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STATE-HOMOMORPHISMS ON MV-ALGEBRAS

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Abstract. Riečan [12] and Chovanec [1] investigated states in MV-algebras. Earlier, Riečan [11] had dealt with analogous ideas in D-posets. In the monograph of Riečan and Neubrunn [13] (Chapter 9) the notion of state is applied in the theory of probability on MV-algebras.

We remark that a different definition of a state in an MV-algebra has been applied by Mundici [9], [10] (namely, the condition (iii) from Definition 1.1 above was not included in his definition of a state; in other words, only finite additivity was assumed).

Below we work with the definition from [13]; but, in order to avoid terminological problems we use the term "state-homomorphism" (instead of "state"). The author is indebted to the referee for his suggestion concerning terminology.

Let $\mathscr{A}$ be an MV-algebra which is defined on a set A with $\operatorname{card} A > 1$. In the present paper we show that there exists a one-to-one correspondence between the system of all state-homomorphisms on $\mathscr{A}$ and the system of all σ-closed maximal ideals of $\mathscr{A}$.

For MV-algebras we apply the notation and the definitions as in Gluschankof [3].

The relations between MV-algebras and abelian lattice ordered groups (cf. Mundici [8]) are substantially used in the present paper.

Keywords: MV-algebra, state homomorphism, σ-closed maximal ideal

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1. Preliminaries

We recall that an MV-algebra is an algebraic system

$$\mathscr{A} = (A; \oplus, *, \neg, 0, 1),$$

where A is a nonempty set, $\oplus$ and $*$ are binary operations, $\neg$ is a unary operation, and $0, 1$ are nulary operations on A such that the conditions (m_1)–(m_9) from [3] are satisfied.

Let us remark that in [1], [11] and [13] another system of axioms for an MV-algebra was applied. Both these systems are equivalent in a natural sense (for a formal description of this equivalence we can apply Marczewski's theory of weak automorphisms of algebraic systems; cf., e.g., Goetz [4]).

In what follows we assume that $\operatorname{card} A > 1$.

Let $x, y \in A$. We put

$$x \vee y = (x * \neg y) \oplus y, \quad x \wedge y = \neg(\neg x \vee \neg y).$$

Then (cf. Mundici [8]) we obtain that $(A; \vee, \wedge)$ is a distributive lattice with the least element 0 and the greatest element 1. This lattice will be denoted by $\ell(\mathscr{A})$.

Let X be a partially ordered set, $x \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \leqslant x_{n+1}$ for each $n \in \mathbb{N}$, and $\sup\{x_n\}_{n\in\mathbb{N}} = x$. Then we write $x_n \nearrow x$.

We denote by $\mathbb{R}$ the additive group of all reals with the natural linear order. For $x, y \in \mathbb{R}$ with $x \leqslant y$ let $[x, y]$ be the corresponding interval in $\mathbb{R}$.

1.1. Definition. Let $\mathscr{A}$ be as above. A state-homomorphism on $\mathscr{A}$ is a mapping $m \to [0, 1]$ which satisfies the following conditions:

(i) $m(1) = 1$.
(ii) If $a, b \in A$ and $a \leqslant \neg b$, then $m(a \oplus b) = m(a) + m(b)$.
(iii) If $a \in A$, $a_n \in A$ for $n \in \mathbb{N}$ and $a_n \nearrow a$, then $m(a_n) \nearrow m(a)$.

According to 9.1.6 and 9.1.7 in [13], the above definition of a state-homomorphism is equivalent to the definition of a state considered in [13]. (We remark that for $x \in A$ the symbol $\neg x$ has the same meaning as the symbol x^* in [13].)

The notion of a congruence relation on $\mathscr{A}$ has the usual meaning (i.e., it is a binary relation on the set A which is compatible with each of the operations $\oplus, *, \neg$).

The system of all congruence relations on $\mathscr{A}$ will be denoted by $\operatorname{Con}\mathscr{A}$; this system is partially ordered in the usual way.

Let $\varrho \in \operatorname{Con}\mathscr{A}$ and $x \in A$. Put $x(\varrho) = \{y \in A \colon y\varrho x\}$. The set $0(\varrho)$ is called an ideal of $\mathscr{A}$.

An ideal $0(\varrho)$ of $\mathscr{A}$ is called maximal if it satisfies the following conditions:

(i) Whenever $\varrho_1 \in \operatorname{Con}\mathscr{A}$ and $0(\varrho) \subseteq 0(\varrho_1) \neq A$, then $0(\varrho) = 0(\varrho_1)$.
(ii) $A \neq 0(\varrho)$.

A subset X of A is said to be σ-closed if, whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in X and a is an element of A such that either $\sup\{x_n\}_{n\in\mathbb{N}} = a$ or $\inf\{x_n\}_{n\in\mathbb{N}} = a$, then $a \in X$.

2. Factor MV-algebras

Let $\mathscr{A}$ be as above and let $\varrho \in \operatorname{Con} \mathscr{A}$. Then we can construct in the usual way the factor MV-algebra $\mathscr{A}/\varrho$ (cf., e.g., [7]). The algebraic system $\mathscr{A}/\varrho$ is an MV-algebra; let us denote its underlying set by A_1. The mapping $x \to x(\varrho)$ of A onto A_1 is a homomorphism of $\mathscr{A}$ onto $\mathscr{A}/\varrho$.

Let $\mathscr{B}$ be an MV-algebra and let φ be a homomorphism of $\mathscr{A}$ onto $\mathscr{B}$. For $x, y \in A$ we put $x\varrho_\varphi y$ if $\varphi(x) = \varphi(y)$. Then ϱ_φ is a congruence relation on $\mathscr{A}$ and the mapping f defined by

$$f(x(\varrho_\varphi)) = \varphi(x)$$

is an isomorphism of the MV-algebra $\mathscr{A}/\varrho_\varphi$ onto $\mathscr{B}$.

For lattice ordered groups we apply the notation and definitions as in [2].

Let G be an abelian lattice ordered group with a strong unit u. Then $\mathscr{A}_0(G, u)$ has the same meaning as in [5].

Without loss of generality we can suppose that $\mathscr{A} = \mathscr{A}_0(G, u)$ (cf. Mundici [8]).

For $\varrho \in \operatorname{Con} A$ we denote by $0(\varrho)_0$ the convex ℓ-subgroup of G which is generated by the set $0(\varrho)$. Further, let ϱ_0 be the congruence relation on G which is generated by the ℓ-ideal $0(\varrho)_0$.

2.1. Lemma. *Let $\varrho \in \operatorname{Con} \mathscr{A}$. Then the following conditions are equivalent:*
(i) $0(\varrho)$ *is a maximal ideal in* $\mathscr{A}$.
(ii) $0(\varrho)_0$ *is a maximal ℓ-ideal in* G.

Proof. This is a consequence of 1.10 in [7]. □

2.2. Lemma. *$\ell(\mathscr{A})$ is a chain if and only if G is linearly ordered.*

Proof. If G is linearly ordered, then it is clear that $\ell(\mathscr{A})$ is linearly ordered as well. If G is not linearly ordered, then there exist g_1 and g_2 in G such that $g_1 > 0$, $g_2 > 0$ and $g_1 \wedge g_2 = 0$. Put $a_1 = g_1 \wedge u$ $(i = 1, 2)$. Then $a_i \in A$, $a_i > 0$ $(i = 1, 2)$ and $a_1 \wedge a_2 = 0$, hence $\ell(\mathscr{A})$ is not linearly ordered. □

In what follows we often speak of $\mathscr{A}$ being linearly ordered meaning that $\ell(\mathscr{A})$ is linearly ordered.

2.3. Lemma. *Let $\varrho \in \operatorname{Con} \mathscr{A}$. Assume that $0(\varrho)$ is a maximal ideal in $\mathscr{A}$. Then the MV-algebra $\mathscr{A}/\varrho$ is linearly ordered.*

Proof. According to 2.1, $0(\varrho)_0$ is a maximal ℓ-ideal in G. Thus G/ϱ_0 is linearly ordered. Now 2.2 and [7], Proposition 2.4 yield that $\mathscr{A}/\varrho$ is linearly ordered. □

For the notion of an archimedean MV-algebra cf., e.g., [6].

2.4. Lemma. *Let ϱ be as in 2.3. Then the MV-algebra $\mathscr{A}/\varrho$ is archimedean.*

P r o o f. By way of contradiction, suppose that $\mathscr{A}/\varrho$ is not archimedean. Then in view of 2.4 in [7] the lattice ordered group G/ϱ_0 is not archimedean. Moreover, according to 2.2 and 2.3, G/ϱ_0 is linearly ordered. Then there exists an ℓ-ideal X in G/ϱ_0 such that $0(\varrho_0) \neq X \neq G/\varrho_0$. Thus the set

$$X_1 = \{x \in G\colon\ x(\varrho_0) \in X\}$$

is an ℓ-ideal in G with $0(\varrho)_0 \subset X_1 \neq G$. Hence $0(\varrho)_0$ is not a maximal ℓ-ideal in G, which contradicts 2.1. □

2.5. Lemma. *Let ϱ be as in 2.3. Then the lattice ordered group G/ϱ_0 is isomorphic to an ℓ-subgroup of the linearly ordered group $\mathbb{R}$.*

P r o o f. It is well-known that each archimedean linearly ordered group is isomorphic to an ℓ-subgroup of $\mathbb{R}$. In the proof of 2.3 we have observed that G/ϱ_0 is linearly ordered. Moreover, the argument performed in the proof of 2.4 shows that G/ϱ_0 is archimedean. □

If ϱ is as in 2.3, then in view of 2.5 and [7], Proposition 2.4 there exists an ℓ-subgroup $\mathbb{R}_1$ of $\mathbb{R}$ and an element $0 < v \in \mathbb{R}_1$ such that $\mathscr{A}/\varrho$ is isomorphic to $\mathscr{A}_0(\mathbb{R}_1, v)$.

It is clear that $\mathscr{A}_0(\mathbb{R}_1, v)$ is a subalgebra of $\mathscr{A}_0(\mathbb{R}, v)$. Further, for each element $v_1 \in \mathbb{R}$ with $v_1 > 0$, the MV-algebra $\mathscr{A}_0(\mathbb{R}, v)$ is isomorphic to $\mathscr{A}_0(\mathbb{R}, v_1)$. In particular, we can put $v_1 = 1$. Thus we obtain

2.6. Lemma. *Let ϱ be as in 2.3. Then there exists an isomorphism ψ of $\mathscr{A}/\varrho$ into the MV-algebra $\mathscr{A}_0(\mathbb{R}, 1)$.*

2.7. Lemma. *Let ϱ be as in 2.3 and let ψ be as in 2.6. Then the following conditions are fulfilled:*

(i_1) $\psi(u) = 1$.

(ii_1) *If $a, b \in A$ and $a \leqslant \neg b$, then $\psi(a \oplus b) = \psi(a) \oplus \psi(b)$.*

P r o o f. The relation (i_1) is an immediate consequence of the fact that ψ is an isomorphism. Let $a, b \in A$ and $a \leqslant \neg b$. The isomorphism ψ yields that $\psi(a) \leqslant \neg\psi(b)$. Since $\neg\psi(b) = 1 - \psi(b)$, we obtain that $\psi(a) + \psi(b) \leqslant 1$, whence

$$\psi(a) \oplus \psi(b) = \psi(a) + \psi(b).$$

Further, in view of 2.6 we have $\psi(a \oplus b) = \psi(a) + \psi(b)$, thus ($\mathrm{ii}_1$) holds. □

2.8. Lemma. *Let ϱ be as in 2.3. Assume that the ℓ-ideal $0(\varrho)$ is σ-closed. Then the following condition is valid:*

(iii$_1$) *If $a_n \in A$ for each $n \in \mathbb{N}$, $a \in A$ and $a_n \nearrow a$, then $a_n(\varrho) \nearrow a(\varrho)$.*

Proof. It is easy to verify that for each $x \in A$, the set $x(\varrho)$ is σ-closed. Let $a_n \nearrow a$. Then $a_n(\varrho) \leqslant a_{n+1}(\varrho) \leqslant a(\varrho)$ for each $n \in \mathbb{N}$. We have to show that

$$\bigvee_{n\in\mathbb{N}} a_n(\varrho) = a(\varrho) \tag{1}$$

is valid in $\mathscr{A}/\varrho$. By way of contradiction, suppose that (1) fails to hold. Thus there is $b \in A$ such that $a_n(\varrho) \leqslant b(\varrho)$ for each $n \in \mathbb{N}$ and $b(\varrho) < a(\varrho)$. We have $a \wedge b \in b(\varrho)$, thus without loss of generality we can suppose that $b \leqslant a$. Then

$$(a_n \vee b) \wedge a \nearrow (a \vee b) \wedge a = a \tag{2}$$

is valid in $\mathscr{A}$ and

$$(a_n \vee b) \wedge a \in b(\varrho)$$

for each $n \in \mathbb{N}$. Since $b(\varrho)$ is σ-closed we obtain from (2) that the element a belongs to $b(\varrho)$, which is a contradiction. □

The mapping ψ considered above was constructed by means of ϱ. Let us now write ψ_φ instead of ψ.

From 2.6, 2.7 and 2.8 we obtain

2.9. Proposition. *Let $\varrho \in \operatorname{Con}\mathscr{A}$. Suppose that the ideal $0(\varrho)$ of $\mathscr{A}$ is maximal and σ-closed. Then the mapping ψ_ϱ is a state-homomorphism in $\mathscr{A}$.*

3. Maximal ideal corresponding to a state-homomorphism

Suppose that m is a state-homomorphism on the MV-algebra $\mathscr{A}$. Let G be as above.

We define a partial binary operation $-$ on A as follows. If $a_1, a_2 \in A$ and $a_1 \leqslant a_2$, then $a_2 - a_1$ in A has the same meaning as $a_2 - a_1$ in G; otherwise, $a_2 - a_1$ is not defined in A.

From 9.16 and 9.1.7 in [13] we obtain

3.1. Lemma. *If $a, b \in A$ and $a \leqslant b$, then $m(b - a) = m(b) - m(a)$.*

Similarly as in the preceding section we consider the interval $[0, 1]$ of $\mathbb{R}$ as the underlying set of the MV-algebra $\mathscr{B} = \mathscr{A}_0(\mathbb{R}, 1)$.

Put $B_1 = m(A)$. In view of 3.1 and according to Proposition 3.1 of [3] we have

3.2. Lemma.
(i) *B_1 is an underlying set of a subalgebra $\mathscr{B}_1$ of $\mathscr{B}$;*
(ii) *m is a homomorphism of $\mathscr{A}$ onto $\mathscr{B}_1$.*

We remark that the corresponding proof in [1] is performed by using different set of operations on an MV-algebra than we are applying in the present paper, but the notions of a congruence relation and of a homomorphism in both settings are the same.

Consider the congruence relation ϱ_m on A which is defined by means of the homomorphism m (cf. Section 2 above). Since $\mathscr{A}/\varrho_m$ is isomorphic to $\mathscr{B}_1$, we obtain

3.3. Lemma. *$\mathscr{A}/\varrho_m$ is linearly ordered and archimedean.*

Thus according to 2.2 and [7], Proposition 2.4 we have

3.4. Lemma. *$G/(\varrho_m)_0$ is linearly ordered and archimedean.*

From 3.4 we infer that $G/(\varrho_m)_0$ has no non-trivial ℓ-ideal. This yields that the ℓ-ideal $0((\varrho_m)_0)$ of G is maximal. Then 2.1 yields

3.5. Lemma. *$0(\varrho_m)$ is a maximal ideal of $\mathscr{A}$.*

3.6. Lemma. *$0(\varrho_m)$ is a σ-closed subset of A.*

P r o o f. a) Let (x_n) be a sequence in $0(\varrho_m)$, $x \in A$ and suppose that the relation

$$\bigvee_{n\in\mathbb{N}} x_n = x$$

is valid in $\mathscr{A}$. Denote $y_n = x_1 \vee x_2 \vee \ldots \vee x_n$ for each $n \in \mathbb{N}$. Then $y_n \leqslant y_{n+1}$ for each $n \in \mathbb{N}$ and

$$\bigvee_{n\in\mathbb{N}} y_n = x,$$

whence $y_n \nearrow x$ in $\mathscr{A}$. Since m is a state-homomorphism on $\mathscr{A}$ we obtain $m(y_n) \nearrow m(x)$. Clearly $y_n \in 0(\varrho_m)$, thus $m(y_n) = 0$ for each $n \in \mathbb{N}$ and hence $m(x) = 0$. Therefore $x \in 0(\varrho_m)$.

b) Let (z_n) be a sequence in $0(\varrho_m)$, $z \in A$. Assume that

$$\bigwedge_{n\in\mathbb{N}} z_n = z$$

holds in $\mathscr{A}$. Then $0 \leqslant z \leqslant z_n$ for each $n \in \mathbb{N}$. Since $0(\varrho_m)$ is a convex sublattice of $\ell(\mathscr{A})$ and $0 \in 0(\varrho_m)$ we obtain $z \in 0(\varrho_m)$. $\square$

3.7. Lemma. *Let G_1 and G_2 be ℓ-subgroups of $\mathbb{R}$ such that $0, 1 \in G_i$ for $i = 1, 2$. Assume that φ is an isomorphism of G_1 onto G_2 with $\varphi(1) = 1$. Then $G_1 = G_2$ and φ is the identity on G_1.*

P r o o f. By way of contradiction, suppose that φ fails to be the identical mapping on G_1. Hence there is $0 < x \in G_1$ such that $\varphi(x) = y \neq x$. Then there exist positive integers n and m such that either (i) $mx < n < my$, or (ii) $my < n < mx$. Suppose that (i) holds. Then $\varphi(mx) < \varphi(n)$. Clearly $\varphi(mx) = my$, $\varphi(n) = n$, whence $my < n$, which is a contradiction. The case (ii) is analogous. □

3.8. Lemma. *Let G_1 and G_2 be ℓ-subgroups of $\mathbb{R}$ such that $0, 1 \in G_i$ for $i = 1, 2$. Put $\mathscr{A}_0 = \mathscr{A}_0(G_1, 1)$, $\mathscr{A}_2 = \mathscr{A}_0(G_2, 1)$. Suppose that φ_0 is an isomorphism of $\mathscr{A}_1$ onto $\mathscr{A}_2$. Then φ_0 is the identical mapping on $\mathscr{A}_1$.*

P r o o f. From the fact that φ_0 is an isomorphism of $\mathscr{A}_1$ onto $\mathscr{A}_2$ we easily obtain that there exists an isomorphism φ of G_1 onto G_2 such that $\varphi(x) = \varphi_0(x)$ for each $x \in A_1$. In particular, we have $\varphi(1) = 1$. Then it suffices to apply 3.7. □

3.9. Lemma ([7], Lemma 1.11). *Let ϱ_1 and ϱ_2 be congruence relations on $\mathscr{A}$ such that $0(\varrho_1) = 0(\varrho_2)$. Then $\varrho_1 = \varrho_2$.*

Let us denote by

$\mathscr{I}$—the set of all σ-closed maximal ideals of $\mathscr{A}$;

$\mathscr{S}$—the set of all state-homomorphisms on $\mathscr{A}$.

Consider a mapping $f_1\colon \mathscr{I} \to \mathscr{S}$ defined by

$$f_1(X) = \psi_\varrho$$

for each $X \in \mathscr{I}$, where ϱ is a congruence relation on $\mathscr{A}$ with $0(\varrho) = X$ (cf. 2.9 and 3.9).

Further, let f_2 be the mapping of $\mathscr{S}$ into $\mathscr{I}$ such that

$$f_2(m) = 0(\varrho_m)$$

for each $m \in \mathscr{S}$ (cf. 3.5 and 3.6).

From the construction of ψ_ϱ we immediately obtain

$$f_2(f_1(X)) = X$$

for each $X \in \mathscr{I}$.

Also, 3.8 and the definition of f_2 yield

$$f_1(f_2(m)) = m$$

for each $m \in M$.

Hence we have

3.10. Theorem. *Under the notation as above, f_1 is a bijection of $\mathscr{I}$ onto $\mathscr{S}$ and $f_2 = f_1^{-1}$.*

The above results show that state-homomorphisms on the MV-algebra $\mathscr{A}$ can be viewed—up to isomorphism—as mappings of the form

$$a \to a \oplus 0(\varrho) \quad (a \in A),$$

where $0(\varrho)$ is a σ-closed maximal ideal of $\mathscr{A}$.

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