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LATERAL COMPLETION OF A PROJECTABLE LATTICE ORDERED GROUP

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A lattice ordered group is said to be laterally complete if each its disjoint subset has the least upper bound.

The notion of the lateral completion of a lattice ordered group was introduced by Conrad in [5] (the term "orthogonal hull" was applied for this notion in [7]). Earlier, lateral completions of complete lattice ordered groups were investigated in [6], [9], [10]. Further, in connection with the lateral completion the following types of lattice ordered groups have been dealt with: (i) representable lattice ordered groups; (ii) lattice ordered groups satisfying the condition (F) (which says that each bounded disjoint set is finite); (iii) lattice ordered groups with a basis; (iv) lattice ordered groups with zero radical (cf. [5]); (v) lattice ordered groups with zero distributive radical (cf. [4]); (vi) archimedean lattice ordered groups (cf. [2]); (vii) strongly pojectable lattice ordered groups (cf. [7], [8]).

Conrad [5] proposed the question whether each lattice ordered group has a uniquely determined lateral completion. This question was affirmatively solved by Bernau [1].

For a lattice ordered group G let G^L be its lateral completion. The symbol $\ell(G)$ will denote the underlying lattice of G.

Bernau's method consists in applying a transfinite process to construct G^L. In each step of this construction, new elements are added to those already given by the preceding steps. The resulting structure (i.e., G^L) is then obtained as a direct limit.

The aim of the present paper is to generalize the main result of [7] concerning lateral completions of strongly projectable lattice ordered groups for the case when the assumption of strong projectability is replaced by the weaker assumption of projectability.

We prove the following results:

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(A) Let G_1 and G_2 be lattice ordered groups. Suppose that (i) G_1 is projectable, and (ii) the lattices $\ell(G_1)$ and $\ell(G_2)$ are isomorphic. Then the lattices $\ell(G_1^L)$ and $\ell(G_2^L)$ are isomorphic as well.

(B) Let G be a projectable lattice ordered group. Then each element of the positive cone of G^L is a join of a disjoint subset of G.

In connection with (A) we remark that if G_1 and G_2 are lattice ordered groups such that the lattices $\ell(G_1)$ and $\ell(G_2)$ are isomorphic, then G_1 need not be isomorphic to G_2.

Further, concerning (B), we remark that without the assumption of projectability the assertion of (B) need not be valid in general. Next, (B) implies that when constructing G^L for a projectable lattice ordered group G it suffices to apply only one step in the process of adding new elements to G. An analogous situation occurs in the case when G satisfies the condition (F) (cf. [5], Theorem 6.1, and [1], Theorem 6.1).

1. Preliminaries

For lattice ordered groups we employ the standard notation; cf. e.g., [3]. Let G be a lattice ordered group.

For $X \subseteq G$, the polar X^δ is defined by

$$X^\delta = \{y \in G\colon\ |y| \wedge |x| = 0 \text{ for each } x \in X\}.$$

If $X = \{x\}$ is a one-element set, then $X^{\delta\delta}$ is said to be a principal polar; we denote

$$\{x\}^{\delta\delta} = [x].$$

G is called pojectable (strongly projectable) if for each $x \in G$ (or each $X \subseteq G$, respectively) $[x]$ (or $X^{\delta\delta}$) is a direct factor of G.

This means that if G is projectable, then for each $x \in G$ we have a direct product decomposition

$$G = [x] \times \{x\}^\delta.$$

Simple examples show that projectablility does not imply strong projectability.

G is said to be σ-complete if each nonempty upper bounded denumerable subset of G possesses the least upper bound in G. This is equivalent to the corresponding dual condition.

The following result is wellknown:

Proposition 1.1. *Each σ-complete lattice ordered group is projectable.*

An indexed system $\{x_i\}_{i\in I}$ of elements of G is called disjoint (or orthogonal) if

(i) $x_i \geqslant 0$ for each $i \in I$, and

(ii) $x_{i(1)} \wedge x_{i(2)} = 0$ whenever $i(1)$ and $i(2)$ are distinct elements of I.

Definition 1.2 (Cf. [5]). Let G be a lattice ordered group. Next, let H be a lattice ordered group such that

(i) G is an ℓ-subgroup of H;
(ii) for each $0 < h \in H$ there is $g \in G$ with $0 < g \leqslant h$;
(iii) H is laterally complete;
(iv) if H_1 is a laterally complete lattice ordered group such that G is an ℓ-subgroup of H_1 and H_1 is an ℓ-subgroup of H, then $H_1 = H$.

Under these assumptions H is called a lateral completion of G.

Each lattice ordered group G has a lateral completion and this is uniquely determined up to isomorphisms (cf. [1]). The lateral completion of G will be denoted by G^L.

2. Auxiliary results

As usual, the positive cone $\{x \in G\colon\, x \geqslant 0\}$ of G will be denoted by G^+. Let $X \subseteq G^+$. We put

$$X^\perp = \{y \in G^+\colon\, x \wedge y = 0 \text{ for each } x \in X\}.$$

Then X will be called a polar of G^+; if X is a one-element set, then $X^{\perp\perp}$ is called a principal polar of G^+.

G^+ is a lattice ordered semigroup; the corresponding lattice will be denoted by $\ell(G^+)$.

Lemma 2.1. *Let G_1 and G_2 be lattice ordered groups and let φ be an isomorphism of $\ell(G_1^+)$ onto $\ell(G_2^+)$. Let $Y \subseteq G_1^+$. Then Y is a polar of G_1^+ if and only if $\varphi(Y)$ is a polar of G_2^+; moreover, Y is principal if and only if $\varphi(Y)$ is principal.*

Proof. This is an immediate consequence of the definition of the polar in the positive cone. $\square$

Let A, B be nonempty subsets of G^+. Consider the following conditions for the pair A, B:

(a_1) A, B are sublattices of $\ell(G^+)$ and for each $g \in G^+$ there are uniquely determined elements $g_A \in A$, $g_B \in B$ such that $g = g_A \vee g_B$.
(a_2) A, B are convex sublattices of $\ell(G^+)$ and for each $g \in G^+$ there are elements $g'_A \in A$ and $g'_B \in B$ such that $g = g'_A \vee g'_B$.

Lemma 2.2. *The conditions* (a_1) *and* (a_2) *are equivalent.*

P r o o f. Let (a_1) be valid and let $a \in A$, $g \in G^+$, $g \leqslant a$. Next, let g_A and g_B be as in (a_1). It is clear that we must have $0_A = 0 = 0_B$ and $a_A = a$, $a_B = 0$. Since $g_B \leqslant a$ we obtain $a = a \vee g_B$ and hence, in view of (a_1), $g_B = 0$. Therefore $g = g_A$ and hence $g \in A$. Thus (a_2) holds.

Conversely, suppose that (a_2) is satisfied. Let $g \in G_B^+$ and let $a_1, a_2 \in A$, $b_1, b_2 \in B$, $g = a_1 \vee b_1 = a_2 \vee b_2$. Since $a_1 \wedge b_2 \in A \cap B$, we obtain that $a_1 \wedge b_2 = 0$. Similarly, $a_2 \wedge b_1 = 0$. Hence

$$a_1 = a_1 \wedge g = a_1 \wedge (a_2 \vee b_2) = a_1 \wedge a_2,$$

implying that $a_1 \leqslant a_2$. Analogously we obtain $a_2 \leqslant a_1$ and thus $a_1 = a_2$. Similarly, $b_1 = b_2$. Thus (a_1) holds. □

Lemma 2.3. *Let* (a_1) *be valid. Let* $g^1, g^2 \in G^+$. *Then* $g^1 \leqslant g^2$ *if and only if* $g_A^1 \leqslant g_A^2$ *and* $g_B^1 \leqslant g_B^2$.

P r o o f. If $g_A^1 \leqslant g_A^2$ and $g_B^1 \leqslant g_B^2$, then clearly $g^1 \leqslant g^2$. Conversely, let $g^1 \leqslant g^2$. By analogous consideration as in the proof of 2.2 we obtain $g_A^1 \wedge g_B^2 = 0$, whence

$$g_A^1 = g_A^1 \wedge g^2 = g_A^1 \wedge (g_A^2 \vee g_B^2) = g_A^1 \wedge g_A^2,$$

yielding that $g_A^1 \leqslant g_A^2$. Similarly, $g_B^1 \leqslant g_B^2$. □

Let us suppose that (a_1) is valid. Then $\ell(G^+)$ will be said to be an internal direct product of A and B and we express this fact by writing

$$\ell(G^+) = (\mathrm{int}) A \times B.$$

In view of 2.3, this notation is appropriate.

By the obvious induction we define the meaning of the notation

$$\ell(G^+) = (\mathrm{int}) A_1 \times A_2 \times \ldots \times A_n;$$

the sublattices A_i are called internal direct factors of $\ell(G^+)$.

The following lemma is obvious.

Lemma 2.4. *Assume that we have a direct product decomposition*

$$G = G_1 \times G_2 \times \ldots \times G_n.$$

Put $A_i = G^+ \cap G_i (i = 1, 2, \ldots, n)$. *Then*

$$\ell(G^+) = (\mathrm{int}) A_1 \times A_2 \times \ldots \times A_n.$$

Lemma 2.5. *Assume that*

$$\ell(G^+) = (\mathrm{int})A_1 \times A_2 \times \ldots \times A_n.$$

If $i \in I$, then let G_i be the set of all $g \in G$ such that $g = x_i^1 - x_i^2$ for some $x_i^1, x_i^2 \in A_i$. Then G_i is a convex ℓ-subgroup of G and

$$G = G_1 \times G_2 \times \ldots \times G_n.$$

P r o o f. This is a consequence of a result of [11]; cf. also [7], Theorem 2.1. $\square$

Proposition 2.6. *Let G_1 and G_2 be lattice ordered groups such that G_1 is projectable and $\ell(G_1), \ell(G_2)$ are isomorphic. Then G_2 is projectable as well.*

P r o o f. Let X_2 be a principal polar of G_2 that is generated by an element x. Then without loss of generality we can suppose that $x \geqslant 0$ (in fact, we have $\{x\}^\delta = \{|x|\}^\delta$). In view of the assumption there exists an isomorphisms φ of $\ell(G_2)$ onto $\ell(G_1)$. Put $\varphi_1(b) = \varphi(t) - \varphi(0)$ for each $t \in G_2$. Hence φ_1 is an isomorphism of $\ell(G_2)$ onto $\ell(G_1)$ such that $\varphi_1(0) = 0$. Put $\varphi_0 = \varphi_1|G_2^+$. Thus φ_0 is an isomorphism of $\ell(G_2^+)$ onto $\ell(G_1^+)$.

Put $X_2^0 = G_2^+ \cap X_2$. Then X_2^0 is a principal polar in G_2^+ generated by the element x. Denote $X_1^0 = \varphi_0(X_2^0)$. According to 2.1, X_1^0 is a principal polar of G_1^+ generated by the element $\varphi_1(x)$. Let X_1 be the principal polar in G_1 generated by $\varphi_1(x)$. Clearly $X_1^0 = X_1 \cap G_1^+$.

Since G_1 is projectable, the relation

$$G_1 = X_1 \times X_1^\delta$$

is valid. Put $A_1 = X_1 \cap G_1^+$, $A_2 = X_1^\delta \cap G_1^+$. In view of 2.4 we have

$$\ell(G_1^+) = (\mathrm{int})A_1 \times A_2,$$

and $A_1 = X_1^0$. Hence $\varphi_0^{-1}(A_1) = X_2^0$ and $\varphi_0^{-1}(A_2) = X_2^{0\perp}$. Moreover,

$$\ell(G_2^+) = (\mathrm{int})X_2^0 \times X_2^{0\perp}.$$

Therefore according to 2.5,

$$G_2 = X_2 \times X_2^\delta.$$

Hence G_2 is projectable. $\square$

In the present section we assume that G is a projectable lattice ordered group.

Let H be the set of all indexed disjoint systems $(x_i)_{i\in I}$ with $\operatorname{card} I \leqslant \operatorname{card} G$. For $h_1 = (x_i)_{i\in I}$ and $h_2 = (y_j)_{j\in J}$ in H we put $h_1 \leqslant h_2$ if

$$x_i = \bigvee_{j\in J} (x_i \wedge y_j)$$

is valid for each $i \in I$.

It is obvious that if for each $i \in I$ there exists $j \in J$ with $x_i \leqslant y_j$, then $h_1 \leqslant h_2$.

Lemma 3.1. *The relation $\leqslant$ is a quasiorder on the set H.*

Proof. It suffices to apply the same steps as in the proof of Lemma 3.1, [7]. □

If h_1 and h_2 are elements of H such that $h_1 \leqslant h_2$ and $h_2 \leqslant h_1$, then we put $h_1 \sim h_2$. The relation $\sim$ is an equivalence on H and the corresponding set $H/\sim$ is a partially ordered set. We denote

$$\overline{H} = H/\sim,$$

$$\overline{h}_1 = \{h_2 \in H\colon\ h_1 \sim h_2\}.$$

Let H_0 be the set of all $h \in H$ such that, whenever $0 < g \in G$, then $g \wedge x_i > 0$ for some $i \in I$.

By applying the Axiom of Choice we obtain

Lemma 3.2. *Let $h = (x_i)_{i\in I} \in H$. Then there exists $h' = (x_j)_{j\in J}$ in H_0 such that $I \subseteq J$.*

If A is a direct factor of G and $g \in G$, then we denote with gA the component of g in A. It is easy to verify that if $g \geqslant 0$, then gA is the greatest element of the set $\{a \in A\colon\ a \leqslant g\}$.

Lemma 3.3. *Let $h = (x_i)_{i\in I} \in H_0$ and $0 \leqslant g \in G$. Then $g = \bigvee_{i\in I} g[x_i]$.*

Proof. For each $i \in I$ we have $g[x_i] \leqslant g$. By way of contradiction, assume that the relation $g = \bigvee_{i\in I} g[x_i]$ fails to hold. Then there is $g' \in G$ such that $g[x_i] \leqslant g'$ for each $i \in I$ and $g \nleqslant g'$. Put $g'' = g \wedge g'$. Thus $g'' < g$ and

$$g[x_i] \leqslant g'' \quad \text{for each } i \in I.$$

This yields that

$$(g[x_i])[x_i] \leqslant g''[x_i] \ ;$$

since $g[x_i] \in [x_i]$ we get $(g[x_i])[x_i] = g[x_i]$, thus

$$g[x_i] \leqslant g''[x_i] \quad \text{for each } i \in I.$$

Since $h \in H_0$ there exists $i \in I$ such that $g^* \wedge x_i > 0$, where $g^* = g - g''$. We have $g^* \wedge x_i \in [x_i]$ and $0 \leqslant g^* \wedge x_i \leqslant g^*$, thus

$$g^* \wedge x_i \leqslant g^*[x_i].$$

Hence

$$g[x_i] = g^*[x_i] + g''[x_i] > g''[x_i] \geqslant g[x_i],$$

which is a contradiction. $\square$

Let us consider two elements of H having the form

$$x = (x_i)_{i \in I}, \quad y = (y_j)_{j \in J}.$$

In view of 3.2 there are $x', y' \in H_0$ such that

$$x' = (x_i)_{i \in I'}, \quad y' = (y_j)_{j \in J'}$$

with $I \subseteq I'$ and $J \subseteq J'$. Put $z = (x_i \wedge y_j)_{(i,j) \in I' \times J'}$.

Lemma 3.4. *The element z belongs to H_0.*

Proof. It is obvious that z belongs to H. Let $0 < g \in G$. Since $x' \in H_0$, there is $i(0) \in I'$ with $g \wedge x_{i(0)} > 0$. Next, since $y' \in H_0$, there is $j(0) \in J'$ such that $(g \wedge x_{i(0)}) \wedge y_{j(0)} > g$. Hence $z \in H_0$. $\square$

Put $z_{ij} = x_i \wedge y_j$ $(i \in I', j \in J')$.

Lemma 3.5. *Let $i \in I$ and $j \in J$. Then*

$$x_i = \bigvee_{j \in J'} x_i[z_{ij}], \quad y_j = \bigvee_{i \in I'} y_j[z_{ij}].$$

Proof. This is a consequence of 3.4 and 3.3. $\square$

Denote

$$x^0 = (x_i[z_{ij}])_{(i,j)\in I\times J'},$$

$$y^0 = (y_j[z_{ij}])_{(i,j)\in I'\times J}.$$

Then clearly $x^0, y^0 \in H$.

Lemma 3.6. *$x \sim x^0$ and $y \sim y^0$.*

Proof. Let $(i,j) \in I \times J'$. Then $x_i[z_{ij}] \leqslant x_i$, whence $x^0 \leqslant x$. Next, 3.5 implies that the relation $x \leqslant x^0$ is valid. Thus $x \sim x^0$. Analogously we obtain that $y \sim y^0$. □

It is clear that the system $\overline{H}$ has the least element. Next, if we apply the notation as above, then from the relations $x^0, y^0 \in H$ we infer that the indexed system $t = (t_{ij})_{(i,j)\in I'\times J'}$ with $t_{ij} = (x_i[z_{ij}]) \vee (y_j[z_{ij}])$ also belongs to H. We obviously have $x^0 \leqslant t$, $y^0 \leqslant t$. Thus we obtain

Lemma 3.7. *The partially ordered set $\overline{H}$ is directed.*

Let us modify the systems x^0 and y^0 as follows. For $(i,j) \in I' \times J'$ we put $x^*_{ij} = x_i[z_{ij}]$ if $(i,j) \in I \times J'$, and $x^*_{ij} = 0$ otherwise. Similarly we set $y^*_{ij} = y_j[z_{ij}]$ if $(i,j) \in I' \times J$ and $y^*_{ij} = 0$ otherwise. Then $x^* = (x^*_{ij})_{(i,j)\in I'\times J'}$ and $y^* = (y^*_{ij})_{(i,j)\in I'\times J'}$ belong to H and

$$x^0 \sim x^*, \quad y^0 \sim y^*.$$

Thus 3.6 yields

Lemma 3.8. *Let $x, y \in H$. There exist $x^*, y^* \in H$ such that $x^* = (x^*_t)_{t\in T}$, $y^* = (y^*_t)_{t\in T}$, $x \sim x^*$, $y \sim y^*$ and $x^*_{t(1)} \wedge y^*_{t(2)} = 0$ whenever t_1 and t_2 are distinct elements of T.*

By the obvious induction we can generalize the previous lemma to the case when the elements x, y are replaced by a finite sequence $x^1, x^2, \ldots, x^n$ of elements of G^+.

Lemma 3.9. *The partially ordered set $\overline{H}$ is a lattice.*

Proof. Let $\overline{x}, \overline{y}, \overline{u}, \overline{v} \in \overline{H}$, $\overline{u} \leqslant \overline{x} \leqslant \overline{v}$, $\overline{u} \leqslant \overline{y} \leqslant \overline{v}$. By 3.8 (generalized to the case of four elements) we can suppose that

$$x = (x_t)_{t\in T}, \quad y = (y_t)_{t\in T}, \quad u = (u_t)_{t\in T}, \quad v = (v_t)_{t\in T}$$

and that, whenever $t(1)$ and $t(2)$ are distinct elements of t, then each of the elements $x_{t(1)}, y_{t(1)}, u_{t(1)}, v_{t(1)}$ is disjoint with each of the elements $x_{t(2)}, y_{t(2)}, u_{t(2)}, v_{t(2)}$.

From the definition of the relation $\leqslant$ in H we obtain that

$$u_t \leqslant x_t \leqslant v_t, u_t \leqslant y_t \leqslant v_t$$

is valid for each $t \in T$.

Put $u_t^0 = x_t \wedge y_t$, $v_t^0 = x_t \vee y_t$ for each $t \in T$, and

$$u^0 = (u_t^0)_{t\in T}, \quad v^0 = (v_t^0)_{t\in T}.$$

We have $u^0, v^0 \in H$ and

$$u \leqslant u^0 \leqslant x \leqslant v^0 \leqslant v, \quad u^0 \leqslant y \leqslant v^0.$$

Therefore $\overline{u^0} = \overline{x} \wedge \overline{y}$ and $\overline{v^0} = \overline{x} \vee \overline{y}$. □

If G_1 and G_2 are projectable lattice ordered groups, then instead of H and $\overline{H}$ we have H_i and $\overline{H}_i$ $(i = 1, 2)$. From the above construction of H and $\overline{H}$ we obviously obtain

Lemma 3.9.1. *Let G_1 and G_2 be projectable lattice ordered groups such that the lattices $\ell(G_1)$ and $\ell(G_2)$ are isomorphic. Then the lattices $\overline{H}_1$ and $\overline{H}_2$ are isomorphic as well.*

Lemma 3.10. *Let $(x_i)_{i\in I}$, $(y_i)_{i\in I} \in H$, $a \in G$, $b \in G$. Suppose that*

(i) *$a = \bigvee\limits_{i\in I} x_i$, $b = \bigvee\limits_{i\in I} y_i$;*

(ii) *if $i(1)$ and $i(2)$ are distinct elements of I, then $x_{i(1)} \wedge y_{i(2)} = 0$.*

Then $a + b = \bigvee\limits_{i\in I} (x_i + y_i)$.

P r o o f. Let $i(1) \in I$. Then $x_{i(1)} + y_i = x_{i(1)} \vee y_i$ for each $i \in I$ with $i \neq i(1)$. Hence

$$\begin{aligned} x_{i(1)} + \Bigl(\bigvee_{i\in I} y_i\Bigr) &= \bigvee_{i\in I}(x_{i(1)} + y_i) = (x_{i(1)} + y_{i(1)}) \vee \Bigl(\bigvee_{i\in I\setminus\{i(1)\}} (x_{i(1)} \vee y_i)\Bigr) \\ &= (x_{i(1)} + y_{i(1)}) \vee \Bigl(\bigvee_{i\in I\setminus\{i(1)\}} y_i\Bigr). \end{aligned}$$

Also, $(x_{i(1)} + y_{i(1)}) \wedge y_i = 0$ whenever $i \in I \setminus \{(i(1))\}$, whence

$$\begin{aligned} a + b &= \Bigl(\bigvee_{i(1)\in I} x_{i(1)}\Bigr) + \Bigl(\bigvee_{i\in I} y_i\Bigr) = \bigvee_{i(1)\in I} \Bigl(x_{i(1)} + \bigvee_{i\in I} y_i\Bigr) \\ &= \bigvee_{i(1)\in I} \Bigl((x_{i(1)} + y_{i(1)}) \vee \Bigl(\bigvee_{i\in I\setminus i(1)} y_i\Bigr)\Bigr) = \bigvee_{i(1)\in I} (x_{i(1)} + y_{i(1)}). \end{aligned}$$

□

Lemma 3.11. *Let $x, y, x', y' \in H$, $x = (x_i)_{i\in I}$, $y = (y_i)_{i\in I}$, $x' = (x'_j)_{j\in J}$, $y' = (y'_j)_{j\in J}$. Assume that*

(i) $x_{i(1)} \wedge y_{i(2)} = 0$ *whenever $i(1)$ and $i(2)$ are distinct elements of I;*
(ii) $x'_{j(1)} \wedge y'_{j(2)} = 0$ *whenever $j(1)$ and $j(2)$ are distinct elements of J;*
(iii) $x \leqslant x'$ *and* $y \leqslant y'$.

Put $u = (x_i + y_i)_{i\in I}$, $u' = (x'_j + y'_j)_{j\in J}$. Then $u, u' \in H$ and $u \leqslant u'$.

Proof. The relations $u \in H$ and $u' \in H$ are obvious. We have to verify that

$$x_i + y_i = \bigvee_{j\in J} ((x_i + y_i) \wedge (x'_j + y'_j)) \tag{1}$$

is valid for each $i \in I$.

We have

$$x_i = \bigvee_{j\in J} (x_i \wedge x'_j), \quad y_i = \bigvee_{j\in J} (y_i \wedge y'_j).$$

By applying 3.10 we obtain

$$x_i + y_i = \bigvee_{j\in J} ((x_i \wedge x'_j) + (y_i \wedge y'_j)).$$

Since

$$(x_i \wedge x'_j) + (y_i \wedge y'_j) = (x_i + y_i) \wedge (x_i + y'_j) \wedge (x'_j + y_i) \wedge (x'_j + y'_j),$$

we get

$$x_i + y_i \leqslant \bigvee_{j\in J} (x_i + y_i) \wedge (x'_j + y'_j) \leqslant x_i + y_i,$$

thus (1) is valid. □

Corollary 3.12. *Let x, y, x' and y' be as in* 3.11 *with the distinction that the condition* (iii) *is replaced by*

(iii$_1$) $x \sim x'$ *and* $y \sim y'$.

Then $u \sim u'$.

Let $\overline{a}, \overline{b} \in H$. There exist x and y in H (expressed as in 3.11) such that $x \sim a$, $y \sim b$ and the condition (i) from 3.11 is valid. We put $\overline{a} + \overline{b} = \overline{u}$, where u is as in 3.11. Then in view of 3.12, the operation $+$ in $\overline{H}$ is correctly defined.

Lemma 3.13. *Let $\overline{a}, \overline{b}, \overline{a'}, \overline{b'} \in \overline{H}$, $\overline{a} \leqslant \overline{a'}$, $\overline{b} \leqslant \overline{b'}$. Then $\overline{a} + \overline{b} \leqslant \overline{a'} + \overline{b'}$.*

Proof. This is a consequence of 3.11. □

Lemma 3.14. *The operation + in $\overline{H}$ is associative.*

Proof. It suffices to apply 3.8 (generalized to the case of three elements). $\square$

Lemma 3.15. *Let $\overline{x}, \overline{y} \in \overline{H}$, $\overline{x} \leqslant \overline{y}$. There exists $\overline{z} \in \overline{H}$ such that $\overline{x} + \overline{z} = \overline{y}$.*

Proof. In view of 3.8 we can suppose that $x = (x_t)_{t \in T}$, $y = (y_t)_{t \in T}$ and that $x_{t(1)} \wedge y_{t(2)} = 0$ whenever $t(1)$ and $t(2)$ are distinct elements of T. This and the relation $x \leqslant y$ yield that $x_t \leqslant y_t$ for each $t \in T$. Hence for each $t \in T$ there is $0 \leqslant z_t \in G$ with $x_t + z_t = y_t$. Then $z = (z_t)_{t \in T} \in H$ and clearly $\overline{x} + \overline{z} = \overline{y}$. $\square$

Lemma 3.16. *The relation $\overline{x} + \overline{H} = \overline{H} + \overline{x}$ holds for each $\overline{x} \in \overline{H}$.*

Proof. Let $\overline{x} \in \overline{H}$. For each $\overline{z} \in \overline{H}$ we have $\overline{x} + \overline{z} \geqslant \overline{x}$. Next, from 3.15 we infer that, whenever $\overline{y} \in \overline{H}$ and $\overline{y} \geqslant \overline{x}$, then $\overline{y} \in \overline{x} + \overline{H}$. Hence $\overline{x} + \overline{H} = \{\overline{y} \in \overline{H} \colon \overline{y} \geqslant \overline{x}\}$. Analogously, $\overline{H} + \overline{x} = \ \overline{y} \in \overline{H} \colon \overline{y} \geqslant \overline{x}\}$. Thus $\overline{x} + \overline{H} = \overline{H} + \overline{x}$. $\square$

Lemma 3.17. *Let $\overline{x}, \overline{y}, \overline{z} \in \overline{H}$, $\overline{x} + \overline{z} = \overline{y} + \overline{z}$. Then $\overline{x} = \overline{y}$.*

Proof. Without loss of generality we can suppose that $\overline{x}, \overline{y}$ and $\overline{z}$ are as in 3.8 (for $n = 3$). Thus

$$x = (x_t)_{t \in T}, \quad y = (y_t)_{t \in T}, \quad z = (z_t)_{t \in T}$$

and, whenever $t(1), t(2)$ are distinct elements of T, then

$$x_{t(1)} \wedge y_{t(2)} = x_{t(1)} \wedge z_{t(2)} = y_{t(1)} \wedge z_{t(2)} = 0.$$

Hence there are $u \in \overline{x} + \overline{z}$, $v \in \overline{y} + \overline{z}$ such that

$$u = (u_t)_{t \in T}, \quad v = (v_t)_{t \in T},$$
$$u_t = x_t + z_t, \quad v_t = y_t + z_t \quad \text{for each } t \in T.$$

But then we have $\overline{u} = \overline{v}$, whence $u \leqslant v$ and $v \leqslant u$. By the obvious calculation we get $u_{t(1)} \wedge v_{t(1)} = 0$ whenever $t(1), t(2)$ are distinct elements of T and thus $u_t = v_t$ for each $t \in T$. Hence $x_t = y_t$ for each $t \in T$. Therefore $\overline{x} = \overline{y}$. $\square$

Analogously, $\overline{z} + \overline{x} = \overline{z} + \overline{y}$ implies that $\overline{x} = \overline{y}$.

If $x = (x_i)_{i \in I} \in H$ is such that $x_i = 0$ for each $i \in I$, then we denote $x = x^0$. It is clear that $\overline{x^0}$ is the least element of $\overline{H}$.

Lemma 3.18. *Let $\overline{x}, \overline{y} \in \overline{H}$, $\overline{x} + \overline{y} = \overline{x^0}$. Then $\overline{x} = \overline{y} = \overline{x^0}$.*

Proof. This is an immediate consequence of the operation + in $\overline{H}$. $\square$

4. Construction of G^L

Let G, H and $\overline{H}$ be as above. In view of the previous section, $\overline{H}$ is a lattice ordered semigroup.

Lemma 4.1. *There exists a lattice ordered group G_1 such that $\overline{H}$ is the positive cone of G_1.*

Proof. This is a consequence of Theorem 3 and Corollary 1 in [3], Chap. 14 (in view of 3.9, 3.14, 3.16, 3.17 and 3.18). $\square$

Lemma 4.2. *G_1 is laterally complete.*

Proof. Let $\{\overline{a}_k\}_{k\in K}$ be a disjoint indexed system in G_1. Hence $\overline{a}_k \in \overline{H}$, $a_k \in H$ for each $k \in K$. If $k(1)$, $k(2)$ are distinct elements of K and x_1, x_2 are elements belonging to $a_{k(1)}$ or $a_{k(2)}$, respectively, then from $\overline{a}_{k(1)} \wedge \overline{a}_{k(2)} = \overline{x^0}$ we obtain that $x_1 \wedge x_2 = 0$. Let

$$a_k = (a_{ki})_{i\in I(k)} \quad (k \in K);$$

without loss of generality we can suppose that $I(k(1)) \cap I(k(2)) = \emptyset$ whenever $k(1)$ and $k(2)$ are distinct elements of K. Put $I = \bigcup\limits_{k\in K} I(k)$ and

$$b = (a_{ki})_{k\in K, i\in I(k)}.$$

Then $b \in H$, whence $\overline{b} \in \overline{H}$. Clearly $a_k \leqslant b$ and hence $\overline{a}_k \leqslant \overline{b}$ for each $k \in K$.

Let $\overline{c} \in \overline{H}$ be such that $\overline{c} \geqslant \overline{a}_k$ for each $k \in K$, where $c = (c_j)_{j\in J}$. Hence $c \geqslant a_k$ for each $k \in K$. Then

$$a_{ki} = \bigvee_{j\in J} (a_{ki} \wedge c_j)$$

for each $k \in K$ and each $i \in I(k)$. Therefore $b \leqslant c$ and so $\overline{b} \leqslant \overline{c}$. Hence $\overline{b} = \bigvee\limits_{k\in K} \overline{a}_k$. $\square$

Let $0 \leqslant g \in G$. Consider the element $x = (x_i)_{i\in I}$ of H such that $I = \{1\}$ and $x_1 = g$. Then we denote $\overline{x} = \overline{g}$.

Lemma 4.3. *Let $0 \leqslant g_i \in G$ $(i = 1, 2)$. Then*

(1) $$\overline{g}_1 + \overline{g}_2 = \overline{g_1 + g_2},\ \overline{g}_1 \vee \overline{g}_2 = \overline{g_1 \vee g_2},\ \overline{g}_1 \wedge \overline{g}_2 = \overline{g_1 \wedge g_2},$$

(2) $$g_1 \neq g_2 \Leftrightarrow \overline{g}_1 \neq \overline{g}_2.$$

Proof. The relations (1) follow from the definitions of the operations $+, \vee$ and $\wedge$ in $\overline{H}$ (as given in Section 3). The equivalence (2) is obvious. $\square$

For $0 \leqslant g \in G$ we will identify g and $\overline{g}$. Hence in view of 4.3, G^+ is a subsemigroup and a sublattice of $\overline{H}$. Therefore in virtue of 4.1 we obtain

Lemma 4.4. *G is an ℓ-subgroup of G_1.*

Lemma 4.5. *Let $0 < v \in G_1$. There exists $0 < x \in G$ with $x \leqslant v$.*

P r o o f. We have $v \in \overline{H}$, hence there is $y \in H$ with $y \in v$. Let $y = (y_i)_{i \in I}$. Since $v \neq 0$, the elements y and x^0 are distinct. Thus there is $i \in I$ with $y_i > 0$. Clearly $y_i \leqslant v$. $\square$

Lemma 4.6. *Let $y \in H$, $y = (y_i)_{i \in I}$. Then $\overline{y} = \bigvee_{i \in I} y_i$.*

The idea of proof is similar to (but simpler than) that applied in the proof of 4.2; the proof will be omitted.

Lemma 4.7. *Let G_2 be an ℓ-subgroup of G_1 such that $G \subseteq G_2$. Let $x = (x_i)_{i \in I} \in H$. Suppose that $\overline{y} \in G_2$ is the least upper bound of the system $\{\overline{x}_i\}_{i \in I}$ in G_2. Then $\overline{y} = \overline{x}$.*

P r o o f. By 4.6, we have $\overline{x} = \bigvee_{i \in I} \overline{x}_i$ in G_1. Thus, since $\overline{x}_i \leqslant \overline{y}$ for each $i \in I$, the relation $x \leqslant y$ is valid. By way of contradiction, suppose that $\overline{x} < \overline{y}$. Then according to 4.5 there is $0 < g \in G$ such that $\overline{y} - \overline{x} > g$. Hence $\overline{y} > \overline{y} - g > \overline{x}$, $\overline{y} - g \in G_2$ and $\overline{y} - g \geqslant \overline{x}_i$ for each $i \in I$, which is a contradiction. $\square$

Lemma 4.8. *Let G_2 be an ℓ-subgroup of G_1 such that G_2 is laterally complete and $G \subseteq G_2$. Then $G_2 = G_1$.*

P r o o f. It suffices to verify that $G_2^+ = G_1^+$. Let $g^1 \in G_1^+$. Thus $g^1 = \overline{x}$ for some $x = (x_i)_{i \in I}$ in H. Then $x_i \in G_2$ for each $i \in I$; since G_2 is laterally complete there exists $\overline{y} \in G_2^+$ with

$$\overline{y} = \overset{2}{\bigvee_{i \in I}} x_i,$$

where $\overset{2}{\bigvee}$ denotes the least upper bound in G_2. In view of 4.7 we have $\overline{y} = \overline{x}$, whence $G_2^+ = G_1^+$. $\square$

Lemma 4.9. *Under the notation as above, $G^2 = G^L$.*

P r o o f. We apply the conditions from Definition 1.2; the assertion follows from 4.2, 4.5 and 4.8. $\square$

P r o o f of (A).

It is easy to verify that if G_1 and G_2 are lattice ordered groups such that the lattices $\ell(G_1^+)$ and $\ell(G_2^+)$ are isomorphic, then the lattices $\ell(G_1)$ and $\ell(G_2)$ are isomorphic as well. Now let the assumptions of (A) be satisfied. According to 2.6, G_2 is projectable. Thus, in view of 3.9.1, $\ell(\overline{H}_1)$ and $\ell(\overline{H}_2)$ are isomorphic. By applying 4.1 we get that $\ell((G_1^L)^+)$ and $\ell((G_2^L)^+)$ are isomorphic. Therefore $\ell(G_1)$ and $\ell(G_2)$ are isomorphic. $\square$

P r o o f of (B).

It suffices to apply 4.9, 4.1 and 4.6. $\square$

From 4.1 we infer that in (A) and (B) the assumption of projectability can be replaced by the assumption of σ-completeness.

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