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ALMOST BUTLER GROUPS

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Abstract. Generalizing the notion of the almost free group we introduce almost Butler groups. An almost B_2-group G of singular cardinality is a B_2-group. Since almost B_2-groups have preseparative chains, the same result in regular cardinality holds under the additional hypothesis that G is a B_1-group. Some other results characterizing B_2-groups within the classes of almost B_1-groups and almost B_2-groups are obtained. A theorem of [BR] stating that a group G of weakly compact cardinality λ having a λ-filtration consisting of pure B_2-subgroup is a B_2-group appears as a corollary.

All groups in this paper are additively written abelian. By a *smooth (ascending) union of a group* G we mean a collection of pure subgroups G_α indexed by an initial segment of ordinals with the property that $G_\beta \leqslant G_\alpha$ when $\beta < \alpha$ and $G_\alpha = \bigcup\limits_{\beta<\alpha} G_\beta$ whenever α is a limit ordinal. For unexplained terminology and notation see [F1].

An exact sequence $E\colon 0 \longrightarrow H \longrightarrow G \xrightarrow{\beta} K \longrightarrow 0$ with K torsion-free is *balanced* if the induced map $\beta_*\colon \operatorname{Hom}(J,G) \longrightarrow \operatorname{Hom}(J,K)$ is surjective for each rank one torsion-free group J. Equivalently, E is balanced if all rank one (completely decomposable) torsion-free groups are projective with respect to E.

A torsion-free group B is said to be a *B_1-group* (*Butler group*) if $\operatorname{Bext}(B,T)=0$ for all torsion groups T, where Bext is the subfunctor of Ext consisting of all balanced-exact extensions.

A subgroup D of a torsion-free group G is said to be *decent* in G if D is pure and, for any finite rank pure subgroup C/D of G/D, there is a finite rank Butler group B of C such that $C = D + B$. The subgroup D is said to be *prebalanced* in G, if the same holds for every rank one pure subgroup C/D of G/D. Our definition of a decent subgroup is slightly stronger than that of [AH] since we demand D to be

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pure. It is easy to verify that decency is transitive. Also, if $A \leqslant B \leqslant G$ and if both A and B/A are decent subgroups of G and G/A, respectively, then B is decent in G.

Another relevant concept in the study of infinite rank Butler groups is the *torsion extension property* (TEP). A (pure) subgroup H of a torsion-free group G is said to have TEP in G, or briefly, H is TEP in G, if every homomorphism $H \to T$ with T torsion extends to a homomorphism $G \to T$.

A torsion-free group G is called a *B_2-group* if G is the union of a smooth ascending chain of pure subgroups $G = \bigcup_{\alpha<\mu} H_\alpha$ where, for each $\alpha + 1 < \mu$, $H_{\alpha+1} = H_\alpha + B_\alpha$ with B_α a Butler group of finite rank. We will call $\{H_\alpha \mid \alpha < \mu\}$ a B-filtration of the group G.

Recall that a pure subgroup K of a torsion-free group G is said to be *preseparative*, if for each $g \in G \setminus K$ there is a countable subset $\{h_0, h_1, \ldots\} \subseteq K$ such that for each $h \in K$ there are $m, n < \omega$, $m \neq 0$, with $\mathbf{t}(g+h) \leqslant \mathbf{t}(mg+h_0) \cup \mathbf{t}(mg+h_1) \cup \ldots \cup \mathbf{t}(mg+h_n)$. In this case we will also say that $\{h_0, h_1, \ldots\}$ is a *preseparative set for g over K*. An equivalent definition of a preseparative subgroup has been given in Bican, Fuchs [15] under the name *$\aleph_0$-prebalanced* subgroup. Let K be a corank one pure subgroup of a torsion-free group G. The types $\mathbf{t}(J)$ of those pure rank one subgroups J of G which are not contained in K generate a lattice ideal $\mathfrak{I}_{G|K}$ in the lattice of all types. The subgroup K is preseparative in G if this ideal is countably generated. If the corank of K in G is greater than one, then K is defined to be preseparative in G if it is preseparative in every pure subgroup H of G containing K as a corank one subgroup. A smooth ascending union $G = \bigcup_{\alpha<\mu} H_\alpha$ of preseparative subgroups with $H_0 = H$ and $|G_{\alpha+1}/G_\alpha| \leqslant \aleph_0$ (equivalently $G_{\alpha+1}/G_\alpha$ of rank one) for each $\alpha < \mu$ is called a *preseparative chain from H to G*. For $H = 0$ we speak about a *preseparative chain of G*.

Recall [AH] that a collection $\mathfrak{C}$ of subgroups of G is called an *axiom-3 family* if $\mathfrak{C}$ satisfies (i) $0, G \in \mathfrak{C}$; (ii) if $\{H_i \mid i \in I\}$ is any set of subgroups in $\mathfrak{C}$, then their sum $\sum_{i\in I} H_i \in \mathfrak{C}$; (iii) if $H \in \mathfrak{C}$ and X is a countable subset of G, then there is a $K \in \mathfrak{C}$ containing H and X such that K/H is countable. If, moreover, each $A \in \mathfrak{C}$ is TEP in G (and consequently G/A is a B_2-group) then such an axiom-3 family has been called *canonical* in [BR] . Looking at the proof of [B2; Theorem 6] we see that with a given B-filtration of a B_2-group G it is associated a canonical axiom-3 family $\mathcal{F}(G)$ of decent, TEP and B_2-subgroups of G in the natural way, given by the closed subsets of the corresponding ordinal number. It is natural to speak about a *canonical axiom-3 family of decent subgroups corresponding to a given B-filtration* of G. It is not too hard to show (use e.g. [B2; Lemma 3]) that if $G = \bigcup_{\alpha<\mu} H_\alpha$ is a B-filtration of G and $G = \bigcup_{\alpha<\lambda} K_\alpha$ is any smooth ascending union consisting of members of the given B-filtration of G, then $\mathcal{F}(K_\beta) \subseteq \mathcal{F}(K_\alpha)$ whenever $\beta \leqslant \alpha$ and $\bigcup_{\beta<\alpha} \mathcal{F}(K_\beta) \subseteq \mathcal{F}(K_\alpha)$, α limit. Moreover, if $H \leqslant K$ are members of $\mathcal{F}(G)$, then one can easily prove the existence of a B-filtration from H to K.

Several recent results (cf. e.g. [FR1], [FR2], [BR], [BRV]) show that Butler groups form an appropriate generalization of free groups. Recall that for an infinite cardinal λ a torsion-free group G is said to be *λ-free* if each subgroup of G of cardinality strictly less than λ is free. Unlike the case of free abelian groups, a (pure) subgroup of a B_1-group (B_2-group) need not be a B_1-group (B_2-group). However, as mentioned above, B_2-groups are characterized in [AH] (see also [FMa]) as torsion-free groups having an axiom-3 family $\mathfrak{C}$ of decent and TEP B_2-subgroups, and consequently every subset X of G is contained in a member of $\mathfrak{C}$ of cardinality $|X| \cdot \aleph_0$. In the light of these facts it is natural to work with some families of subgroups of the given group G and to distinguish between hereditary and non-hereditary families. Thus we are led to the following definitions.

1. Definition. Let λ be an uncountable cardinal. A collection $\mathfrak{C}$ of subgroups of the group G is called a *weak λ-cover* of G if each member of $\mathfrak{C}$ has cardinality less than λ, every subset $\emptyset \neq X \subseteq G$ with $|X| < \lambda$ is contained in a member of $\mathfrak{C}$ of cardinality $|X| \cdot \aleph_0$ and $\mathfrak{C}$ is closed under smooth ascending unions $H = \bigcup_{\alpha<\kappa} H_\alpha$ with $|H| < \lambda$. Moreover, we say that a weak λ-cover $\mathfrak{C}$ of the torsion-free group G is *hereditary*, if for each uncountable $H \in \mathfrak{C}$ the set $\mathfrak{C}_H = \{K \in \mathfrak{C} \mid K \leqslant H, |K| < |H|\}$ is a weak $|H|$-cover of H.

In what follows similar notions and results concerning B_1-groups and B_2-groups will appear several times. For the sake of brevity we shall use the notation B_*-group in the sense that it means either a B_1-group or a B_2-group throughout. In other words, this abbreviation will record two facts at once.

2. Definition. Let λ be an uncountable cardinal. A torsion-free group G is said to be a (*hereditary*) *λ-B_*-group* if it has a (hereditary) weak λ-cover $\mathfrak{C}$ consisting of pure B_*-subgroups. If, moreover, G is of cardinality λ, then G is called a (*hereditary*) *almost B_*-group.*

Recall that a subset C of the regular cardinal λ is called a *cub* (closed and unbounded set) if it is cofinal to λ, i.e. for each $\alpha < \lambda$ there is $\beta \in C$ with $\alpha < \beta$ (C is unbounded) and each limit ordinal $\alpha < \lambda$ such that $\alpha \cap C$ is cofinal to α belongs to C (C is closed). A subset of λ is said to be *stationary*, if it intersects every cub in λ non-trivially. Now we are ready to present our results. We start with the singular cardinality case concerning almost B_2-groups.

κ-Shelah game. Let κ be a regular uncountable cardinal and let G be a torsion-free group of cardinality $|G| > \kappa^+$. We define the *κ-Shelah game* on G in the following way: Player I picks subgroups G_{2i}, $i < \omega$, of cardinality κ and player II picks G_{2i+1} such that $G_i \leqslant G_{i+1}$ for all $i < \omega$. Player II wins if G_{2i+1} is decent and TEP in G_{2i+3} for each $i < \omega$.

3. Lemma. *If κ is a regular uncountable cardinal and G an almost B_2-group of cardinality $\lambda > \kappa^+$, then player II has a winning strategy in the κ-Shelah game.*

Proof. Let $\mathfrak{C}$ be a weak λ-cover of pure B_2-subgroups of G. In view of Lemma 1.2 in [H], the κ-Shelah game is determinated and so we are going to show that player I has no winning strategy. By way of contradiction let us assume that I has a winning strategy s and that he has picked G_0. Take H_0 to be any member of $\mathfrak{C}$ of cardinality κ containing G_0 and assume that H_β, $\beta < \alpha$, have been already defined for some $0 < \alpha < \kappa^+$. For α limit we simply set $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$, while for $\alpha = \beta + 1$ we select H_α to be any member of $\mathfrak{C}$ of cardinality κ containing H_β and all $s(H_{\alpha_0}, \ldots, H_{\alpha_n})$, $\alpha_0 < \ldots < \alpha_n < \alpha$, $n < \omega$. The union $H = \bigcup_{\alpha<\kappa^+} H_\alpha$ belongs to $\mathfrak{C}$ by the hypothesis and [B1; Lemma 12] yields the existence of a cub U in κ^+ such that H_α is TEP in H for each $\alpha \in U$. Moreover, in virtue of [BR; Proposition 5.1] the H_α's can be assumed decent in H.

Now when player I has chosen G_{2i} in the κ-Shelah game, then player II picks G_{2i+1} to be H_α, where α is the least non-limit element of U containing G_{2i}. □

As in the case of free groups we are going to prove the following result.

4. Theorem. *An almost B_2-group of singular cardinality λ is a B_2-group.*

Proof. There is a smooth ascending union $\lambda = \bigcup_{\alpha<\mu} \kappa_\alpha$ with $\kappa_0 > \mu = \operatorname{cof} \lambda$ and κ_α regular whenever α is non-limit. Further, let $\mathfrak{C}$ be a weak λ-cover of B_2-subgroups of G and let $G = \bigcup_{\alpha<\mu} G_\alpha$ be a smooth union with $G_\alpha \in \mathfrak{C}$ and $|G_\alpha| = \kappa_\alpha$.

Set $G^0_\alpha = G_\alpha$ for each $\alpha < \mu$ and assume that G^n_α has been already defined for some $n < \omega$ and all $\alpha < \mu$. For α limit or 0 set $H^n_\alpha = G^n_\alpha$ and for α successor take H^n_α according to the κ_α-Shelah game $G^0_\alpha, H^0_\alpha, G^1_\alpha, H^1_\alpha, \ldots$, the hypotheses of Lemma 3 being obviously satisfied. For each $\alpha < \mu$ let $\{h^j_\alpha \mid j < \kappa_\alpha\}$ be any list of the elements of H^n_α. Moreover, H^n_α has a canonical axiom-3 family $\mathcal{F}(H^n_\alpha)$ of decent and TEP subgroups corresponding to a given B-filtration of H^n_α. The routine set-theoretical arguments lead to the conclusion that we can select G^{n+1}_α in such a way that it has cardinality κ_α, contains $H^n_\alpha \cup \{h^j_\gamma \mid \gamma < \mu, j < \kappa_\alpha\}$ and $G^{n+1}_\alpha \cap H^n_{\alpha+1} \in \mathcal{F}(H^n_{\alpha+1})$.

Now for each α non-limit H^n_α is TEP and decent in H^{n+1}_α by Lemma 9, hence $H^{n+1}_\alpha / H^n_\alpha$ is a B_2-group by [B2; Theorem 12], the B-filtration of H^n_α extends to that of H_α^{n+1} by [DHR; Proposition 3.9] and consequently $\mathcal{F}(H_\alpha^n) \subseteq \mathcal{F}(H_\alpha^{n+1}) \subseteq \mathcal{F}(H_\alpha)$, where $H_\alpha = \bigcup_{n<\omega} H_\alpha^n$. Moreover, for $\alpha < \mu$ arbitrary we have $H_\alpha = H_\alpha \cap H_{\alpha+1} = \bigcup_{n<\omega} (H_\alpha^n \cap H_{\alpha+1}^n) \leqslant \bigcup_{n<\omega} (G_\alpha^{n+1} \cap H_{\alpha+1}^n) \leqslant \bigcup_{n<\omega} (H_\alpha^{n+1} \cap H_{\alpha+1}^{n+1}) = H_\alpha$ and so $H_\alpha \in \bigcup_{n<\omega} \mathcal{F}(H_{\alpha+1}^n) \subseteq \mathcal{F}(H_{\alpha+1})$. Hence there is a B-filtration from H_α to $H_{\alpha+1}$ and consequently it remains to show that the union $G = \bigcup_{\alpha<\mu} H_\alpha$ is smooth.

Let $\alpha < \mu$ be a limit ordinal and let $h \in H_\alpha$ be arbitrary. Then $h \in H_\alpha^n$ for some $n < \omega$ and consequently $h = h_\alpha^j$ for some $j < \kappa_\alpha$. Thus $j < \kappa_\beta$ for some $\beta < \alpha$, the chain $\{\kappa_\alpha \mid \alpha < \mu\}$ being assumed smooth. This yields $h \in G_\beta^{n+1} \leqslant H_\beta$ and the proof is complete. □

Leaving open the case of almost B_1-groups of singular cardinalities we proceed to the regular cardinals.

In [B3] the following construction based on the ideas of [F2] and [FMa] was investigated.

5. Construction. Let H be a preseparative subgroup of a torsion-free group G and let R be a fixed set of representatives of cosets of G/H. For each $g \in R$ we fix a preseparative set $\{h_n^g \mid n < \omega\} \subseteq H$ for g over H. Now if we set $B = \big\langle \langle mg + h_n^g \rangle_* \mid g \in R, m, n < \omega, m \neq 0 \big\rangle$ then it is easy to verify that $G = H + B$ and $|B| = |G/H|$.

Further, if $G = \bigcup_{\alpha<\mu} H_\alpha$ is a smooth ascending union of preseparative subgroups, then for each $\alpha < \mu$ we can construct a subgroup $B_\alpha \leqslant G$ in such a way that $H_{\alpha+1} = H_\alpha + B_\alpha$, $|B_\alpha| = |H_{\alpha+1}/H_\alpha|$ and, obviously, $H_\alpha = \sum_{\varrho<\alpha} B_\varrho + H_0$ for all relevant α's.

Recall that a subset $S \subseteq \mu$ is said to be *closed*, if $L_\beta \cap B_\beta \leqslant H_0 + \langle B_\gamma \mid \gamma \in S, \gamma < \beta\rangle$ for each $\beta \in S$. It was proved in [B3] that for a closed subset $S \subseteq \mu$ the subgroup $G(S) = H_0 + \sum_{\beta \in S} B_\beta$ is pure in G (Lemma 2.3) and preseparative in G (Lemma 2.4). Moreover, every union of closed subsets is closed (Lemma 2.5).

6. Lemma. *Let $G = \bigcup_{\alpha<\mu} H_\alpha$ be a preseparative chain of a torsion-free group G. If $\overline{S} \subseteq \mu$ is a closed subset, then every element $\lambda \in \overline{S}$ lies in a countable closed subset of μ contained in $\overline{S}$.*

Proof. By way of contradiction let us assume that $\lambda \in \overline{S}$ is the first ordinal which is not in a countable closed subset contained in $\overline{S}$. Since $H_\lambda \cap B_\lambda$ is countable, it has a basis $\{x_0, x_1, \ldots\}$ (possibly finite). If we set $\nu(g) = \nu$ for $g \in G$ whenever $g \in H_{\nu+1} \setminus H_\nu$, then we infer from $x_i \in H_\lambda$ that $\lambda_i = \nu(x_i) < \lambda$. We claim that $\lambda_i \in \overline{S}$. If not, then $H_\lambda \cap B_\lambda \leqslant \langle B_\gamma \mid \gamma \in \overline{S}, \gamma < \lambda\rangle$ yields that $x_i = y + z$ with

$y \in \langle B_\gamma \mid \gamma \in \overline{S}, \gamma < \lambda_i \rangle$ and $z \in \langle B_\gamma \mid \gamma \in \overline{S}, \gamma > \lambda_i \rangle$. Assuming z non-zero, z is expressible in the form $z = z_1 + \ldots + z_k$, $0 \neq z_i \in B_{\varrho_i}$, with $\lambda_i < \varrho_1 < \ldots < \varrho_k$ and ϱ_k as small as possible. Now $z_k = x_i - y - z_1 - \ldots - z_{k-1} \in H_{\varrho_{k-1}}$, which contradicts the choice of ϱ_k. Hence $z = 0$ and $x_i = y \in \langle B_\gamma \mid \gamma \in \overline{S}, \gamma < \lambda_i \rangle \subseteq H_{\lambda_i}$, contradicting $\nu(x_i) = \lambda_i$. Thus $\lambda_i \in \overline{S}$, $\lambda_i < \lambda$, $x_i \in B_{\gamma_1} + \ldots + B_{\gamma_n}$, $\gamma_i \in \overline{S}$, $\gamma < \lambda_i$, and the choice of λ yields the existence of a countable closed subset S_i of $\overline{S}$ containing $\lambda_i, \gamma_1, \ldots, \gamma_n$. Now the set $S = \bigcup\limits_{i<\omega} S_i$ is a closed countable subset of $\overline{S}$ and so is $S \cup \{\lambda\}$, since $x_i \in G(S)$ for each $i < \omega$ and consequently $H_\lambda \cap B_\lambda \leqslant G(S)$, $G(S)$ being pure in G and containing the basis $\{x_0, x_1, \ldots\}$ of $H_\lambda \cap B_\lambda$. □

7. Lemma. *Let λ be a regular uncountable cardinal and $G = \bigcup\limits_{\alpha<\lambda} H_\alpha$ a λ-filtration consisting of B_2-groups. Then*

(a) *G has a preseparative chain consisting of B_2-groups of cardinalities strictly less than λ;*

(b) *G is a hereditary almost B_2-group.*

Proof. (a) By [F3; Theorem 8.2] there is a preseparative chain from H_α to $H_{\alpha+1}$ for every $\alpha < \mu$ and the transitivity of preseparativeness yields (a) in view of the fact that the members of the preseparative chain from H_α to $H_{\alpha+1}$ are B_2-groups again by the same reason.

(b) Assume that $G = \bigcup\limits_{\alpha<\lambda} H_\alpha$ is a preseparative chain of G consisting of B_2-groups of cardinalities less than λ. Realizing that the family $\mathfrak{D} = \{G(S) \mid S \subset \lambda, S \text{ closed and bounded}\}$ is a hereditary weak λ-cover of G owing to Lemma 6 and taking into account the closedness of closed subsets under unions we only have to verify that $G(S)$ is a B_2-group whenever $S \subset \lambda$ is closed and bounded, $S \subseteq \mu < \lambda$. Set $S_0 = S$ and assume that for some $\beta \leqslant \mu$ the closed subsets S_γ, $\gamma < \beta$, of μ have been already defined. For β limit the union $S_\beta = \bigcup\limits_{\gamma<\beta} S_\gamma$ is a closed subset of μ. If $\gamma = \beta - 1$ exists and $H(S_\gamma) = H_\mu$ then we stop. Otherwise we take the first ordinal $\delta \in \mu \setminus S_\gamma$. In view of Lemma 6 there is a countable closed subset $S' \subseteq \mu$ containing δ and we can set $S_\beta = S_\gamma \cup S'$. Obviously, $G(S_\beta)/G(S_\gamma)$ is countable and consequently in this way we obtain (by [B3; Lemma 2.4]) a preseparative chain from $G(S)$ to H_μ. Thus $G(S)$ is a B_2-group by [F3; Theorem 8.2]. □

8. Corollary. *Let λ be a regular uncountable cardinal and G a λ-B_2-group with a weak λ-cover $\mathfrak{C}$ consisting of B_2-groups. If $K \leqslant G$ is any subgroup of cardinality λ, then there is a subgroup H of G of cardinality λ that contains K and is an almost B_2-group. Especially, if K is a smooth ascending union of members of $\mathfrak{C}$ then it is an almost B_2-group.*

P r o o f. Let $\{k_\alpha \mid \alpha < \lambda\}$ be any list of elements of K. Set $H_0 = 0$ and assume that for some $\alpha < \lambda$ the members H_β of $\mathfrak{C}$ containing $\{k_\gamma \mid \gamma < \beta\}$ have been already defined for each $\beta < \alpha$. For α limit we simply set $H_\alpha = \bigcup\limits_{\beta<\alpha} H_\beta$, while for $\alpha = \beta + 1$ we take as H_α any member of $\mathfrak{C}$ containing $H_\beta \cup \{k_\beta\}$ of cardinality $|H_\beta| \cdot \aleph_0$. Then $H = \bigcup\limits_{\alpha<\lambda} H_\alpha$ contains K and is an almost B_2-group by Lemma 7. The rest is obvious. □

9. Theorem. *The following conditions are equivalent for an uncountable torsion-free group G:*

(i) *G is an almost B_2-group;*

(ii) *$G = \bigcup\limits_{\alpha<\lambda} H_\alpha$ is a smooth ascending union of B_2-subgroups with $|H_\alpha| < |G|$ for every $\alpha < \lambda$;*

(iii) *G has a preseparative chain consisting of B_2-groups of cardinalities less than $|G|$;*

(iv) *G is a hereditary almost B_2-group.*

P r o o f. If G is of singular cardinality then it is a B_2-group by Theorem 4 and the assertion holds. For $|G| = \lambda$ regular (i) implies (ii) and (iv) implies (i) trivially, while the rest follows easily from the preceding lemma. □

10. Corollary. *An almost B_2-group is a B_2-group if and only if it is a B_1-group.*

P r o o f. By [F3; Theorem 4.1] and Theorem 9. □

The notion of a λ-cover was introduced and investigated in [BRV]. The only difference between this and the weak λ-cover is that the weak λ-cover consists of subgroups of cardinalities strictly less than λ only. Now we are going to extend the notion of a cub and a stationary set in the following natural way.

11. Definition. Let λ be a regular uncountable cardinal and $\mathfrak{C}$ a weak λ-cover of the group G. A collection $\mathfrak{D}$ of members of $\mathfrak{C}$ is called a $\mathfrak{C}$-*cub* provided it is closed under smooth ascending unions $H = \bigcup\limits_{\alpha<\kappa} H_\alpha$ with $H \in \mathfrak{C}$ and every element of $\mathfrak{C}$ is contained in that of $\mathfrak{D}$. Furthermore, a subcollection $\mathfrak{E}$ of $\mathfrak{C}$ is called $\mathfrak{C}$-*stationary* if it intersects each $\mathfrak{C}$-cub non-trivally.

If G is a torsion-free B_1-group of regular cardinality λ and $G = \bigcup\limits_{\alpha<\lambda} G_\alpha$ is any its λ-filtration consisting of B_1-subgroups then there is a cub $C \subseteq \lambda$ such that, for each $\alpha \in C$, G_α is TEP in G_β whenever $\alpha < \beta < \lambda$. This very important result in the theory of infinite rank Butler groups has been proved in [DHR; Theorem 7.1] (for the simplified proof see [F2]). As a special case we obviously get that G has a

λ-filtration $G = \bigcup_{\alpha<\lambda} G_\alpha$ such that the set $\{\alpha < \lambda \mid G_\alpha$ is not TEP in $G_{\alpha+1}\}$ is not stationary. It follows from [BB; Proposition 2.2] that the general condition is also sufficient. Now we are going to show that the special one is sufficient, too.

12. Theorem. *Let G be an almost B_*-group of regular uncountable cardinality λ. The following conditions are equivalent:*

(i) *G is a B_*-group;*

(ii) *for any λ-filtration $G = \bigcup_{\alpha<\lambda} G_\alpha$ of G consisting of B_*-groups the set $E = \{\alpha < \lambda \mid G_\alpha$ is not TEP in some $G_\beta\} \subseteq \lambda$ is not stationary;*

(iii) *there is a λ-filtration $G = \bigcup_{\alpha<\lambda} G_\alpha$ of G consisting of B_*-groups such that the set $E = \{\alpha < \lambda \mid G_\alpha$ is not TEP in some $G_\beta\} \subseteq \lambda$ is not stationary;*

(iv) *for each weak λ-cover $\mathfrak{C}$ of B_*-subgroups of G the set $U = \{H \in \mathfrak{C} \mid H$ is not TEP in some $K \in \mathfrak{C}\}$ is not $\mathfrak{C}$-stationary;*

(v) *there is a weak λ-cover $\mathfrak{C}$ of B_*-subgroups of G such that the set $U = \{H \in \mathfrak{C} \mid H$ is not TEP in some $K \in \mathfrak{C}\}$ is not $\mathfrak{C}$-stationary.*

P r o o f. We start with the B_1-groups case. (i) implies (ii). By [DHR; Theorem 7.1] there is a cub C in λ such that for each $\alpha \in C$, G_α is TEP in G_β for all $\alpha < \beta < \lambda$. Hence $E \cap C = \emptyset$.

The implications (ii) implies (iii) and (iv) implies (v) are obvious.

(iii) implies (iv). Let $G = \bigcup_{\alpha<\lambda} G_\alpha$ be a given λ-filtration of G and let $C \subseteq \lambda$ be a cub disjoint with the set E. If $\mathfrak{C}$ is any weak λ-cover of G consisting of B_1-groups, then we can construct a λ-filtration $G = \bigcup_{\alpha<\lambda} H_\alpha$ from the members of $\mathfrak{C}$ in the natural way. The set $D = \{\alpha < \lambda \mid G_\alpha = H_\alpha\}$ is a cub in λ and $C \cap D$ is a cub in λ, too. Now for each $\alpha \in C \cap D$ we see that $G_\alpha = H_\alpha$ is TEP in any G_β with $\alpha < \beta < \lambda$ and so the regularity of λ yields that $\{G_\alpha \mid \alpha \in C \cap D\}$ is a $\mathfrak{C}$-cub which is obviously disjoint with U.

(v) implies (i). Let $\mathfrak{D} \subseteq \mathfrak{C}$ be a $\mathfrak{C}$-cub such that $\mathfrak{D} \cap U = \emptyset$. Constructing a λ-filtration $G = \bigcup_{\alpha<\lambda} G_\alpha$ of G from the members of $\mathfrak{D}$ in the usual way, we see that G_α is TEP in $G_{\alpha+1}$ for each $\alpha < \lambda$ and an application of [BB; Proposition 2.2] completes the proof of this part.

Proceeding to B_2-groups the implications (i) implies (ii) and (iii) implies (iv) follow from the above part, every B_2-group being a B_1-group, while the implications (ii) implies (iii) and (iv) implies (v) are trivial. To prove the remaining implication (v) implies (i) note that G is a B_1-group by the first part and so Corollary 10 completes the proof. □

Now we proceed to a result on TEP subgroups which is closely related to [BR; Proposition 5.1] and which enables us to prove a stronger version of the implication (ii) $\Longrightarrow$ (i) in the preceding theorem.

13. Proposition. *Let G be a torsion-free group which is expressible as a smooth ascending union of pure subgroups $G = \bigcup_{\alpha<\lambda} G_\alpha$, where λ is a limit ordinal. Then there is a cub C in λ such that for each $\alpha \in C$ either G_α is not* TEP *in $G_{\alpha'}$ where α' is the successor of α in C or it is* TEP *in G_β whenever $\alpha < \beta$ and $\beta \in C$.*

Proof. Note that if $K \leqslant H \leqslant G$ are pure subgroups of G, then if K is TEP in G, it is obviously TEP in H. Thus, if the set $\{\beta < \lambda \mid G_\alpha \text{ is TEP in } G_\beta\}$ is unbounded, then G_α is TEP in G_β whenever $\alpha < \beta < \lambda$. Set $t(0) = 0$ and assume that $t(\beta) < \lambda$ have been already selected for some $\alpha < \lambda$ and all $\beta < \alpha$. For α limit we simply set $t(\alpha) = \bigcup_{\beta<\alpha} t(\beta)$, while for $\alpha = \beta + 1$ we put $t(\alpha) = t(\beta) + 1$ if $G_{t(\beta)}$ is TEP in each G_γ, $t(\beta) < \gamma < \lambda$, and otherwise we take $t(\alpha)$ to be the first ordinal $\gamma < \lambda$ such that $G_{t(\beta)}$ is not TEP in G_γ. Obviously, $C = \{t(\alpha) \mid \alpha < \lambda\}$ is the cub in λ having the required property. □

14. Proposition. *Let G be a smooth ascending union $G = \bigcup_{\alpha<\lambda} G_\alpha$ of pure B_*-subgroups, where λ is a limit ordinal. Then there is a cub C in λ such that for each $\alpha \in C$ the group G_α is* TEP *in G_β for each $\beta \in C$, $\alpha < \beta$, whenever it is* TEP *in $G_{\alpha'}$, where α' is the successor of α in C. If the set $E = \{\alpha \in C \mid G_\alpha \text{ is not TEP in } G_{\alpha'}\}$ is not stationary in λ then G is a B_*-group.*

Proof. The first part follows immediately from Proposition 13. Now if E is not stationary, then there is a cub D in λ such that $D \cap E = \emptyset$. The intersection $C \cap D$ is a cub in λ disjoint to E, hence G_α is TEP in $G_{\alpha'}$ for each $\alpha \in C \cap D$ and its successor α' in $C \cap D$. By [BB; Proposition 2.2] G is a B_1-group and in the case of B_2-groups G has a preseparative chain by Lemma 7 and [F3; Theorem 4.1] applies. □

For the sake of completeness we shall include the following result on B_2-groups (for the free group due independently to J. Gregory, D. W. Kueker, A. Mekler and S. Shelah) which has been proved in fact in [BR]. Moreover, we shall extend it to a similar result for almost B_1-groups. The definition of a weakly compact cardinal was repeated in [BR]. The only fact we will need in the sequel is the following property satisfied by weakly compact cardinals.

Property (P). A regular cardinal λ is said to have the property (P) if for any stationary set $E \subseteq \lambda$ there is a regular cardinal $\kappa < \lambda$ such that $E \cap \kappa$ is stationary in κ.

15. Theorem. *If* $G = \bigcup_{\alpha<\lambda} G_\alpha$ *is a smooth ascending union of pure* B_**-subgroups such that* $|G_\alpha| = |\alpha|\cdot\aleph_0$ *for each* $\alpha < \lambda$ *and* λ *is a regular cardinal having the property* (P)*, then* G *is a* B_**-group.*

P r o o f. Assume first that G_α's are B_1-groups. By Proposition 13 there is a cub C in λ such that for each $\alpha \in C$ the subgroup G_α is TEP in every G_β, $\alpha < \beta < \lambda$, whenever it is TEP in $G_{\alpha'}$, α' being the successor of α in C. In view of Proposition 14 it suffices to show that the set $E = \{\alpha \in C \mid G_\alpha \text{ is not TEP in } G_{\alpha'}\}$ is not stationary.

Assume, by way of contradiction, that E is a stationary subset of λ. By Property (P), there is a regular cardinal $\kappa < \lambda$ such that $E \cap \kappa$ is stationary in κ. Now $G_\kappa = \bigcup_{\alpha<\kappa} G_\alpha$ is a κ-filtration of the B_1-group G_κ consisting of B_1-subgroups and so Theorem 12 yields that the set $E_\kappa = \{\alpha < \kappa \mid G_\alpha \text{ is not TEP in some } G_\beta\}$ is not stationary in κ. Thus, there is a cub D in κ such that $E_\kappa \cap D = \emptyset$. Hence $E \cap \kappa \cap D \neq \emptyset$, $E \cap \kappa$ being stationary in κ, and so for $\alpha \in E \cap \kappa \cap D$ we have $\alpha \in E \cap \kappa$ showing that G_α is not TEP in $G_{\alpha'}$, where α' is the successor of α in C. On the other hand, $\alpha \in D$ means that $\alpha \notin E_\kappa$ and consequently G_α is TEP in every G_β, $\alpha < \beta < \kappa$. If G_α's are B_2-groups, G has a preseparative chain by Lemma 7 and it suffices to use [F3; Theorem 4.1]. □

16. Corollary. *An almost* B_**-group* G *of a weakly compact cardinality* λ *is a* B_**-group.*

P r o o f. If $\mathfrak{C}$ is a weak λ-cover of G consisting of B_*-subgroups, then we can construct, in the natural way, a λ-filtration $G = \bigcup_{\alpha<\lambda} G_\alpha$ of G such that $|G_\alpha| = |\alpha|\cdot\aleph_0$ for each $\alpha < \lambda$ and Theorem 15 applies. □

17. Corollary ([BR; Theorem 5.3]). *Let* λ *be a regular cardinal with the Property* (P) *and let* $G = \bigcup_{\alpha<\lambda} G_\alpha$ *be a* λ*-filtration of* G *consisting of* B_2*-subgroups. Then* G *is a* B_2*-group.*

P r o o f. Without loss of generality we may assume that $G_0 = 0$ and we can construct a refinement of the given λ-filtration to $G = \bigcup_{\alpha<\lambda} H_\alpha$ in such a way that H_α is a B_2-group and $|H_\alpha| = |\alpha| \cdot \aleph_0$ for each $\alpha < \lambda$. Set $H_0 = 0$ and assume that for some $\alpha < \lambda$ we have constructed $H_\beta = G_\alpha$ with the required properties. Let $\mathfrak{C}$ be an axiom-3 family of decent and B_2-subgroups of $G_{\alpha+1}$ and let $\{g_\gamma \mid \gamma < |G_{\alpha+1}|\}$ be any list of elements of $G_{\alpha+1}$. Assuming that for some $\beta \leqslant \gamma$ the subgroup H_γ has been already constructed in such a way that $H_\gamma \subsetneqq G_{\alpha+1}$ and $|H_\gamma| = |\gamma| \cdot \aleph_0$, we can take $H_{\gamma+1}$ to be a member of $\mathfrak{C}$ containing H_γ and the element g_δ with the smallest δ such that $g_\delta \notin H_\gamma$. Taking simply unions for limit ordinals, we see that after an appropriate number of steps we reach $G_{\alpha+1}$. Now it suffices to use Theorem 15. □

Again, we will leave open the question whether B_1-groups are in general almost B_1-groups or not, and we will conclude this note by presenting some criteria under which an almost B_1-group is a B_2-group.

18. Theorem. *A B_1-group G of uncountable cardinality λ is a B_2-group if and only if it is a hereditary almost B_1-group.*

P r o o f. If G is a B_2-group then by [AH] it has an axiom-3 family $\mathfrak{D}$ of decent and TEP B_2-subgroups determined by the so called closed subsets of the ordinal λ. It is easy to verify (see e.g. [B2; Theorem 6]) that the set $\mathfrak{C}$ of all members of $\mathfrak{D}$ of cardinality strictly less than λ is obviously the desired hereditary weak λ-cover of the group G.

To prove the converse let $\mathfrak{C}$ be a hereditary weak λ-cover of G and let λ be the smallest (uncountable) cardinal for which there exists a B_1-group G of cardinality λ satisfying the stated conditions which is not a B_2-group. By [BS] and [DR] any B_1-group of cardinality at most $\aleph_1$ is a B_2-group and so $\lambda \geqslant \aleph_2$. Assuming λ regular we can construct a λ-filtration $G = \bigcup\limits_{\alpha<\lambda} G_\alpha$ of G consisting of members of $\mathfrak{C}$. The choice of λ yields that all G_α's are B_2-groups, $\mathfrak{C}$ being hereditary. Now G is a B_1-group and so by Theorem 12 the set $E = \{\alpha < \lambda \mid G_\alpha \text{ is not TEP in some } G_\beta\}$ is not stationary and an application of Theorem 12 yields that G is a B_2-group, contradicting the hypothesis. Thus λ is necessarily singular. Again, the choice of λ yields that all the members of $\mathfrak{C}$ are B_2-groups and Theorem 4 yields the final contradiction completing the proof. □

19. Corollary. *An almost B_1-group G of uncountable cardinality λ is a B_2-group if and only if it has a hereditary weak λ-cover $\mathfrak{C}$ of B_1-groups such that the set $E = \{H \in \mathfrak{C} \mid H \text{ is not TEP in some } K \in \mathfrak{C}\}$ is not $\mathfrak{C}$-stationary.*

P r o o f. We start with the sufficiency of the condition. Let λ be the smallest cardinal for which there is an almost B_1-group G satisfying the stated conditions which is not a B_2-group. As in the preceding proof we have $\lambda \geqslant \aleph_2$. For λ regular G is a B_1-group by Theorem 12 and Theorem 18 applies. The case of λ singular, as well as the converse implication, have been solved in the preceding proof. □

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