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F-CONTINUOUS GRAPHS

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Abstract. For a nontrivial connected graph F, the F-degree of a vertex v in a graph G is the number of copies of F in G containing v. A graph G is F-continuous (or F-degree continuous) if the F-degrees of every two adjacent vertices of G differ by at most 1. All P_3-continuous graphs are determined. It is observed that if G is a nontrivial connected graph that is F-continuous for all nontrivial connected graphs F, then either G is regular or G is a path. In the case of a 2-connected graph F, however, there always exists a regular graph that is not F-continuous. It is also shown that for every graph H and every 2-connected graph F, there exists an F-continuous graph G containing H as an induced subgraph.

Keywords: F-degree, F-degree continuous

MSC 2000: 05C12

1. Introduction

For a vertex v in a graph G, the *degree* $\deg v$ of v is the number of edges in G incident with v. For a nontrivial connected graph F, the *F-degree* $F\deg v$ of v in G is the number of copies of F in G containing v. Thus the K_2-degree of a vertex is synonymous with its degree. The concept of F-degree was introduced and studied in [2]. If $F\deg v = r$ for every vertex v of G, then G is said to be *F-regular* of degree r.

In [1] an integer-valued parameter f defined on the vertex set of a graph G is called *continuous* if $|f(u) - f(v)| \leqslant 1$ for every two adjacent vertices u and v of G.

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In particular, *degree continuous* graphs have the property that $|\deg u - \deg v| \leqslant 1$ for every two adjacent vertices u and v. Degree continuous graphs were studied by Gimbel and Zhang [5], who showed, among other results, that for every two positive integers r and s with $r \leqslant s$, there exists a degree continuous graph with degree set $\{r, r+1, \ldots, s\}$.

For a nontrivial connected graph F, we define a graph G to be *F-degree continuous* or, more simply, *F-continuous* if the F-degrees of every two adjacent vertices differ by at most 1.

It is an elementary observation that a graph G is F-continuous for some nontrivial connected graph F if and only if every component of G is F-continuous. Hence it suffices to consider only connected graphs G. Also, if G contains no copy of F, then every vertex of G has F-degree 0 and G is trivially F-continuous. Therefore, unless otherwise stated, we assume, for a given graph F, that every graph G under consideration contains a copy of F. The following fact will be useful. We denote the path of order n by P_n.

Lemma 1.1. *Let F be a nontrivial connected graph with the property that for every connected graph G, whenever G contains F as a subgraph, then every vertex of G belongs to a copy of F. Then F is P_2, P_3, or P_4.*

Proof. Obviously, P_2 has the desired property. Suppose next that G is a connected graph containing $F = P_4$ as a subgraph and let v be a vertex of G. Let Q be a shortest path (of length ℓ) in G from v to F. If $\ell = 0$ or $\ell = 3$, then clearly v lies on a copy of P_4. Otherwise, Q together with an appropriate subpath of F gives a path P_4 containing v. The argument for $F = P_3$ is similar.

It remains to show that no graph F different from P_2, P_3, or P_4 has such a property. Assume first that $F = P_k$, where $k \geqslant 5$. Let $P\colon v_1, v_2, \ldots, v_k$ be a path of order k and let G be the tree obtained by adding a new vertex v to P and the edge $vv_{\lfloor \frac{k}{2} \rfloor}$. Then v lies on no copy of F. Assume then that F is not a path. In this case, let ℓ be the length of a longest path in F. A graph G is constructed by identifying an end-vertex of $P_{\ell+1}$ with a vertex of F. Let u be the other end-vertex of $P_{\ell+1}$. Then u lies on no copy of F. □

By Lemma 1.1, it then follows that if P_k $(2 \leqslant k \leqslant 4)$ is a subgraph of a connected graph G, then every vertex of G has a positive P_k-degree. Moreover, only these paths have this property.

In this paper, we present several results concerning F-continuous graphs for various graphs F.

3. P_3-continuous graphs

In this section we consider F-continuous graphs for the case where $F = P_3$, the path of order 3. We begin with the observation that every path P_n $(n \geqslant 3)$ is P_3-continuous. In fact, the P_3-degree of every vertex of P_3 is 1, that is, P_3 is P_3-regular. For $n \geqslant 4$, the end-vertices of P_n have P_3-degree 1, while the P_3-degrees of the two vertices adjacent to an end-vertex are 2. The remaining vertices of P_n have P_3-degree 3.

Next we make a general observation about the P_3-degree of a vertex. Let G be a connected graph containing a path of order 3. By Lemma 1.1, every vertex of G lies on a path of order 3. Denote the neighbourhood of a vertex v (the vertices adjacent to v) by $N(v)$. Then v is the central vertex of $\binom{\deg v}{2}$ copies of P_3 and it is the end-vertex of $\sum_{u \in N(v)} (\deg u - 1)$ copies of P_3. Therefore,

$$P_3 \deg v = \binom{\deg v}{2} + \sum_{u \in N(v)} (\deg u - 1). \tag{1}$$

An immediate consequence of this observation is that every r-regular graph is P_3-regular of degree $3\binom{r}{2}$ and so is P_3-continuous. Hence it follows that all cycles, complete graphs, and hypercubes are P_3-continuous. Next we determine those complete bipartite graphs that are P_3-continuous.

Theorem 2.1. *Among the complete bipartite graphs, only $K_{1,2}$, $K_{1,3}$, $K_{2,3}$ and $K_{r,r}$ $(r \geqslant 2)$ are P_3-continuous.*

Proof. Since $K_{r,r}$ $(r \geqslant 2)$ is an r-regular graph, $K_{r,r}$ is P_3-continuous. Next, let $G = K_{r,s}$, where $1 \leqslant r < s$ and let $u, v \in V(G)$, where $\deg u = r$ and $\deg v = s$.

Assume first that $P_3 \deg v \leqslant P_3 \deg u$. Then

$$\binom{s}{2} + s(r-1) \leqslant \binom{r}{2} + r(s-1).$$

So $(s-r)(r+s-3) \leqslant 0$. This implies that $r + s = 3$, from which it follows that $(r,s) = (1,2)$. Otherwise, $P_3 \deg v = 1 + P_3 \deg u$. In this case, $s(s-3) = (r-1)(r-2)$. Hence $(r,s) = (1,3)$ or $(r,s) = (2,3)$. □

The following lemma describes the P_3-continuous graphs containing vertices with P_3-degree at most 3.

Lemma 2.2. *Let G be a P_3-continuous graph. Then*

(a) *G contains a vertex with P_3-degree 1 if and only if $G = P_n$, where $n \geqslant 3$;*

(b) *G contains a vertex with P_3-degree 2 if and only if $G = P_n$, where $n \geqslant 4$, or $G = K_{1,3}$;*

(c) *G contains a vertex with P_3-degree 3 if and only if $G = P_n$, where $n \geqslant 5$, or $G = C_n$, where $n \geqslant 3$, or $G = K_{1,3}$.*

Proof. Let v be a vertex with $P_3 \deg v = 1$. Necessarily, then, $\deg v \leqslant 2$. If $\deg v = 1$, then v is an end-vertex that is adjacent to a vertex u of degree 2. Let $N(u) = \{v, w\}$. Now $\deg w \leqslant 2$; otherwise, $P_3 \deg u \geqslant 3$, contradicting the P_3-continuity of G. Repeating this procedure, it follows that $G = P_n$, where $n \geqslant 3$. If $\deg v = 2$, then $G = P_3$. This verifies (a).

Next let v be a vertex with $P_3 \deg v = 2$. Then $\deg v \leqslant 2$. If $\deg v = 1$, then v is an end-vertex adjacent to a vertex u of degree 3. Let $N(u) = \{v, w_1, w_2\}$. Now $\deg w_1 = \deg w_2 = 1$; otherwise, $P_3 \deg u \geqslant 4$, contradicting the P_3-continuity of G. Therefore, $G = K_{1,3}$.

Now suppose that $\deg v = 2$, and let $N(v) = \{u, w\}$. Then exactly one of u and w is an end-vertex with P_3-degree 1. By (a), it follows that $G = P_n$, in this case with $n \geqslant 4$. This verifies (b).

Finally, let v be a vertex with degree $P_3 \deg v = 3$. Then $\deg v \leqslant 3$. If $\deg v = 1$, then v is an end-vertex adjacent to a vertex u of degree 4. Consequently, $P_3 \deg u \geqslant \binom{4}{2} = 6$, contradicting the P_3-continuity of G. Hence $\deg v \geqslant 2$.

If $\deg v = 2$, then v is adjacent to two vertices u and w, neither of which is an end-vertex. Necessarily, $\deg u = \deg w = 2$. Continuing in this manner, we see that either $G = C_n$, where $n \geqslant 3$, or $G = P_n$ where $n \geqslant 5$. If $\deg v = 3$, then $G = K_{1,3}$. This verifies (c). □

As a consequence of Lemma 2.2, we are able to determine all P_3-continuous trees.

Corollary 2.3. *The only P_3-continuous trees are P_n, where $n \geqslant 3$, and $K_{1,3}$.*

Proof. Let T be a P_3-continuous tree and let v be an end-vertex of T that is adjacent to w. Let $\deg w = k$. Then

$$\binom{k}{2} \leqslant P_3 \deg w \leqslant 1 + P_3 \deg v.$$

Thus $1 + (k-1) = k \geqslant \binom{k}{2}$, so $k \leqslant 3$. If $k = 2$, then $P_3 \deg v = 1$. By Lemma 2.2 (a), $G = P_n$, where $n \geqslant 3$. If $k = 3$, then $P_3 \deg v = 2$ and either $G = P_n$, where $n \geqslant 4$, or $G = K_{1,3}$ by Lemma 2.2 (b). □

We have already noted that every r-regular graph, $r \geqslant 2$, is P_3-continuous; indeed it is P_3-regular of degree $3\binom{r}{2}$. We now determine the possible P_3-degree sets of all P_3-continuous graphs. Necessarily these sets are of the form $\{r, r+1, r+2, \ldots, s\}$

for positive integers r and s with $r \leqslant s$. We begin by determining the P_3-degree sets of cardinality 2 in a connected P_3-continuous graph.

Theorem 2.4. *If G is a connected P_3-continuous graph with P_3-degree set $\{k, k+1\}$, then $k \in \{1, 2, 5\}$.*

Proof. Since the vertices of G have two distinct P_3-degrees, G is not regular. Since $G \neq P_3$, it follows that the order of G is at least 4. Let u and v be vertices of G with $\deg u = \delta(G) = \delta$ and $\deg v = \Delta(G) = \Delta$, where $\delta < \Delta$. First we show that $P_3 \deg v > P_3 \deg u$. Assume, to the contrary, that

$$P_3 \deg v \leqslant P_3 \deg u. \tag{2}$$

Then, by (1), it follows that

$$\binom{\Delta}{2} + \Delta(\delta - 1) \leqslant P_3 \deg v \leqslant P_3 \deg u \leqslant \binom{\delta}{2} + \delta(\Delta - 1),$$

which yields the inequality $\Delta^2 - 3\Delta \leqslant \delta^2 - 3\delta$ or, equivalently, $(\Delta - \delta)(\Delta + \delta - 3) \leqslant 0$. This implies that $\Delta + \delta = 3$, so $(\delta, \Delta) = (1, 2)$. So $G = P_n$ for $n \geqslant 4$ and $P_3 \deg v > P_3 \deg u$, which contradicts (2). Hence, as claimed, $P_3 \deg v > P_3 \deg u$. Since the P_3-degree set of G is $\{k, k+1\}$, we must have $P_3 \deg v = 1 + P_3 \deg u$. So

$$\binom{\Delta}{2} + \Delta(\delta - 1) \leqslant P_3 \deg v = 1 + P_3 \deg u \leqslant 1 + \binom{\delta}{2} + \delta(\Delta - 1),$$

which produces the inequality

$$(\Delta - \delta)(\Delta + \delta - 3) \leqslant 2. \tag{3}$$

The only pairs (δ, Δ) satisfying (3) are $(1, 2)$, $(1, 3)$, and $(2, 3)$.

If $(\delta, \Delta) = (1, 2)$, then $P_3 \deg u = 1$ and by Lemma 2.2, $G = P_4$, producing the P_3-degree set $\{1, 2\}$. Assume that $(\delta, \Delta) = (1, 3)$. Then $\deg u = 1$. Let w be the neighbour of u. So $2 \leqslant \deg w \leqslant 3$. If $\deg w = 2$, then $P_3 \deg u = 1$ and $G = P_n$ for $n \geqslant 4$ by Lemma 2.2 (b). This, however, is impossible since $\Delta = 3$. Thus $\deg w = 3$. Then $P_3 \deg u = 2$, which implies by Lemma 2.2 (b) that $G = K_{1,3}$. This gives the P_3-degree set $\{2, 3\}$.

If $(\delta, \Delta) = (2, 3)$, then, of course, every vertex of G has degree 2 or 3. Since $P_3 \deg u \leqslant \binom{2}{2} + 2 + 2 = 5$ and $P_3 \deg v \geqslant \binom{3}{2} + 1 + 1 + 1 = 6$, a vertex of degree 3 can only be adjacent to vertices of degree 2 while a vertex of degree 2 can only be adjacent to vertices of degree 3. Thus $k = 5$ and the P_3-continuous graphs with P_3-degree set $\{5, 6\}$ are the subdivision graphs of cubic graphs or cubic multigraphs. □

In Lemma 2.2, we have described P_3-continuous graphs containing vertices with P_3-degree 1, 2, or 3. No vertex of a P_3-continuous graph can have P_3-degree 4, however; suppose, to the contrary, that G is a P_3-continuous graph containing a vertex v with $P_3 \deg v = 4$. By (1), it follows that $1 \leqslant \deg v \leqslant 3$. If $\deg v = 1$, then its neighbour u has degree 5, so $P_3 \deg u \geqslant 10$, contradicting the P_3-continuity of G. Thus $\deg v = 2$ or $\deg v = 3$. In either case, v cannot be adjacent to an end-vertex for such a vertex has P_3-degree at most 2, again contradicting the P_3-continuity of G. Since a vertex v with $P_3 \deg v = 4$ and $\deg v = 3$ in a P_3-continuous graph must be adjacent to an end-vertex, we are left with only one possibility, namely $\deg v = 2$ and one neighbour of v, say u, has degree 3 and the other neighbour of v has degree 2. Since $4 \leqslant P_3 \deg u \leqslant 5$, it follows that u is adjacent to an end-vertex w. However, then, $P_3 \deg w = 2$, again a contradiction.

The following theorem provides us with additional information about the degrees of the vertices of a P_3-continuous graph.

Theorem 2.5. *Every P_3-continuous graph is regular or has maximum degree at most* 3.

P r o o f. Let G be a P_3-continuous graph that is not regular. We show that $\Delta(G) \leqslant 3$. Assume first that $\delta(G) = 1$. Let $\deg u = 1$ and assume that v is adjacent to u. Then $\deg v \leqslant 3$. Therefore, $P_3 \deg u = 1$ or $P_3 \deg u = 2$. By Lemma 2.2, $G = P_n$ for some $n \geqslant 3$ or $G = K_{1,3}$ and so $\Delta(G) \leqslant 3$.

Hence we may assume that $\delta(G) \geqslant 2$. Assume, to the contrary, that $\Delta(G) = \Delta \geqslant 4$. First we show that no vertex of degree 2 can be adjacent to a vertex of degree at least 4; assume, to the contrary, that u and w are adjacent vertices with $\deg u = 2$ and $\deg w \geqslant 4$. Furthermore, we may assume that if v is another neighbour of u, then $\deg v \leqslant \deg w$. Then $P_3 \deg u \leqslant \binom{2}{2} + 2(\deg w - 1) = 2\deg w - 1$, while $P_3 \deg w \geqslant \binom{\deg w}{2} + \deg w$. This implies that $P_3 \deg w - P_3 \deg u \geqslant 3$ as $\deg w \geqslant 4$. Thus a vertex of degree $\Delta \geqslant 4$ can be adjacent only to vertices of degree 3 or more. Let k be the smallest degree of a vertex that is adjacent to a vertex of degree Δ. Say $\deg x = k$ and $\deg y = \Delta$, where $xy \in E(G)$. Then $3 \leqslant k < \Delta$. Therefore, $P_3 \deg y \geqslant \binom{\Delta}{2} + \Delta(k-1)$ and $P_3 \deg x \leqslant \binom{k}{2} + k(\Delta - 1)$, so

$$P_3 \deg y - P_3 \deg x \geqslant \binom{\Delta}{2} + \Delta(k-1) - \binom{k}{2} - k(\Delta - 1) = \frac{1}{2}(\Delta - k)(\Delta + k - 3) \geqslant 2.$$

This is a contradiction. □

With the aid of Theorem 2.5, we now see that only certain P_3-degrees are possible for the vertices of a P_3-continuous graph.

Corollary 2.6. *The only integers that can occur as the P_3-degrees of the vertices of a P_3-continuous graph are* 1, 2, 3, 5, 6, *and* $3\binom{r}{2}$, *where* $r \geqslant 3$.

P r o o f. Let G be a P_3-continuous graph. If G is r-regular, then we have already seen that G is P_3-regular of degree $3\binom{r}{2}$. Thus we may assume that $1 \leqslant \delta(G) = \delta < \Delta(G) = \Delta$, where $\Delta \leqslant 3$ by Theorem 2.5. Hence the only possible pairs for (δ, Δ) for G are (1,2), (1,3), and (2,3). For $(\delta, \Delta) = (1, 2)$, $G = P_n$, which has P_3-degrees 1, 2, and 3 for its vertices. For $(\delta, \Delta) = (1, 3)$, $G = K_{1,3}$, which has P_3-degrees 2 and 3 for its vertices. For $(\delta, \Delta) = (2, 3)$, each P_3-continuous graph is the subdivision of a cubic graph or a cubic multigraph. The P_3-degrees of the vertices of these graphs are 5 and 6. Hence each of the numbers 1, 2, 3, 5, 6 is realizable as the P_3-degree of some vertex in a P_3-continuous graph. □

Corollary 2.7. *The P_3-degree sets of a P_3-continuous graph are* $\left\{3\binom{r}{2}\right\}$ *for* $r \geqslant 2$, $\{1, 2\}$, $\{2, 3\}$, $\{5, 6\}$, *and* $\{1, 2, 3\}$. *Furthermore, the only P_3-continuous graphs are regular graphs, P_n for $n \geqslant 3$, $K_{1,3}$, and the subdivisions of a cubic graph or a cubic multigraph.*

3. Other results concerning F-continuous graphs

By Corollary 2.7, the only P_3-continuous graphs are regular graphs, the paths P_n for $n \geqslant 3$, the star $K_{1,3}$, and the subdivisions of cubic graphs or cubic multigraphs. Certainly, every vertex of $K_{1,3}$ has degree 1 or 3; hence $K_{1,3}$ is not P_2-continuous. If G is a subdivision of a cubic graph or a cubic multigraph, then every vertex of degree 3 in G has P_4-degree 12, while every vertex of degree 2 in G has P_4-degree 6. These observations give the following result.

Corollary 3.1. *If G is a connected graph of order $n \geqslant 2$ that is F-continuous for every nontrivial connected graph F, then either G is regular or $G = P_n$.*

Although the paths P_n, $n \geqslant 2$, are F-continuous for every nontrivial connected graph F, the converse of Corollary 3.1. is not true as there are many nontrivial connected graphs F for which there exist regular graphs that are not F-continuous. Of course, vertex-transitive graphs are F-regular for every nontrivial connected graph F, so they are F-continuous as well. Also, regular graphs that are not K_2-regular clearly do not exist. Since every regular graph is P_3-regular, there is no regular graph that is not P_3-continuous. The paths P_2 and P_3 are also both stars. Indeed, if G is an r-regular graph and $F = K_{1,k}$, $k \geqslant 2$, then every vertex of G has F-degree $(k+1)\binom{r}{k}$ and is consequently F-regular and so F-continuous.

The situation is different, however, if $F = P_4$. Indeed, if v is a vertex of an r-regular graph, then

$$P_4 \deg v = 2r(r-1)^2 - 4\dot{K}_3 \deg v. \tag{4}$$

By (4), if G is a regular graph not all of whose vertices belong to the same number of triangles, then G is not P_4-continuous. Indeed (4) shows us that an r-regular graph G is P_4-continuous if and only if G is K_3-regular. A regular graph that is not P_4-continuous is shown in Fig. 1, where its vertices are labeled with their P_4-degrees.

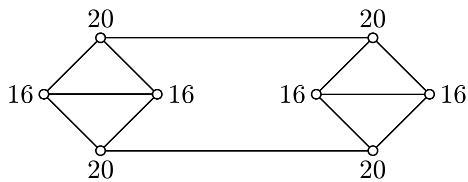

Fig. 1

This suggests the problem of determining those graphs F for which there exists a regular graph G that is not F-continuous. If F is 2-connected, then we have a solution to this problem. Before presenting this solution, it is useful to make a few preliminary remarks. If G is a graph with cycles, then its *circumference* $c(G)$ is the length of its largest cycle, while its *girth* $g(G)$ is the length of its smallest cycle. It was shown by Erdös and Sachs [4] that for every two integers $r \geqslant 2$ and $g \geqslant 3$, there exists an r-regular graph having girth g. An r-regular graph having girth g of minimum order is called an (r, g)-*cage*.

Theorem 3.2. *For every 2-connected graph F, there exists a regular graph that is not F-continuous.*

Proof. Let F have order n, and let H be the graph obtained by identifying three copies F_1, F_2, F_3 of F at the same vertex v, where $\deg_F v = \Delta(F) = \Delta$. Thus $F\deg_H v = 3$ and $F\deg_H x = 1$ for $x \neq v$. Hence H is not F-continuous and $\Delta(H) = 3\Delta$. If either Δ or n is even, let $r = 3\Delta$; otherwise, let $r = 3n+1$. We construct an r-regular graph G that is not F-continuous. Observe that

$$\sum_{u \in V(H)} (r - \deg_H u) = r(3n-2) - \sum_{u \in V(H)} \deg_H u = 2q \tag{5}$$

is even. Let c denote the circumference of F. Hence the circumference of H is c as well. Let J denote an r-regular cage of girth $c+1$. Certainly F is not a subgraph of J. Let $J_1, J_2, \ldots, J_q$ be q copies of J and delete the same edge, say yz, in each copy. Necessarily, the edge yz lies on some cycle (of length at least $c+1$). We now join y and z in each graph $J_i - yz$ $(1 \leqslant i \leqslant q)$ to distinct vertices of H in such a way that the resulting graph G is r-regular. No copy of F contains these two edges since the length of the smallest cycle in G containing these edges exceeds c. Hence the only copies of F in G are F_1, F_2, and F_3. Thus, $F\deg_G v = 3$, $F\deg_G x = 1$ for $x \in V(F_i - v)$, $1 \leqslant i \leqslant 3$, and $F\deg_G x = 0$ for $x \in V(J_i)$, $1 \leqslant i \leqslant q$. Therefore, the graph G has the desired properties. □

Although we have seen that regular graphs exist that are not P_4-continuous, we know of no general construction that shows that regular graphs exist which are not F-continuous when F is not a star. However, we believe that this is the case.

Conjecture 3.3. *For every nontrivial connected graph F different from the star $K_{1,k}$, $k \geqslant 1$, there exists a regular graph that is not F-continuous.*

Fig. 2 shows the graph of Fig. 1 again, but this time the K_3-degrees of its vertices are shown.

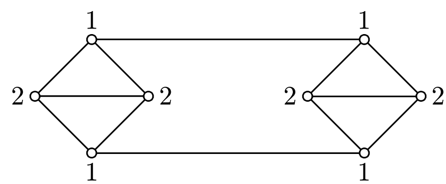

Fig. 2

As we can see from Fig. 2, there exist regular, K_3-continuous graphs that are not K_3-regular. This statement is true if K_3 is replaced by any nontrivial complete graph. For $n \geqslant 4$, the graph of Fig. 3 describes a construction of a regular, K_n-continuous graph that is not K_n-regular. It is obtained by removing an edge from each of two copies of K_{n+1} and joining the corresponding vertices.

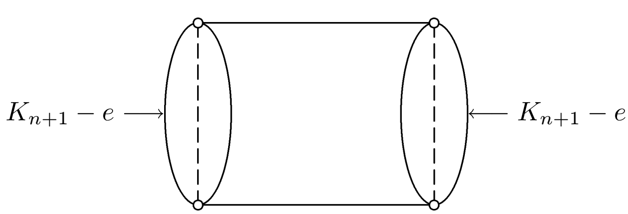

Fig. 3

A regular, C_4-continuous graph that is not C_4-regular is shown in Fig. 4. The C_4-degrees of its vertices are indicated in the figure. We state the following problems.

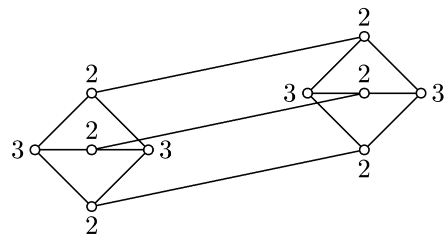

Fig. 4

Problem 3.4. For every nontrivial connected graph F different from the star $K_{1,k}$, $k \geqslant 1$, does there exist a regular, F-continuous graph that is not F-regular?

Problem 3.5. Is it true that every regular graph G that is not vertex-transitive is not F-continuous for some nontrivial connected graph F?

A well known theorem of König [6] states that for every graph H, there exists a regular graph G containing H as an induced subgraph. Certainly, such a graph G is K_2-continuous as well. In the case of 2-connected graphs F, we can extend this result to F-continuous graphs.

Theorem 3.6. *For every graph H and every 2-connected graph F, there exists an F-continuous graph G containing H as an induced subgraph.*

P r o o f. Let H be a g raph and let $\Delta_F = \max\limits_{v \in V(H)} (F \deg_H v)$. If $\Delta_F \leqslant 1$, then let $G = H$, which has the desired properties. So we may assume that $\Delta_F \geqslant 2$. For each vertex v in H, if $F \deg_H v = i$, then we attach $\Delta_F - i$ copies $F_{v,j}$ $(1 \leqslant j \leqslant \Delta_F - i)$ of F to H at v by identifying v and a vertex in each graph $F_{v,j}$ for all j. Denote the resulting graph by G_1. Then H is a induced subgraph of G_1 and every vertex in H is a cut-vertex in G_1.

Since F is 2-connected, every copy of F in G_1 is either a subgraph of H or is some graph $F_{u,j}$ for $u \in V(H)$ and $1 \leqslant j \leqslant \Delta_F - F \deg_H u$. Thus $F \deg_{G_1} v = \Delta_F$ for $v \in V(H)$ and $F \deg_{G_1} v = 1$ for all $v \in V(G_1) - V(H)$. If $\Delta_F = 2$, then G_1 is F-continuous and $G = G_1$ has the desired properties. Otherwise, we construct a graph G_2 from G_1 by attaching $\Delta_F - 2$ copies of F to G_1 at v for each $v \in V(G_1) - V(H)$ as above. Again, H is an induced subgraph of G_2 and every vertex in G_1 is a cut-vertex of G_2. Hence, $F \deg_{G_2} v = \Delta_F$ for all $v \in V(H)$, $F \deg_{G_2} v = \Delta_F - 1$ for all $v \in V(G_1) - V(H)$, and $F \deg_{G_2} v = 1$ for all $v \in V(G_2) - V(G_1)$. If G_2 is F-continuous, then $G = G_2$ has the desired properties. Otherwise, we repeat the procedure described above for each k with $3 \leqslant k \leqslant \Delta_F - 1$ to obtain the graph G_k. In the F-continuous graph $G = G_{\Delta_F - 1}$, the graph H is an induced subgraph of G, as desired. $\square$

The F-degree set of the graph G constructed in the proof of Theorem 3.6 is $\{1, 2, \ldots, \Delta_F\}$. So we have the following consequence of the proof of Theorem 3.6.

Corollary 3.7. *For every 2-connected graph F and integer $s \geqslant 1$, there exists an F-continuous graph G whose F-degree set is $\{1, 2, \ldots, s\}$.*

Proof. Let G_1 be obtained by identifying s copies of F at a vertex u. Then $F \deg_{G_1} u = s$ and $F \deg_{G_1} v = 1$ for all $v \in V(G_1) - \{u\}$. We repeat the procedure in the proof of Theorem 3.6 to construct a sequence $G_1, G_2, \ldots, G_s$ of graphs. Then $G = G_s$ has the desired properties. □

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