Czechoslovak Mathematical Journal

Y. Zhang; Y. Wang

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Czechoslovak Mathematical Journal, Vol. 51 (2001), No. 3, 473–476

Persistent URL: `http://dml.cz/dmlcz/127663`

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UNIQUELY COVERED RADICAL CLASSES OF ℓ-GROUPS

Y. Zhang, Shanghai, Y. Wang, Anshan

(Received January 16, 1996)

Abstract. It is proved that a radical class σ of lattice-ordered groups has exactly one cover if and only if it is an intersection of some σ-complement radical class and the big atom over σ.

Keywords: radical class, atom, unique covering question, quasi-complement radical class, σ-homogeneous

MSC 2000: 06E08

Let g and S be the classes of all ℓ-groups and of all radical classes of ℓ-groups, respectively. Let $\sigma, \tau \in S$. If $\sigma < \tau$ and there does not exist any $\varrho \in S$ such that $\sigma < \varrho < \tau$, then we say that τ is an atom over σ or that τ covers σ. Denote by $A(\sigma)$ the class of all atoms over σ. For $\sigma \in S$, $G \in g$, the symbol $\sigma(G)$ stands for the largest convex ℓ-subgroup of G which belongs to σ. Denote by $T(G)$ the least radical class containing G. Write $R(G) = \{\sigma(G) \mid \sigma \in S\}$ (see [2]).

Let $G \in g$. Let α be an infinite cardinal and $\omega(\alpha)$ be the least ordinal having cardinality α. For any $i \in \omega(\alpha)$, set $G_i = Z$, the additive group of integers under the usual order. Write $G(\alpha) = (\vec{\otimes} G_i) \vec{\otimes} G$ for the lexico-product of these G_i and G, $i \in \omega(\alpha)$, with order from left to right. Both $G(\alpha)$ and $T(G(\alpha))$ are called the regular atoms over G or $T(G)$ (cf. [1], [3]).

Let $\sigma \in S$. Suppose $A(\sigma) \neq \emptyset$. Put $\varepsilon(\sigma, 1) = \sup A(\sigma)$ and for any ordinal α, define inductively

$$\varepsilon(\sigma,\alpha) = \begin{cases} \sup A(\varepsilon(\sigma,\alpha-1)) & \text{when } \alpha \text{ is nonlimit,} \\ \bigvee_{\beta<\alpha} \varepsilon(\sigma,\beta) & \text{when } \alpha \text{ is limit.} \end{cases}$$

Project supported by NSF of China under Grant No. 10071062.

Form $Z(\sigma) = \bigvee_\alpha \varepsilon(\sigma, \alpha)$, where α runs over all ordinals. Call $Z(\sigma)$ the big atom over σ (cf. [1], [3]).

In 1977, J. Jakubík raised the following question of unique covering of radical classes: whether there exists $\alpha \in S$ such that $A(\sigma)$ is a one-element class.

The first author of the present paper answered "yes" in [3] and gave several sufficient and necessary conditions for a radical class to have a unique covering under the condition "non-superatom", i.e. "not containing $A_0 = \sup A(0)$". In this short note, we prove a theorem for all radical classes having a unique covering by using the notion of σ-homogenity. Recall that an ℓ-group G is called homogeneous if $T(G)$ is an atom.

Definition 1. An ℓ-group G is called quasi-homogeneous if there exists a largest radical class which does not contain G. If it is the case, then denote this radical class by T^G and call it the quasi-complement radical class of G or $T(G)$.

Let $\sigma \in S$. Recall that the complement radical class of σ, denoted by σ', is the largest radical class meeting σ in 0.

In the sequel, the appearence of T^G always suggests that G be q.h. A homogeneous ℓ-group G is clearly q.h. and T^G coincides with $T(G)'$ in this case (cf. [3] for detail). The ℓ-group $\{0\}$ is trivialy non-q.h. The cardinal sum of Z and Q (the rationals with the usual order and the usual addition) provides a non-trivial example of non-q.h. ℓ-group.

Definition 2. Let $\sigma \in S$. An ℓ-group G is called σ-homogeneous of $\sigma(G)$ is maximal in $R(G) \setminus \{G\}$. In this case, call $\sigma^G = \sup\{\tau \in S \mid \tau(G) \leqslant \sigma(G)\}$ the associated σ-complement radical class of G or σ-complement of G for short.

Remark.

1. If G is σ-homogeneous, then $G \in g \setminus \sigma$.
2. If G is homogeneous, then for each $\sigma \in S$ with $G \in g \setminus \sigma$, G is σ-homogeneous and $\sigma^G = T^G$.
3. If G is an σ-homogeneous ℓ-group, then $A(\sigma) \neq \emptyset$ (since $\sigma \vee T(G) \in A(\sigma)$).
4. If G is quasi-homogeneous, then there is some $\sigma \in S$ such that G is σ-homogeneous, this is guaranteed by the following proposition.

Proposition 1. *Let $0 \neq G \in g$. G is quasi-homogeneous if and only if $R(G) \setminus \{G\}$ possesses a largest element.*

P r o o f. To show the condition is necessary, let $H \in R(G) \setminus \{G\}$. Then $T(H) < T(G)$ (otherwise, $H = T(H)(G) = T(G)(G) = G$, a contradiction). Thus $T(H) < T^G$ and $H \leqslant T^G(G)$. This asserts that $T^G(G)$ is the largest element in $R(G) \setminus \{G\}$.

Conversely, let H be the largest element in $R(G) \setminus \{G\}$. Put $\tau = \bigvee_{\sigma \in C} \sigma$, where C is the collection of all radical classes which do not contain G. Then $\tau(G) = \bigvee_{\sigma \in C} \sigma(G) \leqslant \bigvee_{\sigma \in C} \sigma(H) = H \neq G$. Hence $G \in g \setminus \tau$. Therefore $\tau = T^G$. $\square$

Proposition 2. *Let $G \in g$, $\sigma \in S$. Then G is σ-homogeneous if and only if $\sigma \vee T(G) \in A(\sigma)$. Therefore $A(\sigma) = \{\sigma \vee T(G) \mid G$ is σ-homogeneous$\}$.*

Proof. We have that $\sigma \vee T(G)$ covers σ if and only if $T(G)$ covers $T(\sigma(G))$ if and only if $\sigma(G)$ is a maximal element in $R(G) \setminus \{G\}$ if and only if G is σ-homogeneous. $\square$

Corollary. *$A(\sigma) = \emptyset$ if and only if there is no σ-homogeneous ℓ-group.*

Proposition 3. *Let G be σ-homogeneous, then the σ-complement of G has exactly one cover.*

Proof. Firstly, $\sigma^G(G) = (\bigvee_{\tau(G) \leqslant \sigma(G)} \tau)(G) = \bigvee \tau(G) \leqslant \sigma(G)$. Since $\sigma^G \geqslant \sigma$, we infer that $\sigma^G(G) = \sigma(G)$. Hence the projectivity of $[\sigma^G, T(G) \vee \sigma^G]$ and $[T(\sigma(G)), T(G)]$ implies that $T(G) \vee \sigma^G$ covers σ^G. On the other hand, if $(T(H) \vee \sigma^G) \in A(\sigma^G)$ and $T(H) \vee \sigma^G \neq T(G) \vee \sigma^G$, then

$$\sigma^G = (T(H) \vee \sigma^G) \wedge (T(G) \vee \sigma^G) = \sigma^G \vee (T(H) \wedge T(G)).$$

Hence

$$\sigma(G) = \sigma^G(G) \geqslant (T(H) \wedge T(G))(G) = T(H)(G).$$

Thus $T(H) \leqslant \sigma^G$, a contradiction. Therefore $A(\sigma^G)$ is a one-element class. $\square$

Corollary 1. *T^G has exactly one cover.*

Corollary 2. *Each regular atom $G(\alpha)$ over a nonzero ℓ-group G is q.h. and has exactly one cover. Moreover, $T^{G(\alpha)} \geqslant A_0$ and is therefore a superatom which is not a polar (cf.* [3]*).*

Theorem 4. *Let $\sigma \in S$. Then $A(\sigma)$ is a one-element class if and only if there is a σ-homogeneous ℓ-group G such that $\sigma = \sigma^G \wedge Z(\sigma)$.*

Proof. For the sufficiency, suppose that $\sigma = \sigma^G \wedge Z(\sigma)$, where G is a σ-homogeneous ℓ-group. We then have $\sigma \vee T(G)$ covers σ. If there exists some $H \in g$ with $\sigma \vee T(H) \in A(\sigma)$ and $\sigma \vee T(H) \neq \sigma \vee T(G)$, then, since $T(H)(G) \neq G$, only two cases may occur: either $T(H)(G) \leqslant \sigma(G)$ or $T(H)(G)$ is not comparable with $\sigma(G)$. For the former, we have $T(H) \leqslant \sigma^G$ and $T(H) \leqslant \sigma^G \wedge Z(\sigma) = \sigma$, which contradicts $T(H) \in A(\sigma)$. For the latter, we have $T(H)(G) \vee \sigma(G) = G$, therefore $T(H) \vee T(\sigma(G)) \geqslant T(G)$. Thus $\sigma \vee T(H) = \sigma \vee T(\sigma(G)) \vee T(H) \geqslant \sigma \vee T(G)$. Hence $\sigma \vee T(H) = \sigma \vee T(G)$, which is not the case. So σ is uniquely covered by $\sigma \vee T(G)$.

To show the condition is also necessary, let $A(\sigma)$ be a one-element class with the unique element ϱ. Then, from Proposition 2, we have a σ-homogeneous ℓ-group G with $\varrho = \sigma \vee T(G)$. Clearly, $\sigma \leqslant \sigma^G \wedge Z(\sigma)$. Now, let $H \in g \setminus \sigma$, then either $H \in g \setminus \sigma^G$ or $H \in \sigma^G$. The first case implies that $H \in g \setminus \sigma^G \wedge Z(\sigma)$. For the second case, we infer that $G \in g \setminus T(H)$ and $T(H)(G) \leqslant \sigma^G(G) = \sigma(G)$, then $\sigma \vee T(H)$ is not comparable with ϱ. This implies that $\sigma \vee T(H)$ is not contained in $Z(\sigma)$, since ϱ is the unique atom over σ, thus $H \in g \setminus Z(\sigma)$ and $H \in g \setminus \sigma^G \wedge Z(\sigma)$, therefore $\sigma = \sigma^G \wedge Z(\sigma)$. This finishes the proof. □

Remark. In general, $\sigma^G \wedge Z(\sigma) \neq \sigma^G$. For instance, let $G = Z \oplus Q$, $\sigma = T(Q)$, then $\sigma^G = T^Z \neq \sigma = \sigma^G \wedge Z(\sigma)$.

Acknowledgements. The author would like to thank the referere for his (or her) helpful comments. The first author received support from the Natural Science Foundation of Hainan Province of China, Grant No. 19702.

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Authors' addresses: Y. Zhang, Department of Mathematics, Shanghai Jiaotong University, Shanghai, 200029, P. R. of China, e-mail: zhangyuh@citiz.net; Y. Wang, Department of Mathematics, Anshan Normal College, 114005, P. R. of China, e-mail: yaowang@public.as.ln.cn.