K is fragmented by ϱ.

Notice also that the claim of Proposition 5 assumes the fragmentability, and not $\mathcal{D}$-fragmentability, of compact subsets of X.

Lemma 2. *Let the assumptions of Proposition* 5 *be fulfilled. If there is no cover from $\mathcal{D}_\sigma$ of the topological space X by sets of ϱ-diameter less than ε, then there are distinct $x_0, x_1 \in X$ such that no open set containing at least one of x_0, x_1 has a cover from $\mathcal{D}_\sigma$ by sets of ϱ-diameter less than ε and $\varrho(x_0, x_1) \geqslant \varepsilon$.*

Moreover, there are open neighbourhoods U_0 and U_1 of x_0 and x_1 such that $\operatorname{dist}_\varrho(\overline{U_0}, \overline{U_1}) \geqslant \varepsilon/2$ and both U_0 and U_1 cannot be covered by a family from $\mathcal{D}_\sigma$ of sets having ϱ-diameter less than ε.

Proof. Let us consider a subspace X_0 of X obtained by omitting all points having a neighbourhood which has a cover from $\mathcal{D}_\sigma$ by sets of ϱ-diameter less than ε.

Now every point of $x \in X \setminus X_0$ has a neighbourhood U_x that has a cover $\mathcal{C}_x$ from $\mathcal{D}_\sigma$ by sets of ϱ-diameter less than ε.

Since X has the property of $\mathcal{D}_\sigma$ refinements and the cross-section property, there is a $\mathcal{D}_\sigma$ cover $\mathcal{E}$ of $X \setminus X_0$ that is a refinement of $\{U_x;\ x \in X \setminus X_0\}$.

Now every element E of $\mathcal{E}$ is a subset of some U_x. By the trace property, the family $\mathcal{H}_E = \{E \cap C;\ C \in \mathcal{C}_x\}$ is in $\mathcal{D}_\sigma$ and it covers E. So, using the property of unions, the family $\bigcup\{\mathcal{H}_E;\ E \in \mathcal{E}\}$ is in $\mathcal{D}_\sigma$, covers $X \setminus X_0$, and its elements have ϱ-diameter less than ε.

Therefore X_0 is nonempty and every its nonempty open subset has no cover from $\mathcal{D}_\sigma$ by sets of ϱ-diameter less than ε.

Therefore there are $x_0, x_1 \in X_0$ such that $\varrho(x_0, x_1) \geqslant \varepsilon$ and no open subset of X which contains at least one of x_0, x_1 has a $\mathcal{D}_\sigma$ cover by sets of ϱ-diameter less than ε.

Since ϱ is lower semi-continuous, we find neighbourhoods V_0, V_1 of x_0, x_1 such that $\varrho(y_0, y_1) > \varepsilon/2$ if $y_0 \in V_0$ and $y_1 \in V_1$. We choose neighbourhoods U_0 and U_1 of x_0 and x_1 such that their closures are contained in V_0 or V_1, respectively. Here we use the regularity of X. □

Proof of Proposition 5. Let $\mathcal{C}_n$ be a complete sequence of covers of X from $\mathcal{D}_\sigma$ such that every cover in this sequence refines the preceding ones. This is possible due to our assumption on $\mathcal{D}$. Let us suppose that X is not $\mathcal{D}_\sigma$-fragmented by the metric ϱ. Then there is a positive ε such that there is no cover of X from $\mathcal{D}_\sigma$ by sets of ϱ-diameter less than ε. We fix such an ε for the rest of our proof.

Using induction, we find, for $(i_1, \ldots, i_n) \in \{0,1\}^n$, $n \in \mathbb{N}$, sets $C_{i_1,\ldots,i_n} \in \mathcal{C}_n$ and open sets $U_{i_1,\ldots,i_n} \subset X$ such that:

(i) $\overline{U_{i_1,\ldots,i_n,i_{n+1}}} \subset U_{i_1,\ldots,i_n}$;

(ii) $\operatorname{dist}(\overline{U_{i_1,\ldots,i_n,0}}, \overline{U_{i_1,\ldots,i_n,1}}) \geqslant \varepsilon/2$;

(iii) $C_{i_1,\ldots,i_n,i_{n+1}} \subset C_{i_1,\ldots,i_n}$;

(iv) $C_{i_1,\dots,i_n} \cap U_{i_1,\dots,i_n}$ does not have a cover from $\mathcal{D}_\sigma$ by sets of ϱ-diameter less than ε.

To this end we proceed as follows. Using Lemma 2, we find U_0 and U_1. As $\mathcal{C}_1$ is in $\mathcal{D}_\sigma$, we may find by the trace property and the property of unions some $C_0, C_1 \in \mathcal{C}_1$ such that $C_{i_1} \cap U_{i_1}$ has no cover from $\mathcal{D}_\sigma$ by sets of diameter less than ε for $i_1 = 0, 1$. In the next steps we proceed similarly, with the difference that instead of X we consider $C_{i_1,\dots,i_n} \cap U_{i_1,\dots,i_n}$ to get $C_{i_1,\dots,i_n,0}, C_{i_1,\dots,i_n,1}$ and $U_{i_1,\dots,i_n,0}, U_{i_1,\dots,i_n,1}$, using Lemma 2, such that (i) to (iv) are satisfied in the corresponding form.

We consider the set $K = \bigcup_{i\in\{0,1\}^{\mathbb{N}}} \bigcap_{n\in\mathbb{N}} \overline{C_{i_1,\dots,i_n} \cap U_{i_1,\dots,i_n}}$. Due to the fact that the sequence $\mathcal{C}_n$ is complete and $C_{i_1,\dots,i_n} \cap U_{i_1,\dots,i_n}$ is a monotone sequence of non-empty sets, we know that $\overline{C_{i_1,\dots,i_n} \cap U_{i_1,\dots,i_n}}$ converge to a compact set $K(i) = \bigcap_{n=1}^{\infty} \overline{C_{i_1,\dots,i_n} \cap U_{i_1,\dots,i_n}}$ for every $i \in \{0,1\}^{\mathbb{N}}$. From (i) we get that $K(i) \subset U_{i_1,\dots,i_n}$ for all $n \in \mathbb{N}$. According to (ii) we get that $\operatorname{dist}(K(i), K(i')) \geqslant \varepsilon/2$ for $i \neq i'$, $i, i' \in \{0,1\}^{\mathbb{N}}$. Thus we may define a map $p\colon K \to \{0,1\}^{\mathbb{N}}$ by the prescription $p(x) = i$ if $x \in K(i)$ and p is a continuous map of K onto $\{0,1\}^{\mathbb{N}}$. The map which assigns $K(i)$ to $i \in \{0,1\}^{\mathbb{N}}$ is usc-K since, for every open set U containing $K(i)$, there is an $n \in \mathbb{N}$ such that $K(i) \subset \overline{C_{i_1,\dots,i_n} \cap U_{i_1,\dots,i_n}} \subset U$. Thus $K(j) \subset U$ if $j|n = i|n$. So K is compact as the image of the compact space $\{0,1\}^{\mathbb{N}}$ under an usc-K map. We finish the proof of Proposition 5 using the following lemma. □

Lemma 3 ([11], Lemma 4.4). *If there is a continuous map p of a compact space K onto $\{0,1\}^{\mathbb{N}}$ such that $\operatorname{dist}(p^{-1}(i), p^{-1}(i')) \geqslant \delta$ for some positive δ and all pairs of distinct $i, i' \in \{0,1\}^{\mathbb{N}}$, then K is not fragmented.*

Proof. The proof of this lemma is easy. It is enough to consider $p\colon K_0 \to \{0,1\}^{\mathbb{N}}$, where K_0 is some minimal compact subset of K with respect to the inclusion for which $p(K_0) = \{0,1\}^{\mathbb{N}}$. Any nonempty open subset of K_0 must then contain the preimage of some nonempty open subset of $\{0,1\}^{\mathbb{N}}$. □

Using Lemma 3, we see that K is not fragmented which is a contradiction to our assumptions, and this also finishes the proof of Proposition 5. □

Notice that, due to the characterizations by complete sequences of covers that we gave above, the preceding proposition can be applied to special classes of generalized analytic spaces.

First of all we draw our attention to the question which $\mathcal{D}_\sigma$-fragmented spaces are $\mathcal{D}$-K-analytic (or $\mathcal{D}_\sigma$-K-complete).

Remark 10. Let us recall that the following properties of a metric space X are equivalent (discrete means discrete in the metric of X).

(a) X is a Souslin subset of its completion;
(b) X is σ-discrete-complete;
(c) X is $\mathcal{S}_\sigma$-complete.

It can be verified e.g. as follows.

Let (a) be fulfilled. By [10, Theorem 2], mentioned already in Remark 6 above, X is $\mathcal{S}$-K-analytic. Due to the metrizability of X there is a σ-discrete, and so σ-scattered, network and the space X is $\mathcal{S}$-analytic by Theorem 5 (a). Theorem 1 ((a′) implies (b′)) shows that X is $\mathcal{S}_\sigma$-complete. So (a) implies (c).

Assume now that (c) holds. Notice that every family of subsets of X from $\mathcal{S}_\sigma$ is point-finite and has a σ-scattered network (or "base" in the terminology of [8]). Such a family is "sb$_d$-σ-decomposable" in the terminology of [8] by [8, Lemma 2], and as such it is discrete σ-decomposable in the metric space X by [8, Lemma 3]. Thus it has a σ-discrete refinement. Replacing now each element of a complete sequence of covers from $\mathcal{S}_\sigma$ by its σ-discrete refinement, we get the validity of (b).

Finally, every σ-discrete-complete metric space is $\mathcal{I}_\sigma$-complete since every discrete family is isolated. Using Theorem 2, (b′) implies (a′), X is $\mathcal{I}$-analytic. By Theorem 4, X is Čech-analytic and so it is in Souslin (Borel) in some compactification of its completion and thus X is Souslin in its completion, and we have proved that (b) implies (a).

Proposition 6. *Let $\mathcal{D}$ fulfil the heredity property and the cross-section property. Let (X,τ) be $\mathcal{D}_\sigma$-fragmented by the metric ϱ. Let the metric topology defined by ϱ be finer than τ and (X,ϱ) be Souslin in its completion. Then (X,τ) is $\mathcal{D}_\sigma$-K-complete.*

If ϱ is moreover lower semi-continuous in τ, then (X,τ) is $\mathcal{D}_\sigma$-complete.

P r o o f. Due to Remark 10, there is a complete sequence of σ-discrete covers $\mathcal{H}_n$ of the metric space (X,ϱ). Thus we can write $\mathcal{H}_n = \bigcup_{k\in\mathbb{N}} \mathcal{H}_n(k)$, where $\mathcal{H}_n(k)$ is $\varepsilon_n(k)$-discrete in (X,ϱ), with $\varepsilon_n(k) > 0$. As X is $\mathcal{D}_\sigma$-fragmented, there are partitions $\mathcal{C}_n(k) \in \mathcal{D}_\sigma$ of (X,τ) to sets of ϱ-diameter less than $\varepsilon_n(k)$. Thus the family $\mathcal{C}_n(k) \wedge \mathcal{H}_n(k)$ is a refinement of $\mathcal{C}_n(k)$ such that for every element N of the partition $\mathcal{C}_n(k)$ there is at most one nonempty element of $\mathcal{C}_n(k) \wedge \mathcal{H}_n(k)$ which is a subset of N. Therefore $\mathcal{C}_n(k) \wedge \mathcal{H}_n(k)$ is $\mathcal{D}_\sigma$ in τ. Hence, the collection $\mathcal{E}_n = \bigcup_{k\in\mathbb{N}} \mathcal{C}_n(k) \wedge \mathcal{H}_n(k)$ is in $\mathcal{D}_\sigma$.

Let now $\mathcal{F}$ be a filter containing some $E_n \in \mathcal{E}_n$ for every $n \in \mathbb{N}$. Then $\bigcap_{F\in\mathcal{F}} \overline{F}^\varrho \neq \emptyset$ because $\mathcal{H}_n$ form a complete sequence of covers of (X,ϱ). Since the τ-closures contain the ϱ-closures, we have obviously that $\bigcap_{F\in\mathcal{F}} \overline{F}^\tau \neq \emptyset$ and the sequence $\mathcal{E}_n$ is a complete sequence of $\mathcal{D}_\sigma$ covers of (X,τ).

We may suppose that each element of $\mathcal{H}_n$ has ϱ-diameter at most $1/n$. If ϱ is lower semi-continuous in τ, the τ-closure of each $H \in \mathcal{H}_n$ has ϱ-diameter at most $1/n$. So, obviously, $\mathcal{H}_n$ form a complete sequence of $\mathcal{D}_\sigma$ covers of X, witnessing that (X,τ) is $\mathcal{D}_\sigma$-complete. □

Now we show that the assumption on X to be Souslin in its completion in Theorem 6 below is necessary.

Proposition 7. *Let (X,τ) be a regular topological space. Let ϱ be a lower semi-continuous metric on (X,τ) and let the topology induced by ϱ on X be finer than τ. Let $\mathcal{D}^\tau$ be a collection of families of subsets of (X,τ) and $\mathcal{D}^\varrho$ a collection of families of subsets of (X,ϱ) such that both $\mathcal{D}^\tau$ and $\mathcal{D}^\varrho$ satisfy the cross-section property and $\mathcal{D}^\tau \subset \mathcal{D}^\varrho$. Let (X,τ) be $\mathcal{D}^\tau_\sigma$-K-complete and let (X,τ) be $\mathcal{D}^\tau_\sigma$-fragmented by ϱ. Then (X,ϱ) is $\mathcal{D}^\varrho_\sigma$-complete.*

P r o o f. As (X,τ) is $\mathcal{D}^\tau_\sigma$-K-complete, there is a complete sequence of $\mathcal{D}^\tau_\sigma$ covers $\mathcal{E}_n$ of (X,τ). As (X,τ) is $\mathcal{D}^\tau_\sigma$-fragmented, there is a sequence of $\mathcal{D}^\tau_\sigma$ covers $\mathcal{C}_n$ of X such that $\operatorname{diam}_\varrho C \leqslant 1/n$ for each element C of $\mathcal{C}_n$. Now $\mathcal{P}_n = \mathcal{E}_n \wedge \mathcal{C}_n \in \mathcal{D}^\tau$ since $\mathcal{D}^\tau$ has the cross-section property. The covers $\mathcal{P}_n$ of X form a complete sequence of (X,τ) since they refine the covers $\mathcal{E}_n$ that form a complete sequence of covers of (X,τ). Let us consider a centered sequence of elements $P_n \in \mathcal{P}_n$ and denote $Q_n = P_1 \cap \ldots \cap P_n$ for each $n \in \mathcal{N}$. By the completeness of $\mathcal{P}_n$ in the regular space (X,τ) we get that the sequence of $\overline{Q_n}$'s converges to the compact set $K = \bigcap_{n=1}^{\infty} \overline{Q_n}^{(X,\tau)} \subset X$ in (X,τ). Using that ϱ is lower semi-continuous in (X,τ), we deduce that $\operatorname{diam}_\varrho \overline{Q_n}^{(X,\tau)} \leqslant \operatorname{diam}_\varrho \overline{P_n}^{(X,\tau)} \leqslant \operatorname{diam}_\varrho P_n \leqslant 1/n$. Hence $K = \{x\} \subset X$ for some $x \in X$. Now $\operatorname{diam}_\varrho \overline{Q_n}^{(Y,\varrho)} \leqslant \operatorname{diam}_\varrho Q_n \leqslant \operatorname{diam}_\varrho P_n \leqslant 1/n$ where (Y,ϱ) is a completion of (X,ϱ). Since $\overline{Q_n}^{(Y,\varrho)}$ form a decreasing sequence of closed nonempty subsets of the complete metric space (Y,ϱ), and the diameters of $\overline{Q_n}^{(Y,\varrho)}$'s tend to zero, we have $\bigcap_{n=1}^{\infty} \overline{Q_n}^{(Y,\varrho)} = \{y\} \subset Y$ and the sequence of $\overline{Q_n}^{(Y,\varrho)}$'s converges to $\{y\}$ in (Y,ϱ). As $\operatorname{diam}_\varrho \overline{Q_n}^{(Y,\varrho)} \cup \overline{Q_n}^{(X,\tau)}$ tends to zero and $\{x,y\} \subset \overline{Q_n}^{(Y,\varrho)} \cup \overline{Q_n}^{(X,\tau)}$ for every $n \in \mathcal{N}$, we get that $x = y \in X$ and $\mathcal{P}_n$ form a complete sequence of covers of (X,ϱ) which shows that (X,ϱ) is $\mathcal{D}^\varrho_\sigma$-complete because $\mathcal{D}^\tau \subset \mathcal{D}^\varrho$. □

Remark 11. We consider also the case of collections $\mathcal{D}^c$ of countable families of subsets of a space in the following theorem. Hence, the cases of K-analytic, analytic, and "countably fragmented" (i.e. $\mathcal{D}^c_\sigma$-fragmented) spaces are included. The property of $\mathcal{D}^c$ refinements of X is thus equivalent to X being hereditarily Lindelöf. Notice that $\mathcal{D}^c$-fragmentability by a finer metric gives that X has a countable network.

Theorem 6. *Let every compact subset of (X, τ) be fragmented by a lower semi-continuous metric ϱ giving a finer topology than the regular topology τ. Let (X, τ) have the property of $\mathcal{D}_\sigma$ refinements, where $\mathcal{D} = \mathcal{S}$, or $\mathcal{D} = \mathcal{I}$, or $\mathcal{D}^c$. Then the following are equivalent.*

(a) (X, τ) *is $\mathcal{D}$-analytic.*

(b) (X, τ) *is $\mathcal{D}$-K-analytic.*

(c) (X, τ) *is $\mathcal{D}_\sigma$-fragmented by ϱ and (X, ϱ) is Souslin in its completion.*

P r o o f. By Theorems 1 and 2, and [3, Theorem 9.3], with the same proof for analytic instead of K-analytic, (X, τ) is $\mathcal{D}_\sigma$-(K-)complete if and only if it is $\mathcal{D}$-(K-)analytic.

Obviously, (a) implies (b). Let (b) be satisfied. Then (X, τ) is $\mathcal{D}_\sigma$-fragmented due to Proposition 5. Using Proposition 7, we get that (X, ϱ) is $\mathcal{D}_\sigma^\varrho$-complete, and so it is Souslin in its completion due to Remark 10 above. Finally, using Proposition 6, we conclude that (c) implies (a). □

In the particular case of spaces $C(K)$ of continuous functions on a compact space, where the Namioka theorem [14, Corollary 4.2] says that every compact set with respect to the topology of pointwise convergence is fragmented by the supremum norm, we get

Corollary. *Let $X \subset C(K)$, where $C(K)$ is the set of all continuous functions on a compact space K, τ_p be the topology of pointwise convergence, and ϱ be the supremum metric on $C(K)$. Then the following are equivalent.*

(a) (X, τ) *is scattered-K-analytic.*

(b) (X, τ) *is scattered-analytic.*

(c) (X, ϱ) *is a Souslin subset of its completion and (X, τ) is σ-fragmented by ϱ.*

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