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EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF HIGHER ORDER NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The higher order neutral functional differential equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big[x(t)+h(t)x(\tau(t))\big]+\sigma f\big(t,x(g(t))\big)=0 \tag{1}$$

is considered under the following conditions: $n\geqslant 2$, $\sigma=\pm1$, $\tau(t)$ is strictly increasing in $t\in[t_0,\infty)$, $\tau(t)<t$ for $t\geqslant t_0$, $\lim\limits_{t\to\infty}\tau(t)=\infty$, $\lim\limits_{t\to\infty}g(t)=\infty$, and $f(t,u)$ is nonnegative on $[t_0,\infty)\times(0,\infty)$ and nondecreasing in $u\in(0,\infty)$. A necessary and sufficient condition is derived for the existence of certain positive solutions of (1).

Keywords: neutral differential equation, positive solution

MSC 2000: 34K11

1. Introduction

In this paper we consider the higher order neutral functional differential equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big[x(t)+h(t)x(\tau(t))\big]+\sigma f\big(t,x(g(t))\big)=0, \tag{1}$$

where $n\geqslant 2$ and $\sigma=+1$ or -1. It is assumed throughout this paper that

(a) $t_0>0$, $\tau\colon[t_0,\infty)\longrightarrow\mathbb{R}$ is continuous and strictly increasing in $t\in[t_0,\infty)$, $\tau(t)<t$ for $t\geqslant t_0$, and $\lim\limits_{t\to\infty}\tau(t)=\infty$;

(b) $h\colon[\tau(t_0),\infty)\longrightarrow\mathbb{R}$ is continuous;

(c) $g\colon[t_0,\infty)\longrightarrow\mathbb{R}$ is continuous, $g(t)>0$ for $t\geqslant t_0$ and $\lim\limits_{t\to\infty}g(t)=\infty$;

(d) $f\colon [t_0,\infty)\times(0,\infty)\longrightarrow\mathbb{R}$ is continuous, $f(t,u)\geqslant 0$ for $(t,u)\in[t_0,\infty)\times(0,\infty)$, and $f(t,u)$ is nondecreasing in $u\in(0,\infty)$ for each fixed $t\in[t_0,\infty)$.

By a solution of (1) we mean a function $x(t)$ which is continuous and satisfies (1) on $[t_x,\infty)$ for some $t_x\geqslant t_0$.

There has been an increasing interest in studying the existence of positive solutions of higher order neutral differential equations. We refer the reader to [1]–[17], [19]–[21]. In particular, the following result is known:

Theorem 0. *Let $k\in\{0,1,2,\dots,n-1\}$. Suppose that one of the following conditions* (i)–(iii) holds:

(i) $|h(t)|[\tau(t)/t]^k\leqslant\lambda<1$ and $h(t)h(\tau(t))\geqslant 0$ ([17]);
(ii) $h(t)\equiv 1$ and $\tau(t)=t-\tau$ $(\tau>0)$ ([11]);
(iii) $1<\mu\leqslant h(t)[\tau(t)/t]^k\leqslant\lambda<\infty$ ([17]).

Then (1) has a solution $x(t)$ satisfying

$$0<\liminf_{t\to\infty}\frac{x(t)}{t^k}\leqslant\limsup_{t\to\infty}\frac{x(t)}{t^k}<\infty \tag{2}$$

if and only if

$$\int_{t_0}^{\infty} t^{n-k-1}f\bigl(t,a[g(t)]^k\bigr)\,\mathrm{d}t<\infty\quad\text{for some } a>0. \tag{3}$$

However, very little is known about the existence of a solution $x(t)$ of (1) satisfying (2) in other cases, such as

$$\liminf_{t\to\infty}h(t)\Bigl[\frac{\tau(t)}{t}\Bigr]^k<1<\limsup_{t\to\infty}h(t)\Bigl[\frac{\tau(t)}{t}\Bigr]^k. \tag{4}$$

The condition (4) seems to be natural and important. Nevertheless, it is not difficult to construct an example illustrating that, while (4) is satisfied, (1) has no solution $x(t)$ with the property (2). Thus we need a condition different from (4).

In this paper we consider the following case:

$$\begin{cases} h(t)\Bigl[\dfrac{\tau(t)}{t}\Bigr]^k>-1,\\[2ex] h(\tau(t))\Bigl[\dfrac{\tau(\tau(t))}{\tau(t)}\Bigr]^k=h(t)\Bigl[\dfrac{\tau(t)}{t}\Bigr]^k,\quad t\geqslant\tau^{-1}(t_0),\end{cases} \tag{5}$$

where $\tau^{-1}(t)$ is the inverse function of $\tau(t)$ and $k\in\{0,1,\dots,n-1\}$. We note here that if (5) holds, then there are constants μ and λ such that

$$-1<\mu\leqslant h(t)\Bigl[\frac{\tau(t)}{t}\Bigr]^k\leqslant\lambda,\quad t\geqslant t_0. \tag{6}$$

(As a general result it is verified that, under the hypothesis (a) on $\tau(t)$, if a continuous function $\varphi(t)$ on $[t_0, \infty)$ satisfies $\varphi(t) > -1$ and $\varphi(\tau(t)) = \varphi(t)$ for $t \geqslant \tau^{-1}(t_0)$, then there are constants μ and λ such that $-1 < \mu \leqslant \varphi(t) \leqslant \lambda$ for $t \geqslant t_0$.) In the case of $k \in \{0, 1, 2, \ldots, n-1\}$, we easily see that

$$x(t) = \frac{b\,t^k}{1 + h(t)[\tau(t)/t]^k} \quad (b > 0)$$

satisfies (2) and is a solution of the unperturbed equation

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big[x(t) + h(t)x\big(\tau(t)\big)\big] = 0,$$

and so it is natural to expect that, if f is small enough in some sense, (1) has a solution $x(t)$ which behaves like the function $b\,t^k\big[1 + h(t)[\tau(t)/t]^k\big]^{-1}$ as $t \to \infty$. In fact, the following theorem will be proved.

Theorem 1. *Let $k \in \{0, 1, 2, \ldots, n-1\}$. Suppose that (5) holds. Then (1) has a solution $x(t)$ satisfying*

$$x(t) = \left[\frac{b}{1 + h(t)\big[\tau(t)/t\big]^k} + o(1)\right]t^k \quad \text{as } t \to \infty \quad \text{for some } b > 0 \tag{7}$$

if and only if (3) holds.

In particular, for the case $k = 0$, Theorem 1 gives the following

Corollary 1. *Suppose that*

$$h(t) > -1 \quad \text{and} \quad h(\tau(t)) = h(t), \quad t \geqslant \tau^{-1}(t_0). \tag{8}$$

Then (1) has a solution $x(t)$ satisfying

$$x(t) = \frac{b}{1 + h(t)} + o(1) \quad \text{as } t \to \infty \quad \text{for some } b > 0$$

if and only if

$$\int_{t_0}^{\infty} t^{n-1} f(t, a)\,\mathrm{d}t < \infty \quad \text{for some } a > 0.$$

Remark 1. Pairs of functions

$$\begin{aligned}
\tau(t) =& t - 2\pi, & h(t) &= 1 + \tfrac{3}{2}\sin t,\\
\tau(t) =& \gamma t, & h(t) &= 1 + \tfrac{3}{2}\sin(2\pi[\log\gamma]^{-1}\log t) \quad (0 < \gamma < 1),\\
\tau(t) =& t^{1/e}, & h(t) &= 1 + \tfrac{3}{2}\sin(2\pi\log(\log t)) \quad (t_0 > 1)
\end{aligned}$$

give typical examples satisfying (8).

Now let us consider the special case $\tau(t) = \gamma t$ $(0 < \gamma < 1)$:

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}[x(t) + h(t)x(\gamma t)] + \sigma f\big(t, x(g(t))\big) = 0. \tag{9}$$

Applying Theorem 1 to equation (9), we obtain the following result.

Corollary 2. *Let $k \in \{0, 1, 2, \ldots, n-1\}$ and $0 < \gamma < 1$. Suppose that*

$$h(t) > -\gamma^{-k} \quad \text{and} \quad h(\gamma t) = h(t), \quad t \geqslant \gamma^{-1} t_0.$$

Then (9) *has a solution $x(t)$ satisfying*

$$x(t) = \left[\frac{b}{1 + \gamma^k h(t)} + o(1)\right] t^k \quad \text{as } t \to \infty \quad \text{for some } b > 0$$

if and only if (3) *holds.*

2. Proof of Theorem 1

First we prove the "only if" part of Theorem 1. The following lemma is a more general result.

Lemma 1. *Let $k \in \{0, 1, 2, \ldots, n-1\}$. Suppose that $h(t)[\tau(t)/t]^k$ is bounded on $[t_0, \infty)$. If there exists a solution $x(t)$ of* (1) *which satisfies* (2), *then* (3) *holds.*

Proof. Put $y(t) = x(t) + h(t)x(\tau(t))$. We get

$$\frac{y(t)}{t^k} = \frac{x(t)}{t^k} + h(t)\left[\frac{\tau(t)}{t}\right]^k \frac{x(\tau(t))}{[\tau(t)]^k}, \tag{10}$$

which implies that $y(t)/t^k$ is bounded. From (1) we have

$$\sigma y^{(n)}(t) = -f\big(t, x(g(t))\big) \leqslant 0 \quad \text{for all large } t. \tag{11}$$

We see that $y^{(i)}(t)$ $(i = 0, 1, 2, \ldots, n-1)$ are eventually monotonic and that $\lim\limits_{t\to\infty} y^{(i)}(t)$ $(i = 0, 1, 2, \ldots, n-1)$ exist in $\mathbb{R} \cup \{-\infty, \infty\}$. Since $y(t)/t^k$ is bounded, we find that $\lim\limits_{t\to\infty} y^{(k)}(t) = c$ for some $c \in \mathbb{R}$ and $\lim\limits_{t\to\infty} y^{(i)}(t) = 0$ for $i = k+1, \ldots, n-1$. Repeated integration of (11) yields

$$y^{(k)}(t) = c + (-1)^{n-k-1}\sigma \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} f\big(s, x(g(s))\big)\,\mathrm{d}s$$

for all large t. Consequently, we obtain

$$\int_T^{\infty} s^{n-k-1} f\big(s, x(g(s))\big)\,\mathrm{d}s < \infty$$

for some $T \geqslant t_0$. By virtue of (2) and the monotonicity of f, we conclude that (3) holds. $\square$

Now we show the "if" part of Theorem 1.

Let $k \in \{0, 1, 2, \ldots, n-1\}$. Suppose that (5) holds. Take a sufficiently large number $T \geqslant \tau(t_0)$ such that

$$h(T)[\tau(T)/T]^k = \max\{h(t)[\tau(t)/t]^k \colon\ t \in [t_0, \infty)\}$$

and

$$T_* \equiv \min\big\{\tau(T), \inf\{g(t) \colon\ t \geqslant T\}\big\} \geqslant t_0 (> 0).$$

Let $C[T_*, \infty)$ denote the Fréchet space of all continuous functions on $[T_*, \infty)$ with the topology of uniform convergence on every compact subinterval of $[T_*, \infty)$. Let $\eta \in C[T, \infty)$ with $\eta(t) \geqslant 0$ for $t \geqslant T$ and $\lim\limits_{t\to\infty} \eta(t) = 0$. We consider the set Y of all functions $y \in C[T_*, \infty)$ which are nonincreasing on $[T, \infty)$ and satisfy

$$y(t) = y(T) \quad \text{for } t \in [T_*, T], \quad 0 \leqslant y(t) \leqslant \eta(t) \quad \text{for } t \geqslant T.$$

It is easy to see that Y is a closed convex subset of $C[T_*, \infty)$.

The next result follows from the Proposition in [18].

Lemma 2. *Suppose that* (2) *holds. Let* $\eta \in C[T, \infty)$ *with* $\eta(t) \geqslant 0$ *for* $t \geqslant T$ *and* $\lim\limits_{t\to\infty} \eta(t) = 0$. *For this* η, *define* Y *as above. Then there exists a mapping* $\Phi \colon Y \longrightarrow C[T_*, \infty)$ *which has the following properties:*

(a) *For each* $y \in Y$, $\Phi[y]$ *satisfies*

$$\lim_{t\to\infty} \Phi[y](t) = 0$$

and

$$\Phi[y](t) + h(t)\left[\frac{\tau(t)}{t}\right]^k \Phi[y](\tau(t)) = y(t), \quad t \geqslant T;$$

(b) Φ *is continuous on* Y *in the* $C[T_*, \infty)$*-topology, i.e., if* $\{y_j\}_{j=1}^{\infty}$ *is a sequence in* Y *converging to* $y \in Y$ *uniformly on every compact subinterval of* $[T_*, \infty)$, *then* $\Phi[y_j]$ *converges to* $\Phi[y]$ *uniformly on every compact subinterval of* $[T_*, \infty)$.

We first prove the "if" part of Theorem 1 for the case $k = 0$.

Proof of the "if" part ($k = 0$). Put

$$\eta(t) = \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s,a)\,\mathrm{d}s, \quad t \geqslant T.$$

We use Lemma 2 for this η. In view of (6), we can take constants $b > 0$, $\delta > 0$ and $\varepsilon > 0$ such that

$$0 < \delta + \varepsilon \leqslant \frac{b}{1+h(t)} \leqslant a - \varepsilon, \quad t \geqslant T_*.$$

We denote the function $\Psi[y](t)$ by

$$\Psi[y](t) = \frac{b}{1+h(t)} + (-1)^{n-1}\sigma\Phi[y](t), \quad t \geqslant T_*, \quad y \in Y.$$

Define a mapping $\mathcal{F}\colon\, Y \longrightarrow C[T_*,\infty)$ as follows:

$$(\mathcal{F}y)(t) = \begin{cases} \displaystyle\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f_0\bigl(s, \Psi[y](g(s))\bigr)\,\mathrm{d}s, & t \geqslant T, \\ (\mathcal{F}y)(T), & t \in [T_*, T], \end{cases}$$

where

$$f_0(t,u) = \begin{cases} f(t,a), & u \geqslant a, \\ f(t,u), & \delta \leqslant u \leqslant a, \\ f(t,\delta), & u \leqslant \delta. \end{cases}$$

It is easy to see that $\mathcal{F}$ maps Y into itself.

From Lemma 2 it follows that the mapping Ψ is continuous on Y, and the Lebesgue dominated convergence theorem shows that $\mathcal{F}$ is continuous on Y.

Since

$$|(\mathcal{F}y)'(t)| \leqslant \int_T^\infty s^{n-2} f(s,a)\,\mathrm{d}s, \quad t \geqslant T, \quad y \in Y,$$

the mean value theorem implies that $\mathcal{F}(Y)$ is equicontinuous on $[T,\infty)$. Since $|(\mathcal{F}y)(t_1) - (\mathcal{F}y)(t_2)| = 0$ for $t_1, t_2 \in [T_*, T]$, we conclude that $\mathcal{F}(Y)$ is equicontinuous on $[T_*,\infty)$. Obviously, $\mathcal{F}(Y)$ is uniformly bounded on $[T_*,\infty)$. Hence, by the Ascoli-Arzela theorem, $\mathcal{F}(Y)$ is relatively compact.

Consequently, we are able to apply the Schauder-Tychonoff fixed point theorem to the operator $\mathcal{F}$ and conclude that there exists an element $\widetilde{y} \in Y$ such that $\widetilde{y} = \mathcal{F}\widetilde{y}$. Set $x(t) = \Psi[\widetilde{y}](t)$. Lemma 2 implies that $x(t)$ satisfies (7) with $k = 0$ and hence there exists a number $\widetilde{T} \geqslant T$ such that $\delta \leqslant x(g(t)) \leqslant a$ for $t \geqslant \widetilde{T}$. Then we have

$f_0\bigl(t, x(g(t))\bigr) = f\bigl(t, x(g(t))\bigr)$ for $t \geqslant \widetilde{T}$. Using Lemma 2 and (5), we observe that

$$\begin{aligned} (12)\qquad & x(t) + h(t)x(\tau(t)) \\ &= \frac{b}{1+h(t)} + \frac{bh(t)}{1+h(\tau(t))} + (-1)^{n-1}\sigma\bigl[\Phi[\widetilde{y}](t) + h(t)\Phi[\widetilde{y}](\tau(t))\bigr] \\ &= b + (-1)^{n-1}\sigma\widetilde{y}(t) \\ &= b + (-1)^{n-1}\sigma\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\bigl(s, x(g(s))\bigr)\,\mathrm{d}s, \quad t \geqslant \widetilde{T}. \end{aligned}$$

By differentiation of (12), we see that $x(t)$ is a solution of (1). The proof is complete. □

The following lemma will be used in the proof of the "if" part of Theorem 1 for the case $k \neq 0$.

Lemma 3. *Let $\ell \in \mathbb{N}$ and let $T > 0$. Suppose that $u \in C[T,\infty)$ is nonnegative and nonincreasing on $[T,\infty)$ and c is a number such that $c \geqslant u(T)T[(\ell-1)!]^{-1}$. Then the function*

$$U(t) = ct^{-1} + t^{-\ell}\int_T^t \frac{(t-s)^{\ell-1}}{(\ell-1)!} u(s)\,\mathrm{d}s$$

satisfies $-2cT^{-2} \leqslant U'(t) \leqslant 0$ for $t \geqslant T$.

P r o o f. We note that

$$\begin{aligned} (13)\qquad -c(\ell-1)! - \int_T^t u(s)\,\mathrm{d}s + tu(t) &\leqslant -c(\ell-1)! - u(t)(t-T) + tu(t) \\ &= u(t)T - c(\ell-1)! \leqslant 0, \quad t \geqslant T. \end{aligned}$$

If $\ell = 1$, then we find by (13) that

$$\begin{aligned} U'(t) &= -ct^{-2} - t^{-2}\int_T^t u(s)\,\mathrm{d}s + t^{-1}u(t) \\ &= t^{-2}\Bigl[-c - \int_T^t u(s)\,\mathrm{d}s + tu(t)\Bigr] \leqslant 0, \quad t \geqslant T \end{aligned}$$

and

$$\begin{aligned} U'(t) &\geqslant -ct^{-2} - t^{-2}\int_T^t u(s)\,\mathrm{d}s \geqslant -cT^{-2} - t^{-2}u(T)(t-T) \\ &\geqslant -cT^{-2} - t^{-2}cT^{-1}t \geqslant -2cT^{-2}, \quad t \geqslant T. \end{aligned}$$

Now we assume that $\ell \geqslant 2$. We see that

$$\begin{aligned} U'(t) &= -ct^{-2} - \ell t^{-\ell-1} \int_T^t \frac{(t-s)^{\ell-1}}{(\ell-1)!} u(s)\, \mathrm{d}s + t^{-\ell} \int_T^t \frac{(t-s)^{\ell-2}}{(\ell-2)!} u(s)\, \mathrm{d}s \\ &= t^{-\ell-1}\left[-ct^{\ell-1} - \ell \int_T^t \frac{(t-s)^{\ell-1}}{(\ell-1)!} u(s)\, \mathrm{d}s + t \int_T^t \frac{(t-s)^{\ell-2}}{(\ell-2)!} u(s)\, \mathrm{d}s \right] \\ &\equiv t^{-\ell-1} V(t), \quad t \geqslant T. \end{aligned}$$

Since

$$\begin{aligned} \int_T^t \frac{(t-s)^{\ell-1}}{(\ell-1)!} u(s)\, \mathrm{d}s &\leqslant u(T) \int_T^t \frac{(t-s)^{\ell-1}}{(\ell-1)!}\, \mathrm{d}s \\ &= u(T) \frac{(t-T)^{\ell}}{\ell!} \leqslant cT^{-1}\ell^{-1}t^{\ell}, \quad t \geqslant T, \end{aligned}$$

we obtain

$$U'(t) \geqslant -ct^{-2} - cT^{-1}t^{-1} \geqslant -2cT^{-2}, \quad t \geqslant T.$$

We claim that $V(t) \leqslant 0$ for $t \geqslant T$. In view of the equality

$$-\ell(t-s) + (\ell-1)t = (\ell-1)s - (t-s),$$

we can rewrite $V(t)$ as

$$\begin{aligned} V(t) &= -ct^{\ell-1} + \int_T^t \frac{(t-s)^{\ell-2}}{(\ell-1)!} u(s)[-\ell(t-s) + (\ell-1)t]\, \mathrm{d}s \\ &= -ct^{\ell-1} + \int_T^t \frac{(t-s)^{\ell-2}}{(\ell-2)!} s u(s)\, \mathrm{d}s - \int_T^t \frac{(t-s)^{\ell-1}}{(\ell-1)!} u(s)\, \mathrm{d}s. \end{aligned}$$

Then we find that

$$\begin{aligned} V^{(i)}(t) = &- c\frac{(\ell-1)!}{\ell-1-i)!} t^{(\ell-1-i} + \int_T^t \frac{(t-s)^{\ell-2-i}}{(\ell-2-i)!} s u(s)\, \mathrm{d}s \\ &- \int_T^t \frac{(t-s)^{\ell-1-i}}{(\ell-1-i)!} u(s)\, \mathrm{d}s, \quad t \geqslant T, \quad 0 \leqslant i \leqslant \ell-2 \end{aligned}$$

and

$$V^{(\ell-1)}(t) = -c(\ell-1)! + tu(t) - \int_T^t u(s)\, \mathrm{d}s, \quad t \geqslant T.$$

From (13) it follows that $V^{(\ell-1)}(t) \leqslant 0$ for $t \geqslant T$ and hence

$$V^{(\ell-2)}(t) \leqslant V^{(\ell-2)}(T) = -c(\ell-1)!T \leqslant 0, \quad t \geqslant T.$$

In exactly the same way, we conclude that

$$V^{(i)}(t) \leqslant 0, \quad t \geqslant T, \quad 0 \leqslant i \leqslant \ell - 2.$$

Consequently, $V(t) \leqslant 0$ for $t \geqslant T$ as claimed. This shows that $U'(t) \leqslant 0$ for $t \geqslant T$. The proof is complete. □

We now show the "if" part of Theorem 1 for the case $k \neq 0$.

P r o o f of the "if" part ($k \neq 0$). Put

$$\varphi(t) = \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} f(s, a[g(s)]^k)\,\mathrm{d}s, \quad c = \frac{\varphi(T)T}{(k-1)!}$$

and

$$\eta(t) = ct^{-1} + t^{-k} \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \varphi(s)\,\mathrm{d}s.$$

Then $\eta(t) \geqslant 0$ for $t \geqslant T$ and $\lim\limits_{t\to\infty} \eta(t) = 0$, and we use Lemma 2 for this η. By using (6), there are constants $b > 0$, $\delta > 0$ and $\varepsilon > 0$ such that

$$0 < \delta + \varepsilon \leqslant \frac{b}{1 + h(t)[\tau(t)/t]^k} \leqslant a - \varepsilon, \quad t \geqslant T_*.$$

We introduce the function $\Psi[y](t)$ by

$$\Psi[y](t) = t^k \left[\frac{b}{1 + h(t)[\tau(t)/t]^k} + (-1)^{n-k-1} \sigma \Phi[y](t) \right], \quad t \geqslant T_*, \quad y \in Y,$$

and define the mapping $\mathcal{F}\colon Y \longrightarrow C[T_*, \infty)$ as follows:

$$(\mathcal{F}y)(t) = \begin{cases} ct^{-1} + t^{-k} \displaystyle\int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \\ \qquad\qquad \times \displaystyle\int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f_k\big(r, \Psi[y](g(r))\big)\,\mathrm{d}r\,\mathrm{d}s, & t \geqslant T, \\ (\mathcal{F}y)(T), \quad t \in [T_*, T], \end{cases}$$

where

$$f_k(t,u) = \begin{cases} f\big(t, a[g(t)]^k\big), & u \geqslant a[g(t)]^k, \\ f(t,u), & \delta[g(t)]^k \leqslant u \leqslant a[g(t)]^k, \\ f\big(t, \delta[g(t)]^k\big), & u \leqslant \delta[g(t)]^k. \end{cases}$$

Lemma 3 implies that

$$-2cT^{-2} \leqslant (\mathcal{F}y)'(t) \leqslant 0, \quad t \geqslant T, \quad y \in Y.$$

Then $(\mathcal{F}y)(t)$ is nonincreasing on $[T,\infty)$ and hence $\mathcal{F}$ maps Y into itself. In a fashion similar to the case $k=0$, we see that $\mathcal{F}$ is continuous on Y and $\mathcal{F}(Y)$ is relatively compact. Then the Schauder-Tychonoff fixed point theorem shows that $\widetilde{y}=\mathcal{F}\widetilde{y}$ for some $\widetilde{y}\in Y$. We set $x(t)=\Psi[\widetilde{y}](t)$. Since $\lim\limits_{t\to\infty}\Phi[\widetilde{y}](t)=0$, we find that $x(t)$ satisfies (7) and $\delta[g(t)]^k\leqslant x(g(t))\leqslant a[g(t)]^k$ for $t\geqslant\widetilde{T}$, where $\widetilde{T}\geqslant T$ is sufficiently large, so that $f_k\bigl(t,x(g(t))\bigr)=f\bigl(t,x(g(t))\bigr)$ for $t\geqslant\widetilde{T}$. In view of (5) and Lemma 2, we find that

$$\begin{aligned}
&x(t)+h(t)x(\tau(t))\\
&=\frac{b}{1+h(t)[\tau(t)/t]^k}t^k+\frac{bh(t)}{1+h(\tau(t))\bigl[\tau(\tau(t))/\tau(t)\bigr]^k}\left[\frac{\tau(t)}{t}\right]^k t^k\\
&\quad+(-1)^{n-k-1}\sigma\bigl[\Phi[\widetilde{y}](t)+h(t)[\tau(t)/t\bigr]^k\Phi[\widetilde{y}](\tau(t))\bigr]t^k\\
&=b\,t^k+(-1)^{n-k-1}\sigma\widetilde{y}(t)t^k\\
&=b\,t^k+(-1)^{n-k-1}\sigma c t^{k-1}\\
&\quad+(-1)^{n-k-1}\sigma\int_T^t\frac{(t-s)^{k-1}}{(k-1)!}\int_s^\infty\frac{(r-s)^{n-k-1}}{(n-k-1)!}f_k\bigl(r,x(g(r))\bigr)\,\mathrm{d}r\,\mathrm{d}s
\end{aligned}$$

for $t\geqslant\widetilde{T}$. Differentiation of the above equality implies that

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\bigl[x(t)+h(t)x(\tau(t))\bigr]=-\sigma f_k\bigl(t,x(g(t))\bigr)=-\sigma f\bigl(t,x(g(t))\bigr),\quad t\geqslant\widetilde{T}.$$

Consequently, $x(t)$ is a solution of (1). This completes the proof. □

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