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EXAMPLES OF BIFURCATION OF PERIODIC SOLUTIONS TO VARIATIONAL INEQUALITIES IN $\mathbb{R}^\kappa$

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Abstract. A bifurcation problem for variational inequalities

$$U(t) \in K,$$
$$(\dot U(t) - B_\lambda U(t) - G(\lambda, U(t)),\ Z - U(t)) \geqslant 0 \ \text{ for all } \ Z \in K,\ \text{a.a.} \ \ t \geqslant 0$$

is studied, where K is a closed convex cone in $\mathbb{R}^\kappa$, $\kappa \geqslant 3$, B_λ is a $\kappa \times \kappa$ matrix, G is a small perturbation, λ a real parameter. The main goal of the paper is to simplify the assumptions of the abstract results concerning the existence of a bifurcation of periodic solutions developed in the previous paper and to give examples in more than three dimensional case.

Keywords: bifurcation, periodic solutions, variational inequality, differential inequality, finite dimensional space

MSC 2000: 37G15, 34C23, 34A40

Introduction

Consider a smooth mapping $F\colon\ I \times \mathbb{R}^\kappa \to \mathbb{R}^\kappa$ ($\kappa \geqslant 3$, I an open interval) such that $F(\lambda, 0) = 0$ for all $\lambda \in I$. Let K be a closed convex cone with its vertex at the origin in $\mathbb{R}^\kappa$. Consider a bifurcation problem for the inequality

$$\text{(I)} \qquad \begin{cases} U(t) \in K, \\ (\dot U(t) - F(\lambda, U(t)),\ Z - U(t)) \geqslant 0 \ \text{ for all } Z \in K,\ \text{a.a.} \ \ t \geqslant 0. \end{cases}$$

By a solution we mean an absolutely continuous function satisfying (I). This paper is closely related to [7] where it is proved that if a Hopf bifurcation for the equation

$$\text{(E)} \qquad \dot U(t) = F(\lambda, U(t))$$

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occurs at some λ_0 then under certain additional assumptions, there exists also a bifurcation point λ_I of (I) at which periodic solutions bifurcate from the branch of trivial solutions. Cf. also [6], [5] for former results of this type. The method used was based on the proof of existence of branches of periodic solutions of the corresponding penalty system. The global bifurcation results given (for equations) in [10] and [1] (see also [4]) form a background for this proof in [5], [6] and [7], respectively. For obtaining bifurcating solutions, certain information about the branches mentioned were necessary. Unfortunately, it is not easy to get this information in concrete situations. In [7], a simple model example in $\mathbb{R}^3$ was discussed. In fact, this example could be solved by a simpler approach developed for the particular case of $\mathbb{R}^3$ in [3], [8]. The aim of the present paper is to simplify the assumptions from [7] and to give examples in $\mathbb{R}^\kappa$, $\kappa > 3$ where the theory from [3], [8] cannot be used. We will modify for a general situation the basic ideas of the verification of the assumptions of the abstract theory applied in [7] only to Model Example. In this way we will obtain a bifurcation Theorem 1.1. While the abstract assumptions in [7] are concerned with general properties of a branch of solutions to a penalty system (see also the assumption (GA) in Section 3 below), now we will deal only with assumptions concerning the "linearized penalty equation" and the "linearized inequality". (Note that these problems are only positively homogeneous but nonlinear again.)

In the examples discussed, the first two equations seem to be independent of the remaining ones but they are coupled by the obstacle given by the cone. Direct use of Theorem 1.2 from [7] would be possible for these examples but the verification of the assumptions in our present setting is easier.

1. Bifurcation Theorem

We denote $(U,V) = \sum\limits_{i=1}^{\kappa} u_i v_i$, $|U|^2 = (U,U)$ for $U = [u_1,\ldots,u_\kappa]$, $V = [v_1,\ldots,v_\kappa]$. Further, we will write $F(\lambda,U) = B_\lambda U + G(\lambda,U)$, where B_λ is a real matrix of the type $\kappa \times \kappa$ depending continuously on a real parameter $\lambda \in I$, I is an open interval in $\mathbb{R}$, $G\colon I \times \mathbb{R}^\kappa \to \mathbb{R}^\kappa$ satisfies the conditions

$$\text{(G)} \qquad |G(\lambda,U)| = O(|U|^2) \text{ uniformly on compact } \lambda \text{ intervals,}$$

$$\text{(L)} \quad \begin{cases} \text{for any } \Lambda_1,\Lambda_2 \in I,\ R>0 \text{ there exists } C>0 \text{ such that} \\ |G(\lambda,U_1) - G(\lambda,U_2)| \leqslant C|U_1-U_2| \text{ for all } \lambda\in[\Lambda_1,\Lambda_2],\ |U_1|,|U_2|\leqslant R. \end{cases}$$

Let $W_1(\lambda),\ldots,W_\kappa(\lambda)$ be a basis of $\mathbb{C}^\kappa$ composed of the elements of the chains corresponding to the eigenvalues of B_λ (i.e. of the corresponding eigenvectors if B_λ

has κ eigenvalues for some λ). Suppose that there are $\lambda_0 \in I$, ω_m, $\omega_M > 0$ such that

$$
(\mu) \qquad \begin{cases} \text{there is a couple of simple eigenvalues } \mu_{1,2}(\lambda) = \alpha(\lambda) \pm \mathrm{i}\omega(\lambda), \\ \alpha, \omega \text{ are continuous real functions, } \omega_m \leqslant \omega(\lambda) \leqslant \omega_M \text{ for all } \lambda \in I, \\ \alpha(\lambda) < 0 \text{ for } \lambda < \lambda_0,\ \alpha(\lambda_0) = 0,\ \alpha(\lambda) > 0 \text{ for } \lambda > \lambda_0, \\ \text{the other eigenvalues have negative real parts for all } \lambda \in I, \\ W_j(\lambda) \text{ depend continuously on } \lambda. \end{cases}
$$

Particularly, the chains corresponding to $\mu_1(\lambda)$ and $\mu_2(\lambda)$ contain only an eigenvector $W_1(\lambda)$ and $W_2(\lambda)$, respectively. We can write $W_j(\lambda) = U_j(\lambda) + \mathrm{i}U_{j+1}(\lambda)$, $W_{j+1}(\lambda) = U_j(\lambda) - \mathrm{i}U_{j+1}(\lambda)$ if $W_j(\lambda)$, $W_{j+1}(\lambda)$ is a pair of complex conjugate elements of some chain, $W_j(\lambda) = U_j(\lambda)$ if $W_j(\lambda)$ is real, where $U_1(\lambda), \ldots, U_\kappa(\lambda)$ is a basis of $\mathbb{R}^\kappa$. Notice that $U_1(\lambda), \ldots, U_\kappa(\lambda)$ are continuously dependent on λ.

Notation 1.1.
$\omega_0 = \omega(\lambda_0)$,
$\mathbb{L}_\lambda = \mathrm{Lin}\{U_1(\lambda), U_2(\lambda)\}$,
$\mathbb{S}_\lambda = \mathrm{Lin}\{U_3(\lambda), \ldots, U_\kappa(\lambda)\}$,
$P_{L_\lambda} V = y_1 U_1(\lambda) + y_2 U_2(\lambda)$ for $V = \sum_{j=1}^{\kappa} y_j U_j(\lambda)$ (projection onto $\mathbb{L}_\lambda$ along $\mathbb{S}_\lambda$).

Denote by P_K the projection onto K, i.e. $P_K U$ for $U \in \mathbb{R}^\kappa$ is the unique point from K satisfying

$$|P_K U - U| = \min_{V \in K} |V - U|.$$

The penalty operator β corresponding to K is defined by

$$\beta = I - P_K.$$

For the proof of the bifurcation theorem, branches of periodic solutions of the following penalty system are studied in [7]:

$$
(\mathrm{PS}) \qquad \begin{cases} \dot U(t) = F(\lambda, U(t)) - \varepsilon(t)\beta U(t), \\ \dot\varepsilon(t) = -\varrho^2 \dfrac{\varepsilon(t)}{1 + |\varepsilon(t)|} + |U(t)|^2. \end{cases}
$$

However, under our present assumptions, only properties of the corresponding "linearized" penalty equation

$$
(\mathrm{LPE}) \qquad \dot U(t) = B_\lambda U(t) - \tau\beta U(t)
$$

and the "linearized" inequality

$$
(\mathrm{LI}) \qquad \begin{cases} U(t) \in K, \\ (\dot U(t) - B_\lambda U(t),\ Z - U(t)) \geqslant 0 \ \text{ for all } Z \in K, \text{ a.a. } \ t \geqslant 0 \end{cases}
$$

will play a role. Let us recall that the problems (LPE), (LI) are in fact nonlinear again, neither our variational inequality nor the penalty system can be linearized.

Remark 1.1. Under the assumption (μ), any solution of the linearized equation

$$\text{(LE)} \qquad \dot{U}(t) = B_\lambda U(t)$$

tends to the plane $\mathbb{L}_\lambda$ for $t \to +\infty$ and if it does not start in $\mathbb{S}_\lambda$ then its projection into $\mathbb{L}_\lambda$ circulates around the origin. A solution of (LPE) is simultaneously a solution of (LE) as far as it lies in K and it is attracted to K by the penalty term when it lies outside of K. Particularly, there is no periodic solution of (LPE) lying in K for all t in the examples below because (LE) has no such solution.

Remark 1.2. For $U \in K$ we will denote by $K_U = \overline{\bigcup_{h>0} \bigcup_{V \in K} h(V-U)}$ the contingent cone to K at the point U, and by P_U the projection to the cone K_U. Let us recall that (LI) is equivalent to the (strongly nonlinear) equation

$$\text{(LI')} \qquad \dot{U}(t) = P_{U(t)} B_\lambda U(t)$$

(see [2]). It follows that any solution of (LI) is simultaneously a solution of (LE) as far as it lies in K^0 (the interior of K). Analogously for (I). Particularly, it follows that there is no periodic solution of (LI) lying in K^0 for all t in the examples below. Of course, any solution of (LE) is simultaneously a solution of (LI) on any interval in which it lies in K.

Notation 1.2.

$U^\tau_{0,\lambda}(\cdot, V)$, $U^\infty_{0,\lambda}(\cdot, V)$ and $U^\infty_\lambda(\cdot, V)$—the solutions of (LPE), (LI) and (I), respectively, with the initial condition V at $t = 0$,

$r^\tau_{0,\lambda}(t, V)$, $\varphi^\tau_{0,\lambda}(t, V)$ (also for $\tau = +\infty$)—polar coordinates of $P_{L_\lambda} P_K U^\tau_{0,\lambda}(t, V)$ with the angle φ measured from $P_{L_\lambda} P_K V$, i.e. continuous functions defined by $\varphi^\tau_{0,\lambda}(0, V) = 0$,

$$\begin{aligned} P_{L_\lambda} P_K U^\tau_{0,\lambda}(t, V) = r^\tau_{0,\lambda}(t, V)[&\cos(\varphi^\tau_{0,\lambda}(t, V) + \varphi_V) \cdot U_1(\lambda) \\ &+ \sin(\varphi^\tau_{0,\lambda}(t, V) + \varphi_V) \cdot U_2(\lambda)] \end{aligned}$$

for $t \in [0, t_0)$ if $|P_{L_\lambda} P_K U^\tau_{0,\lambda}(t, V)| > 0$ on $[0, t_0)$, where φ_V satisfies

$$P_{L_\lambda} P_K V = r^\tau_{0,\lambda}(0, V)[\cos \varphi_V \cdot U_1(\lambda) + \sin \varphi_V \cdot U_2(\lambda)],$$

$t^\tau_{0,\lambda}(V) = \inf\{t_0;\ r^\tau_{0,\lambda}(t, V) > 0 \text{ for } t \in [0, t_0],\ \varphi^\tau_{0,\lambda}(t_0, V) = -2\pi\}$ if $V \notin \mathbb{S}_\lambda$—the time of one circuit of $P_{L_\lambda} P_K U^\tau_{0,\lambda}(\cdot, V)$ around the origin.

Remark 1.3. We have $(P_K U, \beta U) = 0$, $P_K \beta U = 0$ for all $U \in \mathbb{R}^\kappa$. This is why we consider the function $P_K U_{0,\lambda}^\tau(t,V)$ instead of $U_{0,\lambda}^\tau(t,V)$ in the definition of the functions $r_{0,\lambda}^\tau(t,V)$, $\varphi_{0,\lambda}^\tau(t,V)$. In this case, the estimate of $\dot{\varphi}_{0,\lambda}^\tau(t,V)$ in examples becomes simpler because the expression with the penalty term vanishes.

Remark 1.4. Any solution $U_{0,\lambda}^\tau(t,V)$ of (LPE) (τ finite) is continuosusly differentiable. The projection P_K is a lipschitzian mapping (see e.g. [11]) and it follows that $P_K U_{0,\lambda}^\tau(t,V)$ is absolutely continuous on any compact interval. Particularly, $P_K U_{0,\lambda}^\tau(t,V)$ is differentiable a.e.

Further, suppose that there are Λ_m, $\Lambda_M \in I$, $\xi > 0$, $\eta \in (0,1)$, $\Gamma > 0$ such that $\lambda_0 \in [\Lambda_m, \Lambda_M]$ and the following conditions are fulfilled:

$$
\tag{1.1}
\begin{cases}
\text{if } W \in \mathbb{R}^\kappa,\ |W| > 0,\ \tau \in [0,+\infty],\ \lambda \in (\Lambda_m - \xi, \Lambda_M + \xi),\\
U_{0,\lambda}^\tau(T,W) = W \quad \text{with some } T > 0 \text{ then}\\
(a) \qquad U_{0,\lambda}^\tau(t,W) \notin \mathbb{S}_\lambda \text{ for all } t \geqslant 0,\\
(b) \qquad \dot{\varphi}_{0,\lambda}^\tau(t,W) < -\eta \text{ for a.a. } t \geqslant 0,\\
(c) \qquad t_{0,\lambda}^\tau(W) > \Gamma,
\end{cases}
$$

$$
\tag{1.2}
\begin{cases}
\text{if } W \in \mathbb{R}^\kappa,\ |W| > 0,\ \tau \in (0,+\infty],\\
U_{0,\lambda}^\tau(T,W) = W,\ T > 0 \text{ then } \lambda \in (\Lambda_m, \Lambda_M).
\end{cases}
$$

Particularly, the condition (1.1b) means that $P_{L_\lambda} P_K U_{0,\lambda}^\tau(t,W)$ circulates around the origin with a velocity greater than η under the assumptions considered. We will choose a fixed $\eta \in (0,\omega_m)$ such that (1.1) holds, $\eta \neq \frac{\omega_0}{k}$, $k = 1,2,\ldots$, and set

$$
T_M = \frac{2\pi}{\eta} \quad \Big(T_M > \frac{2\pi}{\omega_m}\Big).
$$

Observation 1.1. The assumption (1.1) implies that

$$
t_{0,\lambda}^\tau(t,W) < T_M \text{ for the parameters from (1.1).}
$$

We have $t_{0,\lambda}^0(W) = \frac{2\pi}{\omega(\lambda)}$ if $W \notin \mathbb{S}_\lambda$. Particularly, (1.1$b$), (1.1$c$) imply $\Gamma < \frac{2\pi}{\omega_0} < T_M$.

Theorem 1.1. *Let* (μ), (G), (L) *and* (1.1), (1.2) *be fulfilled. Then there exists a bifurcation point* $\lambda_I \in (\Lambda_m, \Lambda_M)$ *of* (I) *at which periodic (nonstationary) solutions of* (I) *bifurcate from the branch of trivial solutions. Precisely, there exist* $T_n \in (\Gamma, T_M)$, $\lambda_n \in (\Lambda_m, \Lambda_M)$, $V_n \in \mathbb{R}^\kappa$ *such that* $|V_n| \to 0$, $\lambda_n \to \lambda_I$ *and* $U_{\lambda_n}^\infty(\cdot, V_n)$ *are* T_n*-periodic (nonstationary) solutions of* (I). *If* $T_n \to T$, $\frac{V_n}{|V_n|} \to W$ *then the solution* $U_{0,\lambda_I}^\infty(\cdot, W)$ *of* (LI) *is (nonstationary)* T*-periodic.*

Remark 1.5. The assertion of Theorem 1.1 remains valid if we replace (Λ_m, Λ_M) by the closed interval $[\Lambda_m, \Lambda_M]$ in the statement and in the assumption (1.2) (see the proof of Theorem 1.1 in Section 3).

Remark 2.1. Let us consider the following variational inequality in $\mathbb{R}^3$:

$$\tag{2.1} \begin{cases} V(t) \in \tilde{K}, \\ (\dot{V}(t) - \tilde{B}_\lambda V(t) - \tilde{G}(\lambda, V(t), Z - V(t)) \geqslant 0 \text{ for all } Z \in \tilde{K}, \text{ a.a. } t \geqslant 0 \end{cases}$$

with $\tilde{K} = \{V = [v_1, v_2, v_3] \in \mathbb{R}^3\,;\ v_3 \geqslant 0,\ v_3 \geqslant v_1\}$,

$$\tilde{B}_\lambda = \begin{pmatrix} \lambda, & 1, & 0 \\ -1, & \lambda, & 0 \\ 0, & 0, & \nu_1 \end{pmatrix},$$

$\tilde{G}\colon \mathbb{R}^4 \to \mathbb{R}^3$ a mapping satisfying the assumptions (G), (L), $\nu_1 \in [-1, 0)$. This problem was discussed as Model Example in [7] and the existence of a bifurcation point $\lambda_I \in (0, 1)$ was shown. (In fact, the case $\nu_1 = -1$ was considered but all considerations can be repeated for any fixed $\nu_1 \in [-1, 0)$). In [7], some general assumptions concerning the branch of solutions of the penalty system (PS) are verified for this example to apply the abstract theory. Main ideas of this verification (used in [7] only on Model Example) are generalized in the present paper and used for the proof of our Theorem 1.1 by using abstract Theorem 1.2 from [7] (see Section 3). For the use of our present Theorem 1.1, only verification of the conditions (1.1), (1.2) (with $\Lambda_m = 0$, $\Lambda_M = 1$) concerning only the linearized inequality (LI) and linearized penalty equation (LPE) is sufficient. Instead of (1.1a), (1.1c), analogous conditions concerning solutions of (PS) lying on the branch $\mathcal{C}^0_\varrho$ are verified in [7] (conditions (1.14), (1.17)). The assumption (1.1b) is a part of the condition (1.11) in [7] where also analogues for (PS) and (I) are necessary. Our condition (1.2) coincides with (1.12) verified in [7]. All assumptions (1.1), (1.2) can be verified directly in the same way as in Example 2.1 below for the fourth dimensional case.

Note that in addition, in [7] we supposed for simplicity that $g_3(\lambda, U) \geqslant 0$ while in the present approach this condition has no justification.

Let us recall that the problem (2.1) was investigated by another approach (applicable only in the three dimensional space) in [8] where a stability of bifurcating solutions was studied.

Example 2.1. Set $I = \mathbb{R}$, $\kappa = 4$, $K = \{U \in \mathbb{R}^4\,;\ u_3 \geqslant 0, u_4 \geqslant 0, u_3 \geqslant u_1, u_4 \geqslant u_2\}$,

$$B_\lambda = \begin{pmatrix} \lambda, & 1, & 0, & 0 \\ -1, & \lambda, & 0, & 0 \\ 0, & 0, & \nu_1, & 0 \\ 0, & 0, & 0, & \nu_2 \end{pmatrix}$$

where $\nu_1 \in [-\frac{1}{2}, 0)$, $\nu_2 \in [-\frac{1}{2}, 0)$ are given. Let $G\colon \mathbb{R}^5 \to \mathbb{R}^4$ be a mapping satisfying (G), (L), $G(\lambda, U) = [g_1(\lambda, U), g_2(\lambda, U), g_3(\lambda, U), g_4(\lambda, U)]$. We will show that Theorem 1.1 guarantees the existence of a bifurcation point $\lambda_I \in (0,1)$ of our inequality (I) at which periodic solutions bifurcate from the branch of the trivial solutions.

Verification of the assumptions of Theorem 1.1 with $\Lambda_m = 0$, $\Lambda_M = 1$, ξ, $\eta > 0$ small enough: The eigenvalues of B_λ are $\mu_{1,2}(\lambda) = \lambda \pm i$, $\mu_3(\lambda) = \nu_1$, $\mu_4(\lambda) = \nu_2$, the real and imaginary parts of the corresponding eigenvectors form the basis $U_1(\lambda) = U_1 = [1, 0, 0, 0]$, $U_2(\lambda) = U_2 = [0, 1, 0, 0]$, $U_3(\lambda) = U_3 = [0, 0, 1, 0]$, $U_4(\lambda) = U_4 = [0, 0, 0, 1]$. We have $\mathbb{L}_\lambda = \mathbb{L} = \{U \in \mathbb{R}^4;\ u_3 = u_4 = 0\}$, $\mathbb{S}_\lambda = \mathbb{S} = \{U \in \mathbb{R}^4;\ u_1 = u_2 = 0\}$. The assumption (μ) (with $\omega(\lambda) \equiv 1$) holds. For $U = [u_1, u_2, u_3, u_4], u_3 \geqslant 0, u_4 \geqslant 0$ we have

$$
(2.2)\qquad \begin{cases}
P_K U = [u_1, u_2, u_3, u_4] \text{ if } u_3 \geqslant u_1,\ u_4 \geqslant u_2,\\
P_K U = \left[\frac{u_1+u_3}{2},\ u_2,\ \frac{u_1+u_3}{2},\ u_4\right] \text{ if } u_3 < u_1,\ u_4 \geqslant u_2,\\
P_K U = \left[u_1,\ \frac{u_2+u_4}{2},\ u_3,\ \frac{u_2+u_4}{2}\right] \text{ if } u_3 \geqslant u_1,\ u_4 < u_2,\\
P_K U = \left[\frac{u_1+u_3}{2},\ \frac{u_2+u_4}{2},\ \frac{u_1+u_3}{2},\ \frac{u_2+u_4}{2}\right] \text{ if } u_3 < u_1,\ u_4 < u_2,\\
\quad \beta U = \left[\frac{(u_3-u_1)^-}{2}, \frac{(u_4-u_2)^-}{2}, -\frac{(u_3-u_1)^-}{2}, -\frac{(u_4-u_2)^-}{2}\right]\\
\qquad\qquad \text{for all } u_3 \geqslant 0, u_4 \geqslant 0.
\end{cases}
$$

Hence, in the set $\{U \in \mathbb{R}^4;\ u_3, u_4 \geqslant 0\}$, the penalty equation (LPE) has the form

$$
(2.3)\qquad \begin{cases}
\dot u_1 = \lambda u_1 + u_2 - \tau\frac{(u_3-u_1)^-}{2},\\
\dot u_2 = -u_1 + \lambda u_2 - \tau\frac{(u_4-u_2)^-}{2},\\
\dot u_3 = \nu_1 u_3 + \tau\frac{(u_3-u_1)^-}{2},\\
\dot u_4 = \nu_2 u_4 + \tau\frac{(u_4-u_2)^-}{2}.
\end{cases}
$$

The third and the fourth coordinate of $P_K W$ is nonnegative for any W. This together with the form of B_λ and β implies that if $U = B_\lambda W - \tau\beta W$, $\tau \geqslant 0$ and $w_3 < 0$ or $w_4 < 0$ then $u_3 > 0$ or $u_4 > 0$, respectively. It follows that

$$
(2.4)\qquad \begin{cases}
\text{the set } \{U;\ u_3 \geqslant 0, u_4 \geqslant 0\} \text{ is invariant for (LPE) with any } \lambda \in \mathbb{R},\ \tau \geqslant 0,\\
\text{any periodic solution of (LPE) lies in this set and fulfils (2.3) for all } t \geqslant 0.
\end{cases}
$$

We will show that

$$
(2.5)\qquad \begin{cases}
\text{if } W \in \mathbb{R}^4,\ |W| > 0,\ \tau \in (0, +\infty],\ \lambda \leqslant 1,\ U(t) = U^\tau_{0,\lambda}(t, W)\\
= [u_1(t), u_2(t), u_3(t), u_4(t)],\ U(T) = W \ \text{ with some } T > 0\\
\ \text{then } u_3(t) > 0,\ u_4(t) > 0 \text{ for all } t \geqslant 0.
\end{cases}
$$

We already know from (2.4) and from the definition of K (for $\tau \in [0, +\infty)$ and $\tau = +\infty$, respectively) that $u_3(t) \geqslant 0$, $u_4(t) \geqslant 0$. Let us realize that

$$\dot u_3(t) \geqslant \nu_1 u_3(t),\ \dot u_4(t) \geqslant \nu_2 u_4(t) \text{ for a.a. } t \geqslant 0. \tag{2.6}$$

Indeed, this follows directly from (2.3) if $\tau \in [0, +\infty)$. If $\tau = +\infty$ then (2.6) follows from (LI) by setting (for any t fixed) $z_1 = u_1(t)$, $z_2 = u_2(t)$ and $z_4 = u_4(t)$, $z_3 \geqslant u_3(t)$ arbitrary or $z_3 = u_3(t)$, $z_4 \geqslant u_4(t)$ arbitrary, respectively. As a consequence, we obtain from (2.6) that if $u_3(t_0) > 0$ or $u_4(t_0) > 0$ for some t_0 then $u_3(t) > 0$ or $u_4(t) > 0$ for all $t \geqslant t_0$. Hence, for the proof of (2.5) it is sufficient to show that for $\tau \in (0, +\infty]$, u_3 and u_4 cannot vanish identically. Suppose for instance that $u_4(t) = 0$ for all $t \geqslant 0$.

First, let $\tau = +\infty$. Then $V(t) = [u_1(t), u_2(t), u_3(t)]$ is a solution of the variational inequality (2.1) with $\tilde K$, $\tilde B_\lambda$ from Remark 2.1, $\tilde G \equiv 0$. The conditions (1.1*b*), (1.2) with $\Lambda_m = 0$, $\Lambda_M = 1$ hold for this problem (see Remark 2.1) and it follows that the projection of $V(t)$ (and also of $U(t)$) into $\mathbb{L}$ should circulate around the origin. Particularly, $u_2(t)$ should change its sign during any circuit. Simultaneously we should have $u_2(t) \leqslant u_4(t) = 0$ for all $t \geqslant 0$ (because of $U(t) \in K$), which is a contradiction.

Further, let $\tau \in (0, +\infty)$. We suppose that $u_4(t) = 0$ and it follows from the last line in (2.3) that $u_2(t) \leqslant 0$ for all $t \geqslant 0$. Hence, the second equation in (2.3) reads $\dot u_2 = -u_1 + \lambda u_2$ and therefore $V(t) = [u_1(t), u_2(t), u_3(t)]$ satisfies the linearized penalty equation corresponding to (2.1) which is studied in [7], Model Example. According to the conditions (1.1*b*), (1.2) holding for this problem (see Remark 2.1), the projection of $V(t)$ (i.e also of $U(t)$) to $\mathbb{L}$ should circulate around the origin. Particularly, $u_2(t)$ should change the sign, which is a contradiction and the proof of (2.5) is complete. (Notice that we could verify (1.1*b*), (1.2) for the linearized problem to (2.1) directly as for a more complicated situation below.)

In the verification of the assumptions (1.1), (1.2), we will always consider an arbitrary fixed T-periodic solution $U(t) = U^\tau_{0,\lambda}(t, W)$ (with fixed τ, λ, W under consideration) and set $r_K(t) = r^\tau_{0,\lambda}(t, V)$, $\varphi_K(t) = \varphi^\tau_{0,\lambda}(t, V)$. Computing derivatives, we will always suppose that our function $P_K U(t)$ or $\varphi_K(t)$ is differentiable in t under consideration—see Remark 1.4. (In fact, in our example, all solutions discussed are differentiable for all t with the exception of some isolated points t where $U(t)$ intersects the hyperplanes $u_3 = u_1$, $u_4 = u_2$, and we could compute the right derivative at these exceptional points, but it is not necessary.)

Proof of (1.1*a*): Let $U(t_0) = Z \in \mathbb{S}$ for some t_0. Then $Z = [0, 0, z_3, z_4]$, $z_3 > 0$, $z_4 > 0$ by (2.5) and $U(t)$ must coincide with the solution $U^0_{0,\lambda}(t, Z) = [0, 0, \mathrm{e}^{\nu_1(t-t_0)} z_3, \mathrm{e}^{\nu_1(t-t_0)} z_4]$ of (LE) (i.e. (2.3) with $\tau = 0$) for $t \geqslant t_0$ (see Remarks 1.1, 1.2), which is not possible under the assumption $U(0) = U(T) = W$, $|W| > 0$, $T > 0$.

Further, let us set

$$P_L^* V = -y_2 U_1 + y_1 U_2 \text{ for } V = \sum_{j=1}^{4} y_j U_j.$$

We have

$$P_L^* P_K U(t) = r_K(t)\bigl(-\sin\varphi_K(t)\cdot[1,0,0,0] + \cos\varphi_K(t)\cdot[0,1,0,0]\bigr).$$

This together with the definiton of $r_K(t)$, $\varphi_K(t)$ implies

$$\dot{\varphi}_K(t) = \frac{\bigl(\frac{\mathrm{d}}{\mathrm{d}t} P_L P_K U(t), P_L^* P_K U(t)\bigr)}{|P_L P_K U(t)|^2} \text{ for a.a. } t \geqslant 0. \tag{2.8}$$

The formulas (2.2) give $|P_L P_K U(t)|^2 \leqslant u_1^2(t) + u_2^2(t)$ in all cases of our interest and according to (2.8), for *the proof of* (1.1*b*) it is sufficient to show that there are ξ, $\eta > 0$ such that if $\lambda \in (-\xi, 1+\xi)$ then

$$\dot{\varphi}_K(t)\cdot|P_L P_K U(t)|^2 = \Bigl(\frac{\mathrm{d}}{\mathrm{d}t} P_L P_K U(t), P_L^* U(t)\Bigr) \leqslant -\eta(u_1^2(t) + u_2^2(t)) \text{ for a.a. } t \geqslant 0.$$

Proof of (1.1*b*) *for* $\tau \in [0, +\infty)$:

(i) *The case* $u_3(t) \geqslant u_1(t)$, $u_4(t) \geqslant u_2(t)$: $P_K U(t) = U(t)$ and we get (even for all λ)

$$\begin{aligned}\dot{\varphi}_K(t)|P_L P_K U(t)|^2 &= -\dot{u}_1 u_2 + \dot{u}_2 u_1 = -\bigl(\lambda u_1 + u_2\bigr)u_2 + \bigl(-u_1 + \lambda u_2\bigr)u_1 \\ &= -(u_1^2 + u_2^2).\end{aligned}$$

(ii) *The case* $u_3(t) \geqslant u_1(t)$, $u_4(t) < u_2(t)$: Using the inequalities $u_2 u_4 \geqslant 0$, $|u_1 u_4| \leqslant |u_1| u_2$ (see (2.4)) we obtain (see also Remark 1.3)

$$\begin{aligned}&\dot{\varphi}_K(t)\cdot|P_L P_K U(t)|^2 = -\dot{u}_1 \cdot \frac{u_2 + u_4}{2} + \frac{\dot{u}_2 + \dot{u}_4}{2}\cdot u_1 \\ &= -\bigl(\lambda u_1 + u_2\bigr)\frac{u_2 + u_4}{2} + \frac{(-u_1 + \lambda u_2 + \nu_2 u_4)}{2}\cdot u_1 \\ &\leqslant \frac{1}{2}\Bigl[u_1^2\Bigl(-1 + \frac{\lambda - \nu_2}{2}\Bigr) + u_2^2\Bigl(-1 + \frac{\lambda - \nu_2}{2}\Bigr)\Bigr] \leqslant -\eta\bigl(u_1^2 + u_2^2\bigr)\end{aligned}$$

for all $\lambda \leqslant \frac{5}{4}$, $\nu_2 \in [-\frac{1}{2}, 0)$, $\eta = \frac{1}{16}$, which means we can choose $\xi = \frac{1}{4}$ in this case.

(iii) *The case* $u_3(t) < u_1(t)$, $u_4(t) \geqslant u_2(t)$: We have $u_1u_3 \geqslant 0$, $|u_3u_2| \leqslant u_1|u_2|$ (see (2.4)) and we obtain

$$\dot\varphi_K(t)\cdot|P_LP_KU(t)|^2 = -\frac{\dot u_1+\dot u_3}{2}\cdot u_2 + \dot u_2\cdot\frac{u_1+u_3}{2}$$
$$= -\frac{\lambda u_1+u_2+\nu_1u_3}{2}\cdot u_2 + \big(-u_1+\lambda u_2\big)\frac{u_1+u_3}{2}$$
$$\leqslant \frac12\Big[u_1^2\Big(-1+\frac{\lambda-\nu_1}{2}\Big)+u_2^2\Big(-1+\frac{\lambda-\nu_1}{2}\Big)\Big] \leqslant -\eta\big(u_1^2+u_2^2\big)$$

for all $\lambda \leqslant \frac54$, $\nu_1 \in [-\frac12, 0)$, $\eta = \frac{1}{16}$ and we can take $\xi = \frac14$ again.

(iv) *The case* $u_3(t) < u_1(t)$, $u_4(t) < u_2(t)$: According to (2.4) we have $(\lambda-\nu_1)u_2u_3 \leqslant (1+\xi-\nu_1)u_1u_2$, $(\lambda-\nu_2)u_1u_4 \geqslant 0$, $u_3u_4 < u_1u_2$, $u_1u_3 \geqslant 0$, $u_2u_4 \geqslant 0$, $|\nu_1-\nu_2| < \frac12$ and we get

$$\dot\varphi_K(t)\cdot|P_LP_KU(t)|^2$$
$$= -\frac{\lambda u_1+u_2+\nu_1u_3}{2}\cdot\frac{u_2+u_4}{2} + \frac{-u_1+\lambda u_2+\nu_2u_4}{2}\cdot\frac{u_1+u_3}{2}$$
$$= -\frac14[+u_1^2+u_2^2-(\lambda-\nu_1)u_2u_3+(\lambda-\nu_2)u_1u_4+(\nu_1-\nu_2)u_3u_4+u_1u_3+u_2u_4]$$
$$\leqslant -\frac14\Big(1-\frac{1+\xi-\nu_1}{2}-\frac{|\nu_1-\nu_2|}{2}\Big)(u_1^2+u_2^2) = -\eta(u_1^2+u_2^2)$$

for all $\lambda \leqslant 1+\xi$ where $\eta > 0$ if we choose $\xi \in (0, \frac12 - |\nu_1-\nu_2|)$.

Proof of (1.1*b*) *for* $\tau = +\infty$: First, notice that $P_KU(t) = U(t)$ because $U(t) \in K$ for all t. Hence, (2.8) and Remark 1.2 give

$$\dot\varphi_K(t) = \frac{\big(\frac{\mathrm d}{\mathrm dt}P_LU(t), P_L^*U(t)\big)}{|P_LP_KU(t)|^2} = \frac{\big(P_{U(t)}B_\lambda U(t), P_L^*U(t)\big)}{|P_LP_KU(t)|^2} \text{ for all } t \geqslant 0.$$

(i) *The case* $u_3(t) > u_1(t)$, $u_4(t) > u_2(t)$: $K_{U(t)} = \mathbb R^4$, $P_{U(t)}B_\lambda U(t) = B_\lambda U(t)$,

$$\dot\varphi_K(t) = \frac{-\big(\lambda u_1+u_2\big)u_2+\big(-u_1+\lambda u_2\big)u_1}{u_1^2+u_2^2} = -1.$$

(ii) *The case* $u_3(t) = u_1(t)$, $u_4(t) > u_2(t)$: $K_{U(t)} = \{V;\ v_3 \geqslant v_1\}$. If simultaneously $B_\lambda U(t) \in K_{U(t)}$, i.e. $\nu_1u_3(t) \geqslant \lambda u_1(t)+u_2(t)$ then $P_UB_\lambda U = B_\lambda U$ and we obtain the same formula as in (i). If $B_\lambda U(t) \notin K_{U(t)}$, i.e. $\nu_1u_1(t) < \lambda u_1(t)+u_2(t)$ then

$$P_UB_\lambda U = \Big[\frac{\lambda u_1+u_2+\nu_1u_3}{2}, -u_1+\lambda u_2, \frac{\lambda u_1+u_2+\nu_1u_3}{2}, \nu_2u_4\Big],$$
$$\dot\varphi_K(t)\cdot|P_LU(t)|^2 = -\frac{\lambda u_1+u_2+\nu_1u_3}{2}\cdot u_2 + (-u_1+\lambda u_2)u_1$$
$$\leqslant \Big(-1+\frac{\lambda+|\nu_1|}{4}\Big)u_1^2 + \Big(-\frac12+\frac{\lambda+|\nu_1|}{4}\Big)u_2^2 \leqslant -\eta(u_1^2+u_2^2)$$

for all $\lambda \leqslant 1+\xi$ where $\eta > 0$ if $\xi \in (0, \frac{1}{2})$ (and $\nu_1 \in [-\frac{1}{2}, 0)$).

(iii) *The case* $u_3(t) > u_1(t)$, $u_4(t) = u_2(t)$: $K_{U(t)} = \{V;\ v_4 \geqslant v_2\}$. If simultaneously $B_\lambda U(t) \in K_{U(t)}$, i.e. $\nu_2 u_4(t) \geqslant -u_1(t) + \lambda u_2(t)$ then $P_U B_\lambda U = B_\lambda U$ and we obtain the same formula as in (i). If $B_\lambda U \notin K_U$ then

$$P_U B_\lambda U = \Big[\lambda u_1 + u_2, \frac{1}{2}(-u_1 + \lambda u_2 + \nu_2 u_4), \nu_1 u_3, \frac{1}{2}(-u_1 + \lambda u_2 + \nu_2 u_4)\Big],$$

$$\dot\varphi_K(t) \cdot |P_L U(t)|^2 = -(\lambda u_1 + u_2) u_2 + \frac{-u_1 + \lambda u_2 + \nu_2 u_4}{2} \cdot u_1 \leqslant$$

$$\leqslant \Big(-\frac{1}{2} + \frac{\lambda + |\nu_2|}{4}\Big) u_1^2 + \Big(-1 + \frac{\lambda + |\nu_2|}{4}\Big) u_2^2 \leqslant -\eta(u_1^2 + u_2^2)$$

for all $\lambda \leqslant 1+\xi$ where $\eta > 0$ if $\xi \in (0, \frac{1}{2})$.

(iv) *The case* $u_3(t) = u_1(t)$, $u_4(t) = u_2(t)$: $K_{U(t)} = \{V;\ v_3 \geqslant v_1, v_4 \geqslant v_2\}$. If simultaneously $\nu_1 u_3(t) \geqslant \lambda u_1(t) + u_2(t)$, $\nu_2 u_4(t) \geqslant -u_1(t) + \lambda u_2(t)$
or $\nu_1 u_3(t) < \lambda u_1(t) + u_2(t)$, $\nu_2 u_4(t) \geqslant -u_1(t) + \lambda u_2(t)$
or $\nu_1 u_3(t) \geqslant \lambda u_1(t) + u_2(t)$, $\nu_2 u_4(t) < -u_1(t) + \lambda u_2(t)$
then we obtain the same formulas as in the case (i) or (ii) or (iii), respectively. If $\nu_1 u_3(t) < \lambda u_1(t) + u_2(t)$, $\nu_2 u_4(t) < -u_1(t) + \lambda u_2(t)$ then $P_U B_\lambda U = [\frac{\lambda u_1 + u_2 + \nu_1 u_3}{2}, \frac{-u_1 + \lambda u_2 + \nu_2 u_4}{2}, \frac{\lambda u_1 + u_2 + \nu_1 u_3}{2}, \frac{-u_1 + \lambda u_2 + \nu_2 u_4}{2}]$, $\nu_2 u_1 u_4 \leqslant 0$, $-\nu_1 u_2 u_3 \leqslant \frac{1}{4}(u_1^2 + u_2^2)$ and we get

$$\begin{aligned}\dot\varphi_K(t) \cdot |P_L U(t)|^2 &= -\frac{\lambda u_1 + u_2 + \nu_1 u_3}{2} \cdot u_2 + \frac{-u_1 + \lambda u_2 + \nu_2 u_4}{2} \cdot u_1 \\ &\leqslant -\frac{1}{4}(u_1^2 + u_2^2).\end{aligned}$$

Proof of (1.1*c*): It follows from (1.1*b*) that there are $t_1 < t_2$ such that $\varphi_K(t_2) - \varphi_K(t_1) = -\frac{\pi}{2}$, $u_1(t) < 0$, $u_2(t) < 0$ for $t \in (t_1, t_2)$. We get $U(t) \in K$ on (t_1, t_2) and $U(t)$ coincides with the solution of (LE) (i.e. (2.3) with $\tau = 0$) in (t_1, t_2) (see Remarks 1.1, 1.2). It follows by using the assumption (μ) that $t_2 - t_1 > \frac{\pi}{2\omega_M}$ and (1.1*c*) is fulfilled with $\Gamma = \frac{\pi}{2\omega_M}$.

Proof of (1.2): We shall show that

$$\text{(2.9)} \qquad \text{if } \tau \in (0, +\infty],\ \lambda \geqslant 1 \text{ then } \frac{\mathrm{d}}{\mathrm{d}t}\big(|P_L P_K U(t)|^2\big) > 0 \text{ for a.a. } t \geqslant 0,$$

$$\text{(2.10)} \qquad \text{if } \tau \in (0, +\infty],\ \lambda \leqslant 0 \text{ then } \frac{\mathrm{d}}{\mathrm{d}t}\big(|U(t)|^2\big) < 0 \text{ for a.a. } t \geqslant 0.$$

The conditions in (2.9) and (2.10) contradict the periodicity and therefore it follows that a periodic solution of (LPE) can exist only if $\lambda \in (0, 1)$.

Proof of (2.9) *for* $\tau \in (0, +\infty)$: Similarly as in the proof of (1.1*b*) we will distinguish several cases. Let us recall that always $u_3(t) \geqslant 0$, $u_4(t) \geqslant 0$ by (2.4) and $u_1^2 + u_2^2 > 0$ by (1.1*a*).

(i) *The case* $u_3(t) \geqslant u_1(t)$, $u_4(t) \geqslant u_2(t)$: $P_K U(t) = U(t)$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(|P_L P_K U(t)|^2\big) = \big(P_L \dot{U}(t), P_L U(t)\big) = \lambda\big(u_1^2(t) + u_2^2(t)\big) > 0 \text{ if } \lambda > 0.$$

(ii) *The case* $u_3(t) < u_1(t)$, $u_4(t) \geqslant u_2(t)$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(|P_L P_K U(t)|^2\big) = \Big(\frac{\mathrm{d}}{\mathrm{d}t} P_L P_K U(t), P_L P_K U(t)\Big) = \frac{\dot{u}_1 + \dot{u}_3}{2} \cdot \frac{u_1 + u_3}{2} + \dot{u}_2 u_2$$
$$= \frac{\lambda u_1(t) + u_2(t) + \nu_1 u_3(t)}{2} \cdot \frac{u_1 + u_3}{2} + (-u_1 + \lambda u_2) \cdot u_2$$
$$= \frac{\lambda}{4} u_1^2 + \lambda u_2^2 + \Big(-\frac{3}{4} u_1 u_2 + \frac{1}{4} u_2 u_3\Big) + \Big(\frac{\lambda + \nu_1}{4} u_1 u_3 + \frac{\nu_1}{4} u_3^2\Big).$$

Using (2.4) and considering separately $u_2 \geqslant 0$ and $u_2 < 0$ we see that always $-\frac{3}{4}u_1 u_2 + \frac{1}{4} u_2 u_3 \geqslant -\frac{3}{4} u_1 |u_2|$ and for $\lambda \geqslant 1$, $\nu_1 \in [-1, 0)$ we have $\frac{\lambda+\nu_1}{4} u_1 u_3 + \frac{\nu_1}{4} u_3^2 \geqslant \frac{\lambda + 2\nu_1}{4} u_1 u_3 \geqslant 0$. Hence, using the inequality $ab \geqslant -\frac{\delta}{2} a^2 - \frac{1}{2\delta} b^2$ with $\delta = \frac{1}{2}$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(|P_L P_K U(t)|^2\big) \geqslant \frac{\lambda}{4} u_1^2 + \lambda u_2^2 - \frac{3}{4} u_1 |u_2| \geqslant \Big(\frac{\lambda}{4} - \frac{3}{16}\Big) u_1^2 + \Big(\lambda - \frac{3}{4}\Big) u_2^2 > 0.$$

(iii) *The case* $u_3(t) \geqslant u_1(t)$, $u_4(t) < u_2(t)$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(|P_L P_K U(t)|^2\big) = \Big(\frac{\mathrm{d}}{\mathrm{d}t} P_L P_K U(t), P_L P_K U(t)\Big) = \dot{u}_1 u_1 + \frac{\dot{u}_2 + \dot{u}_4}{2} \cdot \frac{u_2 + u_4}{2}$$
$$= (\lambda u_1 + u_2) u_1 + \frac{-u_1 + \lambda u_2 + \nu_2 u_4}{2} \cdot \frac{u_2 + u_4}{2}$$
$$= \lambda u_1^2 + \frac{\lambda}{4} u_2^2 + \Big(\frac{3}{4} u_1 u_2 - \frac{1}{4} u_1 u_4\Big) + \Big(\frac{\lambda + \nu_2}{4} u_2 u_4 + \nu_2 u_4^2\Big).$$

Similarly as in (ii), the last two brackets together are not less than $-\frac{3}{4}|u_1| u_2$ and we obtain (choosing $\delta = 2$) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(|P_L P_K U(t)|^2\big) \geqslant \lambda u_1^2 + \frac{\lambda}{4} u_2^2 - \frac{3}{4}|u_1| u_2 \geqslant \Big(\lambda - \frac{3}{4}\Big) u_1^2 + \Big(\frac{\lambda}{4} - \frac{3}{16}\Big) u_2^2 > 0.$$

(iv) *The case* $u_3(t) < u_1(t)$, $u_4(t) < u_2(t)$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(|P_L P_K U(t)|^2\big) = \frac{\dot{u}_1 + \dot{u}_3}{2} \cdot \frac{u_1 + u_3}{2} + \frac{\dot{u}_2 + \dot{u}_4}{2} \cdot \frac{u_2 + u_4}{2}$$
$$= \frac{\lambda u_1 + u_2 + \nu_1 u_3}{2} \cdot \frac{u_1 + u_3}{2} + \frac{-u_1 + \lambda u_2 + \nu_2 u_4}{2} \cdot \frac{u_2 + u_4}{2}$$
$$= \frac{1}{4}\big[\lambda u_1^2 + \lambda u_2^2 - u_1 u_4 + (u_2 u_3) + (\lambda u_1 u_3 + \nu_1 u_1 u_3 + \nu_1 u_3^2)$$
$$+ (\lambda u_2 u_4 + \nu_2 u_2 u_4 + \nu_2 u_4^2)\big].$$

The expressions closed in brackets are nonnegative for $\lambda \geqslant 1$ and we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|P_L P_K U(t)|^2\right) \geqslant \frac{1}{4}(\lambda u_1^2 + \lambda u_2^2 - u_1 u_2) \geqslant \frac{1}{4}\Big(\lambda - \frac{1}{2}\Big)(u_1^2 + u_2^2) > 0.$$

Proof of (2.9) *for* $\tau = +\infty$: We have $U(t) \in K$, $P_K U(t) = U(t)$.

(i) *The case* $u_3(t) > u_1(t)$, $u_4(t) > u_2(t)$: $K_U = \mathbb{R}^4$, $P_U B_\lambda U = B_\lambda U$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|P_L P_K U(t)|^2\right) = \big(P_L \dot{U}(t), P_L U(t)\big) = \lambda\big(u_1^2(t) + u_2^2(t)\big) > 0 \quad \text{even if} \quad \lambda > 0.$$

(ii) *The case* $u_3(t) = u_1(t)$, $u_4(t) > u_2(t)$: $K_{U(t)} = \{V;\ v_3 \geqslant v_1\}$. If simultaneously $\nu_1 u_3(t) \geqslant \lambda u_1(t) + u_2(t)$ then $P_U B_\lambda U = B_\lambda U$ and we obtain the same formula as in the case (i). If $\nu_1 u_1(t) < \lambda u_1(t) + u_2(t)$ then using the formula

$$P_U B_\lambda U = \Big[\frac{\lambda u_1 + u_2 + \nu_1 u_3}{2}, -u_1 + \lambda u_2, \frac{\lambda u_1 + u_2 + \nu_1 u_3}{2}, \nu_2 u_4\Big]$$

and the inequality $ab \geqslant -\frac{\delta}{2}a^2 - \frac{1}{2\delta}b^2$ with $\delta = \frac{1}{2}$) we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|P_L P_K U(t)|^2\right) = \big(P_L \dot{U}(t), P_L U(t)\big) = \big(P_{U(t)} B_\lambda U(t), P_L U(t)\big)\\
&= \frac{\lambda u_1 + u_2 + \nu_1 u_3}{2} \cdot u_1 + (-u_1 + \lambda u_2) u_2\\
&= \frac{\lambda + \nu_1}{2} u_1^2 + \lambda u_2^2 - \frac{1}{2} u_1 u_2 \geqslant \Big(\frac{\lambda}{4} - \frac{1}{8}\Big) u_1^2 + \Big(\lambda - \frac{1}{2}\Big) u_2^2 > 0 \text{ for } \lambda \geqslant 1.
\end{aligned}$$

(iii) *The case* $u_3(t) > u_1(t)$, $u_4(t) = u_2(t)$: $K_{U(t)} = \{V;\ v_4 \geqslant v_2\}$. If simultaneously $\nu_2 u_4(t) \geqslant -u_1(t) + \lambda u_2(t)$ then we obtain the same formula as in the case (i). If $\nu_2 u_4(t) < -u_1(t) + \lambda u_2(t)$ then similarly as in (ii) (but with $\delta = 2$) we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|P_L P_K U(t)|^2\right) = \big(P_L \dot{U}(t), P_L U(t)\big) = \big(P_{U(t)} B_\lambda U(t), P_L U(t)\big)\\
&= (\lambda u_1 + u_2) u_1 + \frac{-u_1 + \lambda u_2 + \nu_2 u_4}{2} \cdot u_2\\
&= \lambda u_1^2 + \frac{\lambda + \nu_2}{2} u_2^2 + \frac{1}{2} u_1 u_2 \geqslant \Big(\lambda - \frac{1}{2}\Big) u_1^2 + \Big(\frac{\lambda}{4} - \frac{1}{8}\Big) u_2^2 > 0 \text{ for } \lambda \geqslant 1.
\end{aligned}$$

(iv) *The case* $u_3(t) = u_1(t)$, $u_4(t) = u_2(t)$: $K_{U(t)} = \{V;\ v_3 \geqslant v_1,\ v_4 \geqslant v_2\}$. If simultaneously $\nu_1 u_3(t) \geqslant \lambda u_1(t) + u_2(t)$, $\nu_2 u_4(t) \geqslant -u_1(t) + \lambda u_2(t)$
or $\nu_1 u_3(t) < \lambda u_1(t) + u_2(t)$, $\nu_2 u_4(t) \geqslant -u_1(t) + \lambda u_2(t)$
or $\nu_1 u_3(t) \geqslant \lambda u_1(t) + u_2(t)$, $\nu_2 u_4(t) < -u_1(t) + \lambda u_2(t)$

then we obtain the same formulas as in the case (i) or (ii) or (iii), respectively—cf. (iv) in the proof of (1.1b), $\tau = +\infty$. If $\nu_1 u_3(t) < \lambda u_1(t) + u_2(t)$, $\nu_2 u_4(t) < -u_1(t) + \lambda u_2(t)$ then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|P_L P_K U(t)|^2) = \big(P_L \dot{U}(t), P_L U(t)\big) = \big(P_{U(t)} B_\lambda U(t), P_L U(t)\big)$$
$$= \frac{\lambda u_1 + u_2 + \nu_1 u_3}{2}\cdot u_1 + \frac{-u_1 + \lambda u_2 + \nu_2 u_4}{2}\cdot u_2 = \frac{\lambda+\nu_1}{2}\cdot u_1^2 + \frac{\lambda+\nu_2}{2}\cdot u_2^2 > 0.$$

Proof of (2.10) *for* $\tau \in (0,+\infty)$: It follows from (2.4), (2.3) by using (2.5) that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|U(t)|^2)\\
&= \Big(\lambda u_1 + u_2 - \tau\frac{(u_3-u_1)^-}{2}\Big)\cdot u_1 + \Big(-u_1 + \lambda u_2 - \tau\frac{(u_4-u_2)^-}{2}\Big)\cdot u_2\\
&\quad + \Big(\nu_1 u_3 + \tau\frac{(u_3-u_1)^-}{2}\Big)\cdot u_3 + \Big(\nu_2 u_4 + \tau\frac{(u_4-u_2)^-}{2}\Big)\cdot u_4\\
&= \lambda(u_1^2+u_2^2) + \nu_1 u_3^2 + \nu_2 u_4^2 + \frac{\tau}{2}\big[(u_3-u_1)^-(u_3-u_1)\\
&\quad + (u_4-u_2)^-(u_4-u_2)\big] < 0 \text{ if } \lambda \leqslant 0.
\end{aligned}$$

Proof of (2.10) *for* $\tau = +\infty$: Setting $Z = 0$, $Z = 2U(t)$ in (LI) we get $\big(\dot{U}(t), U(t)\big) = \big(B_\lambda U(t), U(t)\big)$ and therefore by using (2.5) we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|U(t)|^2) = \big(\dot{U}(t), U(t)\big) = \lambda\big(u_1^2(t) + u_2^2(t)\big) + \nu_1 u_3^2(t) + \nu_2 u_4^2(t) < 0 \text{ if } \lambda \leqslant 0.$$

Hence, the assumptions (1.1), (1.2) are fulfilled with $\Lambda_m = 0$, $\Lambda_M = 1$ and ξ, η small enough. Theorem 1.1 ensures the existence of a bifurcation point $\lambda_I \in (0,1)$ announced.

Example 2.2. Consider the matrix B_λ as in Example 2.1. Set

$$K = \{U \in \mathbb{R}^4\,;\; u_3 \geqslant 0, u_4 \geqslant 0, u_3 + u_4 \geqslant u_1\}. \tag{2.11}$$

The conditions (1.1), (1.2) can be verified by similar computations as in Example 2.1 and we can get the existence of a bifurcation point $\lambda_I \in (0,1)$ from Theorem 1.1 again. However, this can be derived directly from the properties of the problem (2.1) mentioned in Remark 2.1. If $G\colon \mathbb{R}^5 \to \mathbb{R}^4$ is a mapping satisfying (G), (L) then the corresponding mapping $\tilde{G}\colon \mathbb{R}^4 \to \mathbb{R}^3$ defined by $\tilde{G}(\lambda, v_1, v_2, v_3) = G(\lambda, v_1, v_2, v_3, 0)$ or $\tilde{G}(\lambda, v_1, v_2, v_3) = G(\lambda, v_1, v_2, 0, v_3)$ satisfies (G), (L) as well. If $V(t) = [v_1(t), v_2(t), v_3(t)]$ is a periodic solution of (2.1) then $U(t) = [v_1(t), v_2(t), v_3(t), 0]$ and $U(t) = [v_1(t), v_2(t), 0, v_3(t)]$, respectively, is a periodic solution of (I) with the cone (2.11) and our matrix B_λ. Particularly, any bifurcation point of (2.1) is simultaneously a bifurcation point of our inequality in $\mathbb{R}^4$. Its existence was proved in [7] (see Remark 2.1). Moreover, bifurcating solutions with the third trivial component and bifurcating solutions with the fourth trivial component are obtained. However, the existence of bifurcating solutions $U(t)$ with both nontrivial components u_3 and u_4 does not follow from our theory.

Example 2.3. Consider the matrix B_λ as in Example 2.1. Set

$$K = \{U \in \mathbb{R}^4\,;\ u_3 \geqslant 0, u_4 \geqslant 0, u_3 \geqslant c_1 u_1, u_4 \geqslant c_2 u_2\}$$

where $c_1 \in (0,1]$, $c_2 \in (0,1]$ are given. Then the same assertion about the bifurcation of periodic solutions to (I) as in Example 2.1 can be proved by analogous considerations. However, if at least one $c_j > 1$ is large enough then this approach cannot be used because it is not possible to find Λ_m, Λ_M and ξ, η such that the assumptions (1.1), (1.2) are fulfilled. The circulation of the projection into $\mathbb{L}$ of solutions of (LI) and of (PS) with large τ can be damped down be the boundary of K and by the penalty term, respectively, even for parameters λ for which the periodicity is not excluded. Probably no bifurcation point of (I) exists.

Remark 2.2. We could also replace the cone in Example 1.1 by

$$K = \{U \in \mathbb{R}^4\,;\ u_3 \geqslant 0,\ u_4 \geqslant 0,\ u_3 \geqslant |u_1|,\ u_4 \geqslant |u_2|\}.$$

However, then it is not clear if any periodic solution of (LI) and (LPE) must intersect the interior of K and it would be necessary to find another argument for the proof of (1.1c) than that from Example 1.1. The other considerations from Example 1.1 can be simply modified. The method explained in Example 2.1 can be used for analogous examples in higher dimensional spaces.

3. Proof of Bifurcation Theorem

Notation 3.1.

$U^\tau_{\varrho,\lambda}(\cdot,V))$, $\varepsilon^\tau_{\varrho,\lambda}(\cdot,V))$—the solution of the penalty system (PS) with the initial condition $U(0)=V$, $\varepsilon(0)=\tau$,

$\mathcal{B} = \bigl\{\bigl[\frac{2k\pi}{\omega_0}, 0, 0, \lambda_0\bigr] \in (0,+\infty)\times\mathbb{R}^\kappa\times\mathbb{R}\times I\,;\ k \text{ positive integer}\bigr\}$,

$\mathcal{L}_\varrho = \{[T,V,\tau,\lambda] \in [0,+\infty)\times\mathbb{R}^\kappa\times\mathbb{R}\times I\,;\ U^\tau_{\varrho,\lambda}(T,V)=V,\ \varepsilon^\tau_{\varrho,\lambda}(T,V)=\tau\}$,

$\mathcal{C}_\varrho = \bigl(\mathcal{L}_\varrho \setminus (0,+\infty)\times\{0\}\times\{0\}\times I\bigr)\cup\mathcal{B}$,

$\mathcal{C}^0_\varrho$—the component of $\mathcal{C}_\varrho$ containing $[\frac{2\pi}{\omega_0}, 0, 0, \lambda_0]$,

$t^{k,\tau}_{0,\lambda}(V) = \inf\{t_0\,;\ r^\tau_{0,\lambda}(t,V) > 0 \text{ for } t\in[0,t_0], \varphi^\tau_{0,\lambda}(t_0,V) = -2k\pi\}$ if $V \notin \mathbb{S}_\lambda$—the time of k-circuits of $P_{L_\lambda}P_K U^\tau_{0,\lambda}(\cdot,V)$ around the origin.

Remark 3.1. The properties of the branches $\mathcal{C}_\varrho$, $\mathcal{C}_\varrho^0$ were studied in [7]. At this moment we recall only that $\mathcal{C}_\varrho$ is closed and contains the closure of $\mathcal{L}_\varrho \setminus (0,+\infty) \times \{0\} \times \{0\} \times I$, for $[T,V,\tau,\lambda] \in \mathcal{C}_\varrho$ we have either $|V|>0$, $\tau>0$ or $|V|=\tau=0$, and that the only points in $\mathcal{C}_\varrho$ with $T>0$, $\tau=0$ are $[\frac{2k\pi}{\omega_0},0,0,\lambda_0]$, k positive integer. See [7], Observation 2.2.

Lemma 3.1. *If* $V_n \to 0$, $\frac{V_n}{|V_n|} \to W$, $\tau_n \to \tau \in [0,\infty]$, $\lambda_n \to \lambda \in (\Lambda_m - \xi, \Lambda_M + \xi)$, $U_{0,\lambda}^\tau(T,W) = W$, $T>0$ *then* $t_{0,\lambda_n}^{k,\tau_n}(V_n) \to t_{0,\lambda}^{k,\tau}(W)$ *for any positive integer* k.

Proof. It follows from (1.1b) that there is $\delta>0$ such that

$$\varphi_{0,\lambda}^{k,\tau}(t,W) < -2k\pi \quad \text{for all } t \in (t_{0,\lambda}^{k,\tau}(W), t_{0,\lambda}^{k,\tau}(W)+\delta).$$

Let $t_0 \in (t_{0,\lambda}^{k,\tau}(W), t_{0,\lambda}^{k,\tau}(W)+\delta)$ be fixed. Then $\varphi_{0,\lambda_n}^{k,\tau_n}(t_0,V_n) < -2k\pi$ for n large as a consequence of [7], Lemmas 2.4, 2.5. Hence, there is $t_n' \in (0,t_0)$ such that $\varphi_{0,\lambda_n}^{k,\tau_n}(t_n',V_n) = -2k\pi$ due to the continuity of $\varphi_{0,\lambda_n}^{k,\tau_n}(\cdot,V_n)$. That means $t_{0,\lambda_n}^{k,\tau_n}(V_n) < t_0$. But $t_0 > t_{0,\lambda}^{k,\tau}(W)$ was arbitrarily close to $t_{0,\lambda}^{k,\tau}(W)$ and therefore

$$\limsup\ t_{0,\lambda_n}^{k,\tau_n}(V_n) \leqslant t_{0,\lambda}^{k,\tau}(W).$$

Let t_{l_n} be an arbitrary subsequence of $t_{0,\lambda_n}^{k,\tau_n}(V_n)$, $t_{l_n} \to t'$. As a consequence of [7], Lemmas 2.4, 2.5 we obtain $\varphi_{0,\lambda_{l_n}}^{k,\tau_{l_n}}(t_{l_n},V_{l_n}) \to \varphi_{0,\lambda}^{k,\tau}(t',W)$, i.e. $\varphi_{0,\lambda}^{k,\tau}(t',W) = -2k\pi$. It follows that $t'>0$, $t_{0,\lambda}^{k,\tau}(W) \leqslant t'$. This holds for an arbitrary converging subsequence and therefore $\liminf\ t_{0,\lambda_n}^{k,\tau_n}(V_n) \geqslant t_{0,\lambda}^{k,\tau}(W)$, which together with the above estimate gives $t_{0,\lambda_n}^{k,\tau_n}(V_n) \to t_{0,\lambda}^{k,\tau}(W)$. □

In the paper [7], the following general assumptions were considered: there exist $\varrho_0>0$, $\gamma>0$, $t_M>0$, $\Lambda_1,\Lambda_2 \in I$ such that

(GA) if $[T,V,\tau,\lambda] \in \mathcal{C}_\varrho^0$, $\varrho \in (0,\varrho_0)$ then $\gamma < T < t_M$, $\lambda \in (\Lambda_1,\Lambda_2)$,

(NS) $\begin{cases} \text{for any } U \in \partial K,\ |U|>0,\ \lambda \in [\Lambda_1,\Lambda_2] \\ \text{there is } Z \in K \text{ such that } (B_\lambda U, Z-U) > 0. \end{cases}$

The condition (NS) means that the linearized inequality (LI) has no stationary solution in ∂K for $\lambda \in [\Lambda_1,\Lambda_2]$; for $\Lambda_1 = \Lambda_m - \xi$, $\Lambda_2 = \Lambda_M + \xi$ this is a consequence of our assumption (1.1b) in the setting of Section 1. The existence of a bifurcation point of (I) was proved under the assumptions (μ), (G), (L), (GA), (NS) (see [7], Theorem 1.2). We will see that our Theorem 1.1 follows from [7] by using the following assertion.

Lemma 3.2. *If* (1.1), (1.2) *hold then the assumption* (GA) *is fulfilled with* $\gamma = \frac{1}{3}\Gamma$, $t_M = T_M + \gamma$, $\Lambda_1 = \Lambda_m - \xi$, $\Lambda_2 = \Lambda_M + \xi$.

P r o o f. (Cf. the considerations in [7], Model Example.) Set

$$\mathcal{C}_\varrho^M = \text{the component of } \{[T,V,\tau,\lambda] \in \mathcal{C}_\varrho^0;\ \gamma \leqslant T \leqslant t_M\} \text{ containing } \Big[\frac{2\pi}{\omega_0}, 0, 0, \lambda_0\Big]$$

(see Observation 1.1). We will see later that in fact $\mathcal{C}_\varrho^M = \mathcal{C}_\varrho^0$. First, let us show that there is $\varrho_0 > 0$ such that

$$\text{(3.1)} \qquad \text{if } \varrho \in (0,\varrho_0),\ [T,V,\tau,\lambda] \in \mathcal{C}_\varrho^M \text{ then } \lambda \in (\Lambda_1,\Lambda_2).$$

Suppose by contradiction that this is not true. Since $\lambda_0 \in [\Lambda_m,\Lambda_M]$ and $[\frac{2\pi}{\omega_0},0,0,\lambda_0] \in \mathcal{C}_\varrho^M$ for any $\varrho > 0$, we obtain by using the connectedness of $\mathcal{C}_\varrho^M$ that there are $\varrho_n \to 0$, $[T_n,V_n,\tau_n,\lambda] \in \mathcal{C}_{\varrho_n}^M$, $T_n \in [\gamma, t_M]$, $T_n \to T$, $\tau_n \to \tau \in [0,+\infty]$ and either $\lambda = \Lambda_1$ or $\lambda = \Lambda_2$. It follows from [7], Observation 2.3, Lemma 2.1 and the T_n-periodicity of $U^{\tau_n}_{\varrho_n,\lambda_n}(t,V_n)$, $T_n \leqslant t_M$ that $|V_n| \to 0$. We can suppose $\frac{V_n}{|V_n|} \to W$. We have $U^{\tau_n}_{\varrho_n,\lambda_n}(T_n,V_n) = V_n$ and the limiting process in this equation divided by $|V_n|$ (see [7], Lemma 2.4 or 2.5) implies that $U^{\tau}_{0,\lambda}(T,W) = W$, $T \geqslant \gamma$. This contradicts (1.2) or the fact that for $\tau = 0$, (LPE) coincides with (LE) and therefore it has a periodic solution only if $\lambda = \lambda_0$ under the assumption (μ).

Further, we will show that $\varrho_0 > 0$ can be chosen such that

$$\text{(3.2)} \qquad \begin{cases} \text{if } \varrho \in (0,\varrho_0),\ [T,V,\tau,\lambda] \in \mathcal{C}_\varrho^M,\ [T,V,\tau,\lambda] \neq [\frac{2\pi}{\omega_0},0,0,\lambda_0], \\ \text{then } |T - t^{\tau}_{0,\lambda}(V)| < \gamma. \end{cases}$$

To prove (3.2), let us show first that for ϱ_0 small enough

$$\text{(3.3)} \quad \begin{cases} \text{if } \varrho \in (0,\varrho_0),\ [T,V,\tau,\lambda] \in \mathcal{C}_\varrho^M,\ [T,V,\tau,\lambda] \neq [\frac{2m\pi}{\omega_0},0,0,\lambda_0],\ m = 1,2,\ldots, \\ \text{then there is a positive integer } k \text{ such that } |T - t^{k,\tau}_{0,\lambda}(V)| < \gamma. \end{cases}$$

Indeed, otherwise there would be $[T_n,V_n,\tau_n,\lambda_n] \in \mathcal{C}_{\varrho_n}^M$ such that $\varrho_n \to 0$, $|V_n| > 0$, $|T_n - t^{k,\tau_n}_{0,\lambda_n}(V_n)| \geqslant \gamma$ for all k positive integer. We have $|V_n| \to 0$ by [7], Observation 2.3, Lemma 2.1 and the T_n-periodicity and we can suppose $[T_n,V_n,\tau_n,\lambda_n] \to [T,0,\tau,\lambda]$, $\frac{V_n}{|V_n|} \to W$. We have $U^{\tau_n}_{\varrho_n,\lambda_n}(T_n,V_n) = V_n$. The limiting process ([7], Lemma 2.4 or 2.5) gives $U^{\tau}_{0,\lambda}(T,W) = W$, $T \geqslant \gamma$, and it follows by using (1.1), (1.2) and (3.1) that $T = t^{k,\tau}_{0,\lambda}(W)$ with some positive integer k. Lemma 3.1 ensures that $t^{k,\tau_n}_{0,\lambda_n}(V_n) \to t^{k,\tau}_{0,\lambda}(W)$ and therefore $|T_n - t^{k,\tau_n}_{0,\lambda_n}(V_n)| \leqslant |T_n - T| + |T - t^{k,\tau_n}_{0,\lambda_n}(V_n)| \to 0$, which is a contradiction and (3.3) is proved.

Further, denote

$$\mathcal{C}^1_\varrho = \{[T,V,\tau,\lambda] \in \mathcal{C}^M_\varrho;\ \tau > 0,\ |T - t^\tau_{0,\lambda}(V)| < \gamma\} \cup \Big\{\Big[\frac{2\pi}{\omega_0}, 0, 0, \lambda_0\Big]\Big\}.$$

Let us show that

$$\operatorname{dist}\Big(\mathcal{C}^1_\varrho, \Big[\frac{2k\pi}{\omega_0}, 0, 0, \lambda_0\Big]\Big) > 0 \text{ for all } k > 1. \tag{3.4}$$

If this were not true then we would have $[T_n, V_n, \tau_n, \lambda_n] \in \mathcal{C}^M_\varrho$, $[T_n, V_n, \tau_n, \lambda_n] \to [\frac{2k\pi}{\omega_0}, 0, 0, \lambda_0]$ with some $k > 1$, $\frac{V_n}{|V_n|} \to W$, $|T_n - t^{\tau_n}_{0,\lambda_n}(V_n)| < \gamma$. The limiting process ([7], Lemma 2.5) gives $U^0_{0,\lambda_0}(T, W) = W$. Hence, Lemma 3.1 implies $t^{\tau_n}_{0,\lambda_n}(V_n) \to t^0_{0,\lambda_0}(W) = \frac{2\pi}{\omega_0}$, $\frac{2k\pi}{\omega_0} - \frac{2\pi}{\omega_0} > \Gamma$ by Observation 1.1, which is a contradiction.

Let us realize that

$$\Big[\frac{2\pi}{\omega_0}, 0, 0, \lambda_0\Big] \in \operatorname{int} \mathcal{C}^1_\varrho \quad \text{where } int \text{ means the interior with respect to } \mathcal{C}^M_\varrho. \tag{3.5}$$

Indeed, otherwise $[T_n, V_n, \tau_n, \lambda_n] \in \mathcal{C}^M_\varrho$ would exist such that $[T_n, V_n, \tau_n, \lambda_n] \to [\frac{2\pi}{\omega_0}, 0, 0, \lambda_0]$, $|T_n - t^{\tau_n}_{0,\lambda_n}(V_n)| \geqslant \gamma$, $\frac{V_n}{|V_n|} \to W$. We would have $U^0_{0,\lambda}(\frac{2\pi}{\omega_0}, W) = W$ by [7], Lemma 2.5 and therefore $t^{\tau_n}_{0,\lambda_n}(V_n) \to t^0_{0,\lambda_0}(W) = \frac{2\pi}{\omega_0}$ by Lemma 3.1, a contradiction.

Let us show that $\mathcal{C}^1_\varrho$ is open in $\mathcal{C}^M_\varrho$ for any $\varrho \in (0, \varrho_0)$ if ϱ_0 is small enough. If $\mathcal{C}^1_\varrho$ is not open then according to (3.5), there exists $[T, V, \tau, \lambda] \in \mathcal{C}^1_\varrho$, $\tau > 0$, $|V| > 0$ (see also Remark 3.1) such that in any neighbourhood there is $[T', V', \tau', \lambda'] \in \mathcal{C}^M_\varrho \setminus \mathcal{C}^1_\varrho$. It follows that if $\mathcal{C}^1_{\varrho_n}$ are not open for some $\varrho_n \to 0$ then there are $[T_n, V_n, \tau_n, \lambda_n] \in \mathcal{C}^1_{\varrho_n}$, $\tau_n > 0$, $|V_n| > 0$, $[T'_n, V'_n, \tau'_n, \lambda'_n] \in \mathcal{C}^M_{\varrho_n} \setminus \mathcal{C}^1_{\varrho_n}$, $|V'_n| > 0$ such that $|T_n - T'_n| < |V_n|$, $|V_n - V'_n| < |V_n|$, $|\tau_n - \tau'_n| < |V_n|$, $|\lambda_n - \lambda'_n| < |V_n|$. We have $|V_n| \to 0$ by [7], Observation 2.3, Lemma 2.1, we can suppose $T_n \to T$, $\frac{V_n}{|V_n|} \to W$ and obtain also $T'_n \to T$, $\frac{V'_n}{|V'_n|} \to W$. We have $|T_n - t^{\tau_n}_{0,\lambda_n}(V_n)| < \gamma$, $|T'_n - t^{k_n,\tau'_n}_{0,\lambda'_n}(V'_n)| < \gamma$ with some $k_n > 1$ by the definition of $\mathcal{C}^1_{\varrho_n}$ and (3.3). If $k_n \to +\infty$ then there are k'_n such that $t^{\tau'_n}_{0,\lambda'_n}(Z_n) \to 0$ for $Z_n = U^{\tau'_n}_{0,\lambda'_n}(t^{k'_n,\tau'_n}_{0,\lambda'_n}(V'_n), V'_n)$. We have $|Z_n| \to 0$ ([7], Observation 2.3, Lemma 2.1) and we can suppose $\frac{Z_n}{|Z_n|} \to Y$. We get $U^\tau_{0,\lambda}(t^\tau_{0,\lambda}(Y), Y) = Y$ by [7], Lemmas 2.4, 2.5 and therefore $t^{\tau'_n}_{0,\lambda'_n}(Z_n) \to t^\tau_{0,\lambda}(Y) \geqslant \Gamma$ by Lemma 3.1 and (1.1c), a contradiction. Hence, k_n must be bounded and we can suppose that our sequence is chosen such that $k_n = k$ with some k. Then we obtain by using (1.1c)

$$\begin{cases} T = \lim T'_n \geqslant \lim t^{k,\tau'_n}_{0,\lambda'_n}(V'_n) - \gamma = t^{k,\tau}_{0,\lambda}(W) - \gamma \geqslant t^\tau_{0,\lambda}(W) + (k-1)\Gamma - \gamma, \\ T = \lim T_n \leqslant \lim t^{\tau_n}_{0,\lambda_n}(V_n) + \gamma = t^\tau_{0,\lambda}(W) + \gamma. \end{cases}$$

We have $(k-1)\Gamma - \gamma \geqslant \Gamma - \gamma > \gamma$, which is a contradiction and therefore $\mathcal{C}^1_\varrho$ must be open in $\mathcal{C}^M_\varrho$.

Let us show that $\mathcal{C}^1_\varrho$ is also closed in $\mathcal{C}^M_\varrho$ for any $\varrho \in (0, \varrho_0)$ if ϱ_0 is small enough. If $\mathcal{C}^1_\varrho$ is not closed then (according to (3.4), (3.5)) there exists $[T, V, \tau, \lambda] \in \mathcal{C}^M_\varrho \setminus \mathcal{C}^1_\varrho$, $\tau > 0$, $|V| > 0$ such that in any neighbourhood there is $[T', V', \tau', \lambda'] \in \mathcal{C}^1_\varrho$. Particularly, if $\mathcal{C}^1_{\varrho_n}$ are not closed for some $\varrho_n \to 0$ then there are $[T_n, V_n, \tau_n, \lambda_n] \in \mathcal{C}^M_{\varrho_n} \setminus \mathcal{C}^1_{\varrho_n}$, $\tau_n > 0$, $|V_n| > 0$, $[T'_n, V'_n, \tau'_n, \lambda'_n] \in \mathcal{C}^1_{\varrho_n}$, $\tau'_n > 0$, $|V'_n| > 0$ such that $|T_n - T'_n| < |V_n|$, $|V_n - V'_n| < |V_n|$, $|\tau_n - \tau'_n| < |V_n|$, $|\lambda_n - \lambda'_n| < |V_n|$. We have $|V_n| \to 0$ by [7], Observation 2.3, Lemma 2.1, we can suppose $T_n \to T$, $\frac{V_n}{|V_n|} \to W$ and obtain also $T'_n \to T$, $\frac{V'_n}{|V'_n|} \to W$. We have $|T_n - t^{k_n,\tau_n}_{0,\lambda_n}(V_n)| < \gamma$ with some $k_n > 1$ by (3.3) and $|T'_n - t^{\tau'_n}_{0,\lambda'_n}(V'_n)| < \gamma$. Analogously to the proof of the openess, we can show that k_n must be bounded and we can suppose that our sequence is chosen such that $k_n = k$ with some k. Then we obtain by using (1.1c)

$$\begin{cases} T = \lim T_n \geqslant \lim t^{k,\tau_n}_{0,\lambda_n}(V_n) - \gamma = t^{k,\tau}_{0,\lambda}(W) - \gamma \geqslant t^{\tau}_{0,\lambda}(W) + (k-1)\Gamma - \gamma, \\ T = \lim T'_n \leqslant \lim t^{\tau'_n}_{0,\lambda'_n}(V'_n) + \gamma = t^{\tau}_{0,\lambda}(W) + \gamma. \end{cases}$$

This is a contradiction and $\mathcal{C}^1_\varrho$ must be closed in $\mathcal{C}^M_\varrho$.

Hence, if ϱ_0 is small enough then $\mathcal{C}^1_\varrho$ is nonempty, closed and open in $\mathcal{C}^M_\varrho$ for any $\varrho \in (0, \varrho_0)$, that means $\mathcal{C}^1_\varrho = \mathcal{C}^M_\varrho$ and (3.2) is proved.

We obtain from (3.2), (1.1c) and Observation 1.1 that

$$\text{(3.6)} \qquad \begin{cases} \text{if } \varrho \in (0, \varrho_0),\ [T, V, \tau, \lambda] \in \mathcal{C}^M_\varrho,\ [T, V, \tau, \lambda] \neq [\frac{2\pi}{\omega_0}, 0, 0, \lambda_0], \\ \text{then } \gamma < \Gamma - \gamma < t^\tau_{0,\lambda}(V) - \gamma < T < t^\tau_{0,\lambda}(V) + \gamma < T_M + \gamma = t_M. \end{cases}$$

It follows by using the connectedness of $\mathcal{C}^0_\varrho$ that $\mathcal{C}^M_\varrho = \mathcal{C}^0_\varrho$. The assertion of Lemma 3.2 follows now from (3.1) and (3.6). □

Proof of Theorem 1.1. It follows from Lemma 3.2 and [7], Theorem 1.2 that there is a bifurcation point $\lambda_I \in [\Lambda_m - \xi, \Lambda_M + \xi]$ with the properties announced in our Theorem 1.1. Particularly, $U^\infty_{0,\lambda_I}(\cdot, W)$ is nonstationary periodic for some W, $|W| > 0$, and therefore $\lambda_I \in (\Lambda_m, \Lambda_M)$ by (1.2). At this moment, also the assertion of Remark 1.5 is obvious. □

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