Giovanni Emmanuele; Kamil John

The space of compact operators contains c_0 when a noncompact operator is suitably factorized

Czechoslovak Mathematical Journal, Vol. 50 (2000), No. 1, 75–82

Persistent URL: `http://dml.cz/dmlcz/127550`

Czechoslovak Mathematical Journal, 50 (125) (2000), 75–82

THE SPACE OF COMPACT OPERATORS CONTAINS c_0 WHEN A NONCOMPACT OPERATOR IS SUITABLY FACTORIZED*

G. Emmanuele, Catania and K. John, Praha

(Received February 21, 1997)

Keywords: spaces of linear operators, copies of c_0, approximation properties

MSC 2000: 46A32, 46B25

In this note we generalize certain results on when the space $K(X,Y)$ of compact operators contains an isomorphic copy of the sequence space c_0, a fact strictly connected to the nonexistence of a projection from the space $L(X,Y)$ onto the subspace $K(X,Y)$ as showed in the papers [3], [6]. One of the first results in this direction was obtained by Kalton in [7] who proved that if there is a non compact operator with a domain space X possessing an unconditional finite dimensional expansion of the identity and taking values in an arbitrary Banach space Y then c_0 embeds into $K(X,Y)$. Diestel and Morrison [1] have proved the same statement under the assumption that Y has an unconditional basis. Other results of the same nature obtained by Feder in [5], have been generalized by the authors in the recent paper [4]; in particular, it was there shown that if $L_{w^*}(X^*,Y)$ contains a noncompact operator, if the space Y has the compact approximation property and if $Y \subset Y_1$ where the space Y_1 has an unconditional expansion of the identity, then again $c_0 \subset K_{w^*}(X^*,Y)$ (here $L_{w^*}(X^*,Y)$ denotes the space of w^*-w continuous operators). Another similar result is contained in [2] where the first author proved that if there is a non compact operator factorizing through a reflexive Banach space with an unconditional basis then again c_0 embeds into $K(X,Y)$.

* The work of the first-named author was supported by M.U.R.S.T. of Italy (40 %, 1994), the work of the second-named author was supported by the grants of AV ČR No. A1019504 and of GA ČR No. 201/94/0069.

In this note we show that all these results, as well as other facts from [5], actually are consequence of our Proposition 1. It describes a quite general procedure useful to construct copies of c_0 inside $K(X, Y)$ when starting from the existence of non compact operators.

We observe that the proof of our Proposition 1 below actually is a refinement of the techniques used in the previous papers; but even if not original at all, it allows us to cover (in the separable case) the old quoted results and to furnish some new facts; among them Theorem 1 is, in our opinion, the main new application.

Before finishing this Introduction we remark that in [2] and [6] it was independently shown that if a noncompact operator $T \in L(X, Y)$ factorizes through a Banach space which has an unconditional basis then $c_0 \subset K(X, Y)$; this seems to be the only old result not covered by the present ones.

Before presenting the main result we need a definition.

Definition. We shall say that $\{K_n\} \subset K(X)$ is an unconditional compact approximating sequence if $\|A_n x - x\| \longrightarrow 0$ and the sum $\sum\limits_n (A_{n+1} - A_n)x$ is weakly unconditionally Cauchy for all $x \in X$. Moreover, we shall say that such a sequence is shrinking if $\{K_n^*\}$ is an (unconditional) compact approximating sequence for X^*. A Banach space X is said to have the (shrinking) unconditional compact approximation property if there is a (shrinking) unconditional compact approximating sequence for X.

In fact the above definition is possible for nets also, but in this section **sequences** are substantial.

We shall use also the following refinement of a fact due to Kalton [7]:

Fact (K). *Let $\widehat{X} \subset X^*$ be total on X, let $\widehat{Y} \subset Y^*$ be a norming subspace and suppose that the sequence $\{T_n\} \subset K(X, Y)$ has the property that $\widehat{y}(T_n x) \xrightarrow{n} 0$ for all $x \in X$ and all $\widehat{y} \in \widehat{Y}$. Suppose further that the unit ball B_X of X is $w(X, \widehat{X})$ compact, the unit ball $B_{\widehat{Y}}$ of $\widehat{Y}$ is $w(\widehat{Y}, Y)$ compact and that the T_n's are $w(X, \widehat{X})$-$w(Y, \widehat{Y})$ continuous. Then $T_n \longrightarrow 0$ in the weak topology of the space $L(X, Y)$.*

P r o o f. Suppose $T \in K(X, Y)$ is $w(X, \widehat{X})$-$w(Y, \widehat{Y})$ continuous. Then consider the compact topological space $K = B_X \times B_{\widehat{Y}}$ where on B_X we consider the $w(X, \widehat{X})$ topology and on $B_{\widehat{Y}}$ the $w(\widehat{Y}, Y)$-topology. Let $f(x, \widehat{y}) = \widehat{y}T(x)$ define a real function on K. Using the fact that on $\overline{T(B_X)}$ the norm topology and the Hausdorff topology $w(Y, \widehat{Y})$ coincide, it is not difficult to prove that f is continuous. So we may consider T_n as continuous functions on the compact Hausdorff space K equipped with the described topology. Then our convergence assumption and the Lebesgue theorem imply that $T_n \longrightarrow 0$ in the weak topology of the normed space $C(K)$ and thus also in the weak topology of the Banach space $K(X, Y)$. $\square$

We are now ready to give our

Proposition 1. *Let $T \in L(X,Y)$ be an operator and let $T = BA$ be a factorization of T through a Banach space E. Suppose that E is isomorphic to a subspace of a Banach space E_1 by an isomorphism J. Suppose also that $\widehat{X}$ is a total subspace of X^*. Finally, let the following conditions* (i)–(vi) *be satisfied:*

(i) *there are a Banach space Y_1 containing isomorphically Y by an isomorphism I and a bounded linear operator $\widetilde{B}\colon E_1 \to Y_1$, that is an extension of the operator $B\colon E \to Y$ in the sense that $\widetilde{B}J(e) = IB(e)$ for all $e \in E$,*

(ii) *there are a norming subspace $\widehat{Y_1}$ of Y_1^* and continuous operators $B_n \in K(E_1)$ for all $n \in N$, such that*

$$\widehat{y}_1\left(\sum_{i=1}^{n} \widetilde{B}B_iJA(x)\right) \overset{n}{\longrightarrow} \widehat{y}_1(\widetilde{B}JA(x)) \text{ for all } x \in X \text{and all } \widehat{y}_1 \in \widehat{Y_1}$$

and such that $\sum B_n$ is a weakly unconditionally Cauchy (WUC) series in the space $L(E_1)$,

(iii) *there is a sequence $\{A_n\} \subset K(E)$ of continuous operators such that, for all $x \in X$, $IBA_nA(x) \longrightarrow IBA(x)$ in the $w(Y_1, \widehat{Y_1})$-topology,*

(iv) *$IBA_iA\colon X \to Y_1$ is $w(X,\widehat{X})$-$w(Y_1,\widehat{Y_1})$ continuous,*

(v) *$\widetilde{B}B_iJA\colon X \to Y_1$ is $w(X,\widehat{X})$-$w(Y_1,\widehat{Y_1})$ continuous,*

(vi) *the unit balls of the spaces X and $\widehat{Y_1}$ are compact in the $w(X,\widehat{X})$ and $w(\widehat{Y_1}, Y_1)$ topologies, respectively.*

Then certain convex blocks of $\{IBA_iA\}$ are (WUC) and, in each point $x \in X$, they converge to $IT(x)$ in the $w(Y_1,\widehat{Y_1})$ topology.

Moreover, if the operator T is not compact then the sequence space c_0 is isomorphically contained in $\overline{\text{span}}\{BA_iA\} \subset K(X,Y)$.

Proof. The conditions (ii) and (iii) give that for all $x \in X$ and all $\widehat{y}_1 \in \widehat{Y_1}$ we have

$$\widehat{y}_1(IBA_nA(x)) \overset{n}{\longrightarrow} \widehat{y}_1(IBA(x)) \tag{1}$$

and

$$\widehat{y}_1\left(\sum_{i=1}^{n} \widetilde{B}B_iJA(x)\right) \overset{n}{\longrightarrow} \widehat{y}_1(\widetilde{B}JA(x)).$$

Thus, since $\widetilde{B}$ extends B in the sense quoted in the assumption (ii), we get easily

$$\widehat{y}_1(IBA_nA(x)) - \widehat{y}_1\left(\sum_{i=1}^{n} \widetilde{B}B_iJA(x)\right) \overset{n}{\longrightarrow} 0. \tag{2}$$

Now from (iv)–(v) we see that the operators

$$IBA_nA - \sum_{i=1}^{n} \widetilde{B}B_iJA\colon\ X \to Y_1$$

are $w(X,\widehat{X})$-$w(Y_1,\widehat{Y_1})$ continuous; so we may deduce from (vi), (2) and Fact (K) that

$$U_n = IBA_nA - \sum_{i=1}^{n} \widetilde{B}B_iJA \xrightarrow{n} 0$$

in the weak topology of the space $K(X,Y_1)$.

Now we proceed as in [9, p. 32]. Since $U_n \xrightarrow{w} 0$, we can find disjoint convex combinations (blocks) U_j' of $\{U_n\}$, such that $\sum\limits_{j=1}^{\infty} \|U_j'\| < \infty$. Let Y_j' be the blocks of $\{Y_n\} = \{BA_nA\}$ built with the same coefficients and let us put $Z_j = Y_{j+1}' - Y_j'$. Computing, we get that

$$IZ_j = U_{j+1}' - U_j' + C_j',$$

where C_j''s are disjoint blocks of $\{C_n\} = \{\widetilde{B}B_nJA\}$ with coefficients between 0 and 1. Now we claim that $\sum\limits_{j=1}^{\infty} IZ_j$ is a weakly unconditionally Cauchy (WUC) series. To see this let $Z^* \in K(X,Y_1)^*$. Then we have

$$\sum_{j=1}^{\infty} |Z^*(IZ_j)| \leqslant 2\|Z^*\| \cdot \sum_{j=1}^{\infty} \|U_j'\| + \sum_{n=1}^{\infty} |Z^*(C_n)| < \infty$$

using the fact that $\sum\limits_{j=1}^{\infty} C_n$ is a WUC series thanks to (ii). Indeed, (ii) means that $\|\sum\limits_{n=1}^{m} \pm B_n\| \leqslant K$ for all m and all $\pm$ and thus $\{\|\sum\limits_{n=1}^{m} \pm C_n\|;\ m \in N\}$ is also bounded, meaning that $\sum C_n$ is WUC. But I is an isomorphism; so we conclude that also $\sum\limits_{j=1}^{\infty} Z_j$ is a WUC series. Further we observe that $\sum\limits_{j=1}^{\infty} Z_j$ is not norm convergent. Indeed, (1) may be rewritten

$$\widehat{y}_1(IY_n(x)) \xrightarrow{n} \widehat{y}_1(IT(x)) \qquad \text{for} \quad \widehat{y}_1 \in \widehat{Y}_1, x \in X$$

which implies that also for convex blocks Y_j' we have

$$\widehat{y}_1(IY_n'(x)) \xrightarrow{n} \widehat{y}_1(IT(x)). \tag{3}$$

Now assume that T is not compact; it easily follows that the sequence $\{Y_n'\} \subset K(X,Y)$ does not converge in the norm topology since otherwise, by (3), $\{IY_n'\}$ would converge (in the norm) to the non compact operator IT. The famous Bessaga-Pełczyński Theorem (see [8]) now ensures that a subsequence of $\{Z_j\}$ is equivalent to the unit vector basis of c_0, which finishes the proof. □

As a special case we might formulate

Proposition 1a. *Let $T \in L(X,Y)$ be an operator and let $T = BA$ be a factorization of T through a Banach space E. Suppose that E is isomorphic to a subspace of a Banach space E_1 by an isomorphism J and that, further, $\widehat{E} \subset E^*$ and $\widehat{E}_1 \subset E_1^*$ are subspaces such that J is $w(E,\widehat{E})$-$w(E_1,\widehat{E}_1)$ continuous. Suppose also that $\widehat{Y}$ is a subspace of Y^*, $\widehat{X}$ a total subspace of X^*. Finally, let the following conditions* (i)–(vi) *be satisfied:*

(i) *there are a Banach space Y_1 containing isomorphically Y by an isomorphism I, a norming subspace $\widehat{Y}_1$ of Y_1^* such that I is $w(Y,\widehat{Y})$-$w(Y_1,\widehat{Y}_1)$ continuous and a $w(E_1,\widehat{E}_1)$-$w(Y_1,\widehat{Y}_1)$ continuous bounded linear operator $\widetilde{B}\colon E_1 \to Y_1$, that is an extension of the operator $B\colon E \to Y$ in the sense that $\widetilde{B}J(e) = IB(e)$ for all $e \in E$,*

(ii) *there are $w(E_1,\widehat{E}_1)$-continuous operators $B_n \in K(E_1)$ for all $n \in N$, such that $\sum\limits_{n=1}^{\infty} \widehat{z}_1(B_n(z_1)) = \widehat{z}_1(z_1)$ for all $z_1 \in E_1$ and all $\widehat{z}_1 \in \widehat{E}_1$ and such that $\sum B_n$ is a weakly unconditionally Cauchy (WUC) series in the space $L(E_1)$,*

(iii) *there is a sequence $\{A_n\} \subset K(E)$ of $w(E,\widehat{E})$-continuous operators such that, for all $z \in E$, $A_n(z) \longrightarrow z$ in the $w(E,\widehat{E})$-topology,*

(iv) *$A\colon X \to E$ is $w(X,\widehat{X})$-$w(E,\widehat{E})$ continuous and bounded,*

(v) *$B\colon E \to Y$ is $w(E,\widehat{E})$-$w(Y,\widehat{Y})$ continuous and bounded,*

(vi) *the unit balls of the spaces X and $\widehat{Y}_1$ are compact in the $w(X,\widehat{X})$ and $w(\widehat{Y}_1,Y_1)$ topologies, respectively.*

Then certain convex blocks of $\{IBA_iA\}$ are (WUC) and, in each point $x \in X$, they converge to $IT(x)$ in the $w(Y_1,\widehat{Y}_1)$ topology.

Moreover, if the operator T is not compact then the sequence space c_0 is isomorphically contained in $\overline{\operatorname{span}}\{BA_iA\} \subset K(X,Y)$.

Remark 1. As we shall see below the condition (i) is usually automatically verified by considering Y embedded into an injective superspace Y_1. A version of the Proposition where X is a quotient of an $l_1(\Gamma)$ is also possible.

Remark 2. Note that the following condition (ii)$'$ implies the conditions (ii) in the Propositions 1 and 1a.

(ii)$'$ *There are $w(E_1,\widehat{E}_1)$-continuous operators $B_n \in K(E_1)$ for all $n \in N$, such that $\sum\limits_{n=1}^{\infty} B_n(e_1) = e_1$ where the countable sum converges unconditionally in the norm for all $e_1 \in E_1$.*

Moreover, if $\sum\limits_n B_n(e_1)$ converges unconditionally to e_1 for all $e_1 \in E_1$ then (ii)$'$ together with the other assumptions of the Propositions also imply that certain convex blocks of $\{BA_iA\}$ are, for each point $x \in X$, unconditionally converging to $T(x)$. This applies also in the Corollaries 1–4 and in the Theorem 1. Indeed, the set $\{\sum_{n=1}^{m} \pm B_n(e_1);\ m \in N\}$ is bounded for all $e_1 \in E_1$. The uniform boundedness principle then yields that the set $\{\|\sum_{n=1}^{m} \pm B_n\|;\ m \in N\}$ is bounded again.

Corollary 1([4]). *Let $T \in L_{w^*}(X^*, Y)$ be a noncompact operator. Suppose that Y has the compact approximation property and that Y is a subspace of a separable Banach space Y_1 such that Y_1 has the unconditional compact approximation property. Then $c_0 \subset K_{w^*}(X^*, Y)$.*

P r o o f. It is enough to choose $E = Y$ and $B = \mathrm{Id}_Y$ in the Proposition. □

Similarly we get the more general and new

Corollary 2. *Let $T = BA\colon X^* \to Y$ be a factorization of a noncompact operator T through a Banach space E such that $A\colon X^* \to E$ is w^*-w continuous and $B \in L(E, Y)$. Suppose that E has the compact approximation property and that E is a subspace of a separable Banach space E_1 such that E_1 has an unconditional compact approximation property. Then $c_0 \subset K(X, Y)$.*

P r o o f. We choose in the Proposition for Y_1 any injective Banach space containing Y, e.g. $l_\infty(B_{Y^*})$, $\widehat{Y}_1 = Y_1^*$ and $\widehat{Y} = Y^*$. □

Similar statement may be given e.g. for the case when A is w^*-w^* continuous and B is w^*-w continuous.

Corollary 3. *Let $T = BA\colon X^* \to Y$ be a factorization of a noncompact operator T through a Banach space E^* such that $A\colon X^* \to E^*$ is w^*-w^* continuous and $B \in L(E^*, Y)$ is w^*-w continuous. Suppose that E has the compact approximation property and that E is a quotientspace of a separable Banach space E_1 such that the imbedding $J\colon E^* \to E_1^*$ is w^*-w^* continuous and such that E_1 has an unconditional compact approximation property. Suppose further that I is an imbedding of the space Y into the Banach space Y_1 such that the operator B has an extension to a w^*-w continuos operator $\widetilde{B}_1\colon E_1^* \to Y_1$ in the sense that $\widetilde{B}J(e) = IB(e)$ for all $e \in E^*$. Then $c_0 \subset K(X, Y)$.*

The next theorem is in fact a consequence of our Proposition 1. Because it has a less technical formulation, we prefer to state it separately.

Theorem 1. *Let $T \in L(X, Y)$ be a noncompact operator and let $T = BA$ be a factorization of T through a Banach space E. Suppose that* *either*

(1) *E is isomorphic to a quotient space of a Banach space E_1, the space E^* has the compact approximation property and the space E_1 has the shrinking unconditional compact approximation property*

or

(2) *E is isomorphic to a subspace of a Banach space E_1, the space E^{**} has the compact approximation property and the space E_1^* has the shrinking unconditional compact approximation property.*

Then the sequence space c_0 is isomorphically contained in $K(X,Y)$.

Proof. *Case 1.* We shall apply the Proposition to the noncompact operator $T^* = A^*B^*\colon Y^* \to X^*$. Let $Q\colon E_1 \to E$ be the surjection operator. It is well known that we may choose a linear surjection $q\colon l_1(\Gamma) \to X$. The lifting property of $l_1(\Gamma)$ yields an operator $S\colon l_1(\Gamma) \to E_1$ such that $Aq = QS$. In the Proposition we may now substitute for the space Y the space X^*, for the isomorphic embedding $J\colon E \to E_1$ the w^*-w^* continuous embedding $Q^*\colon E^* \to E_1^*$, for the isomorphic embedding $I\colon Y \to Y_1$ the w^*-w^* continuous embedding $q^*\colon X^* \to l_\infty(\Gamma)$, for $\widetilde{B}$ the mapping S^*. Further we substitute $l_1(\Gamma)^{**}$ for $\widehat{E}_1$, X for $\widehat{Y}$ and Y for $\widehat{X}$. Then (i)–(iv) are easily seen to be satisfied. The condition (vi) means in our case that the closed unit balls B_{Y^*} and $B_{X^{**}}$ are w^*-compact. To check (iv) it is sufficient to observe that the operators $q^*A^*A_i^*B^*$ are w^*-w continuous. But this follows immediately because these operators are w^*-w^* continuous and compact. Similarly we observe that (v) holds. Proposition 1 now gives that $c_0 \subset \overline{\operatorname{span}}\{A^*A_i^*B^*\}$ which means that $c_0 \subset \overline{\operatorname{span}}\{BA_iA\}$.

Case 2. In this case E^* is isomorphic to a quotient of the space E_1^* and we may apply the case (1) to the noncompact operator $T^* = A^*B^*\colon Y^* \to X^*$. □

Remark 4. The case (2) in the above Theorem may also be obtained by applying the Proposition directly to the factorization of $T^{**}\colon X^{**} \to Y^{**}$ through the space E^{**} where $E^{**} \subset E_1^{**}$. We also embed Y into an injective Banach space.

Remark 5. If in the Theorem 1 the operator $A\colon X \to E$ is weakly compact or if $l_1 \not\subseteq E^*$ then the assumption concerning E in (1) may be that only E has the compact approximation property and in (2) that only E^* has the compact approximation property. Indeed, first we notice that we may assume that A^* is unconditionally convergent (otherwise A^* would fix a copy of c_0 and thus $c_0 \subset X^*$ and this in turn would imply that $c_0 \subset K(X,Y)$). If now $l_1 \not\subseteq E^*$ then by Pełczyński (see [10]) E^* has the property (V) and thus A^* is weakly compact.

The last consequence of the previous results is a slight generalization of a result due to Feder

Corollary 4 ([5]). *Let X be isomorphic to a factor space of a Banach space X_1, X_1 having the shrinking unconditional compact approximation property. Let the space X^* have the compact approximation property and let $L(X,Y)$ contain a noncompact operator M. Then $c_0 \subset K(X,Y)$ isomorphically.*

Proof. We apply Theorem 1 (1) (after taking $X = E$, $X_1 = E_1$) to the operator $T = M\,\mathrm{Id}_X$; it then yields a copy of c_0 inside $\overline{\mathrm{span}}\{MA_i\} \subset K(Z,Y)$. $\square$

References

[1] *J. Diestel, T. J. Morrison*: The Radon-Nikodym property for the space of operators. Math. Nachr. *92* (1979), 7–12.

[2] *G. Emmanuele*: Dominated operators on $C[0,1]$ and the (CRP). Collect. Math. *41(1)* (1990), 21–25.

[3] *G. Emmanuele*: A remark on the containment of c_0 in spaces of compact operators. Math. Proc. Cambridge Philos. Soc. *111* (1992), 331–335.

[4] *G. Emmanuele, K. John*: Uncomplementability of spaces of compact operators in larger spaces of operators. Czechoslovak Math. J. To appear.

[5] *M. Feder*: On subspaces of spaces with an unconditional basis and spaces of operators. Illinois J. Math. *24* (1980), 196–205.

[6] *K. John*: On the uncomplemented subspace $K(X,Y)$. Czechoslovak Math. J. *42* (1992), 167–173.

[7] *N. J. Kalton*: Spaces of compact operators. Math. Ann. *208* (1974), 267–278.

[8] *J. Lindenstrauss, L. Tzafriri*: Classical Banach Spaces, Sequence Spaces. EMG 92 Springer Verlag, 1977.

[9] *J. Lindenstrauss, L. Tzafriri*: Classical Banach Spaces, Function Spaces. EMG 97 Springer Verlag, 1979.

[10] *A. Pełczyński*: A connection between weak unconditional convergence and weak sequential completeness in Banach spaces. Bull. Acad. Polon. Sci. *6* (1958), 251–253.

Authors' addresses: G. Emmanuele, Department of Mathematics, University of Catania, Viale A. Doria 6, 95125 Catania, Italy, e-mail: `Emmanuele@Dipmat.Unict.It`; K. John, Mathematical Institute, Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail `Kjohn@math.cas.cz`.