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ON A GENERALIZATION OF A GREGUŠ FIXED POINT THEOREM

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Abstract. Let C be a closed convex subset of a complete convex metric space X. In this paper a class of selfmappings on C, which satisfy the nonexpansive type condition (2) below, is introduced and investigated. The main result is that such mappings have a unique fixed point.

Keywords: convex metric space, nonexpansive type mapping, fixed point

MSC 2000: 47H10, 54H25

1. Introduction

Let X be a Banach space and C a closed convex subset of X. Recently, Greguš proved the following result.

Theorem 1. (Greguš [7]). *Let* $T\colon C \to C$ *be a mapping satisfying*

$$\|Tx - Ty\| \leqslant a\|x-y\| + p\|Tx - x\| + p\|Ty - y\| \tag{1}$$

for all $x, y \in C$, *where* $0 < a < 1$, $p \geqslant 1$ *and* $a + 2p = 1$. *Then* T *has a unique fixed point.*

In recent years, many theorems which are closely related to Greguš's Theorem have appeared ([1]–[9]).

The purpose of this paper is to define and to investigate a class of mappings (not necessarily continuous) which are defined on metric spaces and satisfy the following contractive condition:

$$\begin{aligned} d(Tx,Ty) \leqslant{} & a\max\{d(x,y), c[d(x,Ty)+d(y,Tx)]\} \\ & + b\max\{d(x,Tx), d(y,Ty)\} \end{aligned} \tag{2}$$

where

$$0 < a < 1, \quad a + b = 1, \quad c \leqslant \frac{4-a}{8-a}.$$

We shall prove a fixed point theorem which is a double generalization of the above theorem of Greguš. Firstly, the nonexpansive nature of the mapping is generalized, and secondly, the underlying space is more general than Banach spaces. An example is constructed to show that our theorem is a genuine generalization of the theorems of Greguš [7] and Li [8].

We recall the following definition of the convex metric space.

Definition 2. (Takahashi [10]) Let X be a metric space and $I = [0,1]$ the closed unit interval. A continuous mapping $W\colon X \times X \times I \to X$ is said to be a convex structure on X if for all $x, y \in X$ and $\lambda \in I$, $d[u, W(x,y,\lambda)] \leqslant \lambda d(u,x) + (1-\lambda) d(u,y)$ for all $u \in X$. X together with a convex structure is called a convex metric space. A subset $K \subseteq X$ is convex, if $W(x,y,\lambda) \in K$ whenever $x, y \in K$ and $\lambda \in I$.

Clearly a Banach space, or any convex subset of it, is a convex metric space with $W(x,y,\lambda) = \lambda x + (1-\lambda) y$.

2. Main result

Now we are in position to state our main result.

Theorem 3. *Let C be a closed convex subset of a complete convex metric space X and $T\colon C \to C$ a mapping satisfying (2) for all $x, y \in C$. Then T has a unique fixed point.*

Proof. Let $x \in C$ be arbitrary and let $\{x_n\}$ be the sequence defined by

$$x_0 = x, \quad x_{n+1} = T x_n \quad (n = 0, 1, 2, \ldots).$$

From (2) we have

$$\begin{aligned}
&d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \\
&\leqslant a \max\{d(x_n, x_{n+1}), c[d(x_n, x_{n+2}) + 0]\} + b \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} \\
&\leqslant a \max\{d(x_n, x_{n+1}), c[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]\} \\
&\quad + b \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}.
\end{aligned}$$

If we suppose that $d(x_{n+1}, x_{n+2}) > d(x_n, x_{n+1})$ then we obtain

$$d(x_{n+1}, x_{n+2}) < a d(x_{n+1}, x_{n+2}) + b d(x_{n+1}, x_{n+2}) = (a+b) d(x_{n+1}, x_{n+2}),$$

which is a contradiction since $a+b=1$. Therefore

$$d(x_{n+1},x_{n+2})\leqslant d(x_n,x_{n+1})\leqslant\ldots\leqslant d(x,Tx). \tag{3}$$

Using (2), (3) and the triangle inequality we get

$$\begin{aligned} d(Tx_n,T^3x_n)\leqslant{}& a\max\{d(Tx_{n-1},T^3x_{n-1}),c[d(Tx_n,T^3x_n)+2d(x,Tx)]\}\\ &+b\,d(x,Tx). \end{aligned} \tag{4}$$

We shall show that for some $k\in\mathbb{N}$

$$d(Tx_k,T^3x_k)\leqslant\left(1+\frac{2a(2-a)}{8-5a+a^2}\right)d(x,Tx). \tag{5}$$

Assume first that for some $n=k$ we have from (4)

$$d(Tx_k,T^3x_k)\leqslant ac[d(Tx_k,T^3x_k)+2d(x,Tx)]+bd(x,Tx). \tag{6}$$

Then we get

$$d(Tx_k,T^3x_k)\leqslant\frac{1-a+2ac}{1-ac}d(x,Tx).$$

Since $0<a<1$ and $0\leqslant c\leqslant\frac{4-a}{8-a}$, it follows that

$$\frac{1-a+2ac}{1-ac}\leqslant\frac{8-a-a^2}{8-5a+a^2}=1+\frac{2a(2-a)}{8-5a+a^2}.$$

So (5) holds.

If we suppose that (6) does not follow from (4) for any n, then we have

$$d(Tx_n,T^3x_n)\leqslant ad(Tx_{n-1},T^3x_{n-1})+bd(x,Tx).$$

By induction we obtain

$$\begin{aligned} d(Tx_n,T^3x_n)&\leqslant a[ad(Tx_{n-2},T^3x_{n-2})+bd(x,Tx)]+bd(x,Tx)\\ &\leqslant\ldots\leqslant a^nd(Tx_0,T^3x_0)+b(1+a+a^2+\ldots)d(x,Tx)\\ &\leqslant 2a^nd(x,Tx)+b\frac{1}{1-a}d(x,Tx)=(2a^n+1)d(x,Tx), \end{aligned} \tag{7}$$

where we have used that $b=1-a$. Since $0<a<1$ we can choose n such that

$$2a^n\leqslant\frac{2a(2-a)}{8-5a+a^2}.$$

Then we see from (7) that for such $k = n$ the inequality (5) holds. So we have shown (5).

Let k be such that (5) holds and put $y = x_k$. Then

$$d(Ty, T^3y) \leqslant (1+q)d(x, Tx); \quad q = \frac{2a(2-a)}{8-5a-a^2}. \tag{8}$$

Since C is convex, by Definition 2 the element $W(T^2y, T^3y, \frac{1}{2}) = z$ is in C. Then, using Definition 2, (3) and (8) we have

$$d(z, T^2y) \leqslant \frac{1}{2}d(T^2y, T^3y) \leqslant \frac{1}{2}d(x, Tx),$$

$$d(z, T^3y) \leqslant \frac{1}{2}d(T^2y, T^3y) \leqslant \frac{1}{2}d(x, Tx),$$

$$d(z, Ty) \leqslant \frac{1}{2}[d(Ty, T^2y) + d(Ty, T^3y)] \leqslant \left(1 + \frac{q}{2}\right)d(x, Tx), \tag{9}$$

$$d(z, Tz) \leqslant \frac{1}{2}[d(Tz, T^2y) + d(Tz, T^3y)]. \tag{10}$$

Now we show that

$$d(z, Tz) \leqslant \left(1 - \frac{a^3(1-a)}{64}\right) d(x, Tx). \tag{11}$$

Set

$$M = M(x, z) = \max\{d(x, Tx), d(z, Tz)\} \tag{12}$$

and suppose $M > 0$. Then (2), (3) and (9) imply

$$d(Tz, T^3y) \leqslant a \max\left\{\frac{1}{2}M, c\left[\frac{1}{2}M + d(Tz, T^2y)\right]\right\} + bM, \tag{13}$$

$$d(Tz, T^2y) \leqslant a \max\left\{\left(1 + \frac{q}{2}\right)M, c\left[\frac{1}{2}M + d(Tz, Ty)\right]\right\} + bM. \tag{14}$$

Consider now four possible cases.

Case I. Assume that we have from (13)

$$d(Tz, T^3y) \leqslant \frac{1}{2}aM + bM = \left(1 - \frac{a}{2}\right)M \tag{15}$$

and from (14)

$$d(Tz, T^2y) \leqslant a\left(1 + \frac{q}{2}\right)M + bM = \left(1 + \frac{aq}{2}\right)M. \tag{16}$$

Then by (8), (10), (15) and (16) we get

$$(17)\quad d(z,Tz) \leqslant \frac{1}{2}\left[2-\frac{a}{2}(1-q)\right]M \leqslant \left(1-\frac{a}{4}\cdot\frac{8-9a+3a^2}{8-5a+a^2}\right)M \leqslant \left(1-\frac{a}{8}\right)M.$$

Since $(1-\frac{a}{8}) < 1-\frac{a^3(1-a)}{64} < 1$ and M is defined by (12), we conclude that (17) implies (11).

Case II. Assume now that (13) implies (15) and (14) implies the inequality

$$d(Tz,T^2y) \leqslant ac\left[\frac{1}{2}M + d(Tz,Ty)\right] + bM.$$

Using the triangle inequality and (8) we get

$$d(Tz,Ty) \leqslant d(Tz,T^3y) + d(T^3y,Ty) \leqslant d(Tz,T^3y) + (1+q)M.$$

So we have

$$(18)\qquad d(Tz,T^2y) \leqslant ac\left[\left(\frac{3}{2}+q\right)M + d(Tz,T^3y)\right] + bM.$$

Since $c<\frac{1}{2}$ and $q<\frac{1}{2}$(see (8)), we have $ac(\frac{3}{2}+q) < a$ and so by (15) and (18)

$$(19)\qquad d(Tz,T^2y) < \left[a+\frac{a}{2}\left(1-\frac{a}{2}\right)+b\right]M = \left[1+\frac{a}{2}\left(1-\frac{a}{2}\right)\right]M.$$

Now from (10), (15) and (19) we get

$$d(z,Tz) \leqslant \frac{1}{2}\left(1-\frac{a}{2}+1+\frac{a}{2}-\frac{a^2}{4}\right)M = (1-\frac{a^2}{8})M < \left[1-\frac{a^3(1-a)}{64}\right]M.$$

Hence we conclude that (11) holds.

Case III. Assume now that (13) implies

$$(20)\qquad d(Tz,T^3y) \leqslant ac\left[\frac{1}{2}M + d(Tz,T^2y)\right] + bM$$

and that (16) holds. Then (10), (16) and (20) imply

$$(21)\qquad d(z,Tz) \leqslant \frac{1}{2}\left[1+\frac{aq}{2}+ac\left(\frac{1}{2}+1+\frac{aq}{2}\right)+1-a\right]M.$$

Since

$$aq < q = \frac{2a(2-a)}{8-5a+a^2} < \frac{4+a^2}{2(6-a)};\quad c \leqslant \frac{4-a}{8-a},$$

from (21) we get

$$\begin{aligned}d(z,Tz) &< \frac{1}{4}\left[4-2a+aq+a(3+q)\frac{4-a}{8-a}\right]M = \frac{1}{4}\left[4+a\frac{-4-a}{8-a}+\frac{aq(12-2a)}{8-a}\right]M\\ &< \frac{1}{4}\left[4+a\frac{-4-a}{8-a}+a\frac{4+a^2}{2(6-a)}\cdot\frac{12-2a}{8-a}\right]M\\ &= \left[1-\frac{a^2(1-a)}{4(8-a)}\right]M < \left[1-\frac{a^3(1-a)}{64}\right]M.\end{aligned}$$

Hence, and using (12), we conclude that (11) holds.

Case IV. Assume now that (13) implies (18) and (14) implies (20). Adding (18) and (20) we obtain

$$d(Tz,T^2y)+d(Tz,T^3y) \leqslant ac[(2+q)M+d(Tz,T^2y)+d(Tz,T^3y)]+2bM$$

and hence

$$d(Tz,T^2y)+d(Tz,T^3y) \leqslant \frac{ac(2+q)+2(1-a)}{1-ac}M.$$

Now from (10)

$$d(z,Tz) \leqslant \frac{ac(1+\frac{q}{2})+1-a}{1-ac}M. \tag{22}$$

Since

$$q=\frac{2a(2-a)}{8-5a+a^2} < \frac{2a(1-\frac{a}{8})}{4-a}; \quad c \leqslant \frac{4-a}{8-a},$$

from (22) we have

$$\begin{aligned}d(z,Tz) &< \frac{1}{1-ac}\left[ac\left(1+\frac{a(1-\frac{a}{8})}{4-a}\right)+1-a\right]M\\ &\leqslant \frac{8-a}{8-5a+a^2}\cdot\frac{a(4-\frac{a^2}{8})+8-9a+a^2}{8-a}M\\ &= \frac{8-5a+a^2-\frac{a^3}{8}}{8-5a+a^2}M = \left[1-\frac{a^3}{8(8-5a+a^2)}\right]M.\end{aligned}$$

Hence, as

$$-\frac{a^3}{8(8-5a+a^2)} < -\frac{a^3}{8\cdot 8} < -\frac{a^3(1-a)}{64},$$

we obtain (11). Therefore, (11) holds in each case.

Since for any $x\in C$ there exists $z=z(x)$ such that (11) holds, we have

$$\inf\{d(x,Tx)\colon\ x\in C\}=0. \tag{23}$$

Now we show that

$$\max\{d(Tx,Ty),d(x,y)\} \leqslant \frac{3-a}{1-a}\max\{d(x,Tx),d(y,Ty)\}. \tag{24}$$

Let $R = R(x,y) = \max\{d(x,Tx),d(y,Ty)\}$. Then (2) and the triangle inequality yield

$$\begin{aligned} d(Tx,Ty) \leqslant a\max\{&[d(x,Tx)+d(Tx,Ty)+d(y,Ty)], \\ &c[d(x,Tx)+2d(Tx,Ty)+d(y,Ty)]\} + bR \\ &\leqslant (2a+b)R + ad(Tx,Ty) = (1+a)R + ad(Tx,Ty). \end{aligned}$$

Hence

$$d(Tx,Ty) \leqslant \frac{1+a}{1-a}R. \tag{25}$$

Then we have

$$d(x,y) \leqslant d(x,Tx)+d(Tx,Ty)+d(y,Ty) \leqslant 2R + \frac{1+a}{1-a}R. \tag{26}$$

From (25) and (26) we get (24).

Now by (23) we can choose a sequence $\{x_n\}$ in C such that

$$d(x_n,Tx_n) \leqslant \frac{1}{n} \quad (n = 1,2,\ldots).$$

From (24) we have

$$\max\{d(Tx_n,Tx_m),d(x_n,x_m)\} \leqslant \frac{3-a}{1-a}\cdot\frac{1}{n} \quad \text{for } 1 \leqslant n \leqslant m.$$

Therefore, both $\{x_n\}$ and $\{Tx_n\}$ are Cauchy sequences, and moreover they have a common limit, say $u \in C$. From (2) we obtain

$$\begin{aligned} d(Tx_n,Tu) \leqslant a\max\{&d(x_n,u),c[d(x_n,Tu)+d(u,Tx_n)\} \\ &+ b\max\{d(x_n,Tx_n),d(u,Tu)\}. \end{aligned}$$

Taking the limit when n tends to infinity we get

$$d(u,Tu) \leqslant (ac+b)d(u,Tu) = [1-a(1-c)]d(u,Tu)$$

Hence $d(u,Tu) = 0$, since $1-a(1-c) < 1$. So we have proved that u is a fixed point of T. The uniqueness of the fixed point follows from (2). $\square$

Remark. If $c=0$ then Theorem 3 reduces to the theorem of Fisher [5]. Such result also appears as a corollary of the corresponding fixed point theorems in [1], [4], [6] and [9].

Theorem 4. *Let C be as in Theorem 3 and let $T\colon\ C\to C$ be a mapping satisfying*

$$d(Tx,Ty)\leqslant ad(x,y)+b\max\{d(x,Tx),d(y,Ty)\}+c[d(x,Ty)+d(y,Tx)] \tag{27}$$

for all $x,y\in C$, where

$$0\leqslant a<1,\quad b\geqslant 0,\quad c\geqslant 0,\quad a+c>0 \tag{28}$$

and

$$a+b+\frac{7}{3}c=1. \tag{29}$$

Then T has a unique fixed point.

Proof. Set $a+\frac{7}{3}c=a_1$. Then $a_1+b=1$ and we have

$$\begin{aligned}
&ad(x,y)+b\max\{d(x,Tx),d(y,Ty)\}+c\,\frac{7}{3}\cdot\frac{3}{7}\cdot\Big[d(x,Ty)+d(y,Tx)\Big]\\
&\quad\leqslant\Big(a+\frac{7}{3}c\Big)\max\Big\{d(x,y),\frac{3}{7}\Big[d(x,Ty)+d(y,Tx)\Big]\Big\}+b\max\Big\{d(x,Tx),d(y,Ty)\Big\}\\
&\quad=a_1\max\Big\{d(x,y),\frac{3}{7}\Big[d(x,Ty)+d(y,Tx)\Big]\Big\}+b\max\Big\{d(x,Tx),d(y,Ty)\Big\}.
\end{aligned}$$

Since $\frac{3}{7}<\frac{4-a}{8-a}$ we see that (27), (28), (29) imply (2) with $a_1+b=1$. Therefore, we can apply Theorem 3 in the case $a>0$.

If $a=0$, then $a+c>0$ implies $c>0$ and then (29) implies

$$0<b+2c=1-\frac{1}{3}c<1.$$

So Theorem 4 in the case $a=0$ reduces to a special case of the Theorem 1 of [2]. $\square$

Corollary 5. (Li [8]) *Let C be a closed convex subset of a convex metric space X and let $T\colon\ C\to C$ be a mapping satisfying*

$$d(Tx,Ty)\leqslant ad(x,y)+b[d(x,Tx)+d(y,Ty)]+c[d(x,Ty)+d(y,Tx)] \tag{30}$$

for all $x,y\in C$, where $0\leqslant a<1$, $b\geqslant 0$, $c\geqslant 0$, $a+c>0$ and

$$a+2b+3c\leqslant 1. \tag{31}$$

If X has the property that every sequence of non-empty closed convex subsets of X with diameters tending to zero has non-empty intersection, then T has a unique fixed point in C.

P r o o f. It is clear that the inequalities (27) and (28) are more general than the corresponding inequalities (30) and (31). Since the property of X stated in Corollary 5 is equivalent to the completeness of X, we see that all assumptions of Theorem 4 are satisfied. $\square$

The following simple example shows that our Theorems 3 and 4 are genuine generalizations of the theorems of Greguš [7] and Li [8].

Example. Let $C = [-3, 5]$ be the subset of real numbers, and let $T\colon C \to C$ be a mapping defined by

$$Tx = \frac{x}{7} \quad \text{if } -2 \leqslant x \leqslant 5; \qquad Tx = 5 \quad \text{if } -3 \leqslant x < -2.$$

It is clear that if $x, y \in [-3, -2)$ or $x, y \in [-2, 5]$, then $d(Tx, Ty) \leqslant \frac{1}{7} d(x, y)$. Let now $x \in [-2, 5]$ and $y \in [-3, -2)$. Then we have

$$d(Tx, Ty) \leqslant 5 + \frac{2}{7} < \frac{6}{7} \cdot 7 \leqslant \frac{6}{7} d(y, Ty) = \frac{6}{7} \max\{d(x, Tx), d(y, Ty)\}.$$

Therefore T satisfies the condition (27) with $a = \frac{1}{7}$, $b = \frac{6}{7}$ and $c = 0$, and condition (2) with $a = \frac{1}{7}$, $b = \frac{6}{7}$ and any $0 \leqslant c \leqslant \frac{27}{55}$. Since C is compact, all hypotheses of Theorems 3 and 4 are satisfied and $u = 0$ is the unique fixed point of T. On the other hand, T does not satisfy (30) with $a + 2b + 3c \leqslant 1$, and hence the contractive condition of Greguš, since for all $x \in [-1, 0]$ and $y \in [-3, -2)$ we have

$$\begin{aligned} & ad(x, y) + b\Big[d(x, Tx) + d(y, Ty)\Big] + c\Big[d(x, Ty) + d(y, Tx)\Big] \\ & \quad \leqslant (a + 2b + 3c) \max\Big\{d(x, y), \frac{1}{2}\Big[d(x, Tx) + d(y, Ty)\Big], \frac{1}{3}\Big[d(x, Ty) + d(y, Tx)\Big]\Big\} \\ & \quad \leqslant \max\Big\{3, \frac{1}{2}(\frac{6}{7} + 8), \frac{1}{3}(6 + 3)\Big\} = 5 - \frac{4}{7} < 5 \leqslant d(Tx, Ty) \end{aligned}$$

for any $a, b, c \geqslant 0$ with $a + 2b + 3c \leqslant 1$.

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