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AN AXIOMATIC APPROACH TO METRIC PROPERTIES OF CONNECTED GRAPHS

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Let G be a nontrivial connected graph and let d denote its distance function. As is wellknown, d is a metric on $V(G)$. In [4], and axiomatic characterization of the set of all geodesics (i.e. shortest paths) in G was given. In [8], an axiomatic characterization of the set of all steps in G (i.e. the set of all ordered triples (u, v, x) of vertices in G with the property that $d(u, v) = 1$ and $d(v, x) = d(u, x) - 1$) was given. In the present paper, a certain connection between an axiomatic characterization of the set of all nontrivial geodesics in G and that of the set of all steps in G will be studied.

0. In this paper the letters i, j, k, m and n are reserved for denoting non-negative integers. By a graph we mean a finite undirected graph with no loop or multiple edge. In the whole paper we assume that a nontrivial connected graph G is given. Its vertex set, its edge set and its distance function will be denoted by V, E and d, respectively. Hence V is a finite set with at least two elements.

As usual, if $i \geqslant 0$, then V^{i+1} denotes the set of all ordered $(i+1)$-tuples

(1) $$(u_0, \ldots, u_i),$$

where $u_0, \ldots, u_i \in V$. Instead of (1) we will shortly write

(2) $$u_0 \ldots u_i.$$

If $j \geqslant 0$, then we denote by Σ_j the set

$$\bigcup_{i=j}^{\infty} V^{i+1}.$$

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If $\alpha = v_0 \dots v_k$ and $\beta = w_0 \dots w_m$, where $k \geqslant 0$, $m \geqslant 0$ and $v_0, \dots, v_k, w_0, \dots, w_m \in V$, then we denote

$$\alpha\beta = v_0 \dots v_k w_0 \dots w_m.$$

Let $\gamma = x_0 \dots x_n$, where $n \geqslant 0$ and $x_0, \dots, x_n \in V$. We denote

$$\overline{\gamma} = x_n \dots x_0,\ a(\gamma) = x_0 \quad \text{and} \quad c(\gamma) = x_n.$$

If $n \geqslant 1$, then we denote $b(\gamma) = x_1$. Moreover, we denote

$$\Sigma = \Sigma_0 \cup \{*\},$$

where $\delta * = \delta = * \delta$ for every $\delta \in \Sigma_0$. Define $** = *$ and $\overline{*} = *$.

Let $u_0, \dots, u_i \in V$, where $i \geqslant 0$. As usual, we say that (2) is a walk in G if

$$\{u_j, u_{j+1}\} \in E \quad \text{for each } j,\ 0 \leqslant j < i;$$

we say that (2) is a path in G if it is a walk in G and the vertices $u_0, \dots, u_i$ are mutually distinct; we say that (2) is a nontrivial path in G if it is a path in G and $i \geqslant 1$. Let Π denote the set of all paths in G, and let Π_N denote the set of all nontrivial paths in G.

By a *geodesic* ([2]) or a shortest path ([1]) in G we mean such a path (2) in G that $d(u_0, u_i) = i$.

Theorem 0 ([4]). *Let $R \subseteq \Pi$. Then R is the set of all geodesics in G if and only if it fulfils the following Axioms* X0, X1, X2+, X3, X4+, X5, X6 *and* X7*:*

X0 *if* $\{u, v\} \in E$, *then* $uv \in R$ $(\forall u,\ v \in V)$;

X1 *if* $\alpha \in R$, *then* $\overline{\alpha} \in R$ $(\forall \alpha \in \Sigma)$;

X2 + *if* $u\alpha x \in R$, *then* $u\alpha \in R$ $(\forall u,\ x \in V,\ \forall \alpha \in \Sigma)$;

X3 *if* $\alpha u \beta x \gamma,\ u\delta x \in R$, *then* $\alpha u \delta x \gamma \in R$ $(\forall u,\ x \in V,\ \forall \alpha, \beta, \gamma, \delta \in \Sigma)$;

X4 + *if* $uv\alpha x,\ xy \in R$, $u\varrho yx \notin R$ *for all* $\varrho \in \Sigma$ *and* $uv\sigma y \notin R$ *for all* $\sigma \in \Sigma$, *then* $v\alpha xy \in R$ $(\forall u, v, x, y \in V,\quad \forall \alpha \in \Sigma)$;

X5 *if* $u \neq x$, *then there exists* $\tau \in \Sigma$ *such that* $u\tau x \in R$ $(\forall u, x \in V)$;

X6 *if* $uv\alpha x \in R$, *then* $ux \notin R$: $(\forall u, v, x \in V,\quad \forall \alpha \in \Sigma)$;

X7 *if* $uv\alpha x,\ vu\beta y,\ u\beta yx \in R$, *then* $v\alpha xy \in R$ $(\forall u, v, x,\ y \in V,\quad \forall \alpha,\ \beta \in \Sigma)$.

Note that Theorem 0 was generalized in [6] and modified in [7].

We will need two propositions.

Proposition 1. *Let $R \subseteq \Pi$ and let R fulfil Axioms* X1, X2+ *and* X6. *Then R fulfils Axiom* X4+ *if and only if it fulfils the following Axiom* X4*:*

X4 *if $uv\alpha x,\ xy \in R,\ u \neq y \neq v,\ u\varrho yx \notin R$ for all $\varrho \in \Sigma$ and $uv\sigma y \notin R$ for all $\sigma \in \Sigma$, then $v\alpha xy \in R$* $(\forall u, v, x, y \in V, \quad \forall \alpha \in \Sigma)$.

P r o o f. Obviously, if R fulfils X4+, then it fulfils X4. Conversely, let R fulfil X4. Consider arbitrary $u, v, x, y \in V$ and an arbitrary $\alpha \in \Sigma$ such that $uv\alpha x,\ xy \in R$, $u\varrho yx \notin R$ for all $\varrho \in \Sigma$ and $uv\sigma y \notin R$ for all $\sigma \in \Sigma$. By X1, $yx \in R$. If $u = y$, then $yv\alpha x \in R$, which contradicts X6. Hence $u \neq y$.

Suppose $y = v$. Then $uy\alpha x \in R$. Combining X2+ and X1, we get $x\overline{\alpha}y \in R$. By X6, $\alpha = *$. Hence $uyx \in R$. This means that there exists $\varrho \in \Sigma$ such that $u\varrho yx \in R$, which is a contradiction. Hence $y \neq v$. By X4, $v\alpha xy \in R$. This means that R fulfils X4+, which completes the proof. □

Proposition 2. *Let $R \subseteq \Pi$ and let R fulfil Axiom* X0. *Then R fulfils Axiom* X2+ *if and only if it fulfils the following Axioms* X2 *and* X8*:*

X2 *if $uv\alpha x \in R$, then $uv\alpha \in R$* $(\forall u, v, x \in V, \quad \forall \alpha \in \Sigma)$;

X8 $u \in R \quad (\forall u \in V)$.

P r o o f. Since G has no isolated vertex, the result is obvious. □

Combining Theorem 0 with Propositions 1 and 2, we obtain the following characterization of the set of all nontrivial geodesics in G:

Theorem A. *Let $R \subseteq \Pi_N$. Then R is the set of all nontrivial geodesics in G if and only if it fulfils Axioms* X0–X7.

Consider $T \subseteq V^3$. We will say that T is *associated* with G if

$\{u, v\} \in E$ if and only if $u \neq v$ and there exists $x \in V$ such that either $uvx \in T$ or $vux \in T$

for all ordered pairs $uv \in V^2$.

Following [8], by a *step* in G we mean an ordered triple $uvx \in V^3$ with the properties

$$d(u, v) = 1 \quad \text{and} \quad d(v, x) = d(u, x) - 1.$$

The next theorem was proved in [8]:

Theorem B. *Let $T \subseteq V^3$, and let T be associated with G. Then T is the set of all steps in G if and only if it fulfils the following Axioms* Y0–Y7*:*

Y0 *if* $uvx \in T$, *then* $vuu \in T$ $(\forall u, v, x \in V)$;

Y1 *if* $uvx,\ vuy \in T$, *then* $x \neq y$ $(\forall u, v, x, y \in V)$;

Y2 *if* $uvx,\ xyv \in T$, *then* $xyu \in T$ $(\forall u, v, x, y \in V)$;

Y3 *if* $uvx,\ xyv \in T$, *then* $uvy \in T$ $(\forall u, v, x, y \in V)$;

Y4 *if* $uvx,\ xyy \in T$, *then either* $xyu \in T$ *or* $yxv \in T$ *or* $uvy \in T$ $(\forall u, v, x, y \in V)$;

Y5 *if* $u \neq x$, *then there exists* $z \in V$ *such that* $uzx \in T$ $(\forall u, x \in V)$;

Y6 *if* $uvx,\ uyv \in T$, *then* $y = v$ $(\forall u, v, x, y \in V)$;

Y7 *if* $uvx,\ vuy,\ xyy \in T$, *then* $xyu \in T$ $(\forall u, v, x, y \in V)$.

We denote by $\mathcal{R}$ the set of all $R \subseteq \Pi_N$ such that R fulfils X0–X5. We denote by $\mathcal{T}$ the set of all $T \subseteq V^3$ such that T is associated with G and it fulfils Y0–Y5. A one-to-one mapping Φ from $\mathcal{R}$ onto $\mathcal{T}$ will be found in Theorem 1. Moreover, if $R \in \mathcal{R}$ and $T = \Phi(R)$, then we will prove that R fulfils X6 and X7 if and only if T fulfils Y6 and Y7 (Theorem 2).

Remark 1. The set of all geodesics in G is closely connected with the interval function of G in the sense of [3]. An axiomatic characterization of the interval function of G was given in [5].

1. In this part of the paper, some consequencies of Axioms Y0–Y5 will be found. Let $T \subseteq V^3$. If $u_0, \dots, u_i,\ v \in V$, where $i \geqslant 1$, then instead of

$$u_0u_1v, \dots, u_{i-1}u_iv \in T$$

we will write

$$u_0 \dots u_iTv.$$

Consider $u, v, w, x \in V$ and $\alpha, \beta \in \Sigma$. It is obvious that

(3) $$u\alpha vTx \quad \text{and} \quad v\beta wTx \quad \text{if and only if} \quad u\alpha v\beta wTx.$$

Let $i \geqslant 1$ and let $u_0, \dots, u_i \in V$. We will say that (2) is a process in T if

$$u_0 \dots u_iTu_i.$$

As follows from (3),

(4) if $i \geqslant 2$ and $u_0 u_1 \dots u_i$ is a process in T, then $u_1 \dots u_i$ is a process in T, too.

We denote by $\mathcal{T}_0$ the set of all $T \subseteq V^3$ such that T is associated with G.
Part of the next lemma was proved in [8].

Lemma 1. *Consider $T \in \mathcal{T}_0$. Assume that T fulfils Axioms* Y2 *and* Y3. *Let $u_0, \dots, u_i$, $v, w \in V$, where $i \geqslant 1$, and let*

(5) $$u_0 \dots u_i T v.$$

If

(6_0) $$wvu_0 \in T,$$

then

(6_j) $$u_{j-1}u_j w,\ wvu_j \in T$$

for each j, $1 \leqslant j \leqslant i$. If

(7_i) $$vwu_i \in T,$$

then

(7_k) $$u_{k-1}u_k w,\ vwu_{k-1} \in T$$

for each k, $1 \leqslant k \leqslant i$.

P r o o f. First, let $wvu_0 \in T$. We will prove that (6_j) holds for each j, $0 \leqslant j \leqslant i$. We proceed by induction on j. The case $j = 0$ is obvious. Let $j \geqslant 1$. By the induction hypothesis, $wvu_{j-1} \in T$. By (5), $u_{j-1}u_j v \in T$. Combining Y2 and Y3, we get (6_j).

Next, let $vwu_i \in T$. We will prove that (7_{i-j}) holds for each j, $0 \leqslant j \leqslant i$. The case $j = 0$ is obvious. Let $j \geqslant 1$. By the induction hypothesis, $vwu_{i-j+1} \in T$. By (5), $u_{i-j}u_{i-j+1}v \in T$. Combining Y2 and Y3, we get (7_{i-j}). Thus the lemma is proved. □

Corollary 1. *Consider $T \in \mathcal{T}_0$. Assume that T fulfils Axioms* Y2 *and* Y3. *Let $u_0, \ldots, u_i$, $v, w \in V$, where $i \geqslant 1$, and let* (5) *hold. If*

$$\text{either } wvu_0 \in T \text{ or } vwu_i \in T,$$

then $u_0 \ldots u_i T w$.

Lemma 2. *Consider $T \in \mathcal{T}_0$. Assume that T fulfils Axioms* Y0, Y2 *and* Y3. *Let $u_0, \ldots, u_i \in V$, where $i \geqslant 1$, and let $u_i \ldots u_0$ be a process in T. Then $u_0 \ldots u_i$ is a process in T as well.*

P r o o f. Since $u_i \ldots u_0$ is a process in T, we have $u_i \ldots u_0 T u_0$. We want to prove that

$$u_0 \ldots u_i T u_i. \tag{8}$$

We proceed by induction on i. If $i = 1$, then (8) immediately follows from Y0. Let $i \geqslant 2$. Since

$$u_{i-1} \ldots u_0 T u_0,$$

the induction hypothesis implies that

$$u_0 \ldots u_{i-1} T u_{i-1}.$$

Moreover, we have $u_i u_{i-1} u_0 \in T$. By virtue of Corollary 1,

$$u_0 \ldots u_{i-1} T u_i.$$

Since $u_i u_{i-1} u_0 \in T$, Y0 implies that $u_{i-1} u_i u_i \in T$, and thus (8) holds, which completes the proof. □

A very special version of the next lemma was proved (in a connection with characterizing geodetic graphs) in [9].

Lemma 3. *Consider $T \in \mathcal{T}_0$. Assume that T fulfils Axioms* Y0, Y2, Y3 *and* Y4. *Let $u_0, \ldots, u_i$, $v \in V$, where $i \geqslant 1$, and let* (5) *hold. If $u_i = u_0$, then*

$$u_0 \ldots u_i T w \text{ for each } w \in V. \tag{9}$$

P r o o f. Let $u_i = u_0$. Suppose, to the contrary, that (9) does not hold. Since G is connected, there exist distinct $x, y \in V$ such that $\{x, y\} \in E$,

$$u_0 \ldots u_i T x \tag{10}$$

and there exists j, $0 \leqslant j \leqslant i-1$, such that $u_j u_{j+1} y \notin T$. By virtue of (10), $u_j u_{j+1} x \in T$. Recall that $T \in \mathcal{T}_0$. Since $\{x, y\} \in E$, Y0 implies that $xyy \in T$. According to Y4,

$$\text{either } yxu_{j+1} \in T \text{ or } xyu_j \in T. \tag{11}$$

Put $u_{i+1} = u_{-i+1} = u_1, \ldots, u_{2i-1} = u_{-1} = u_{i-1}$. Then $u_{j+i} = u_{j-i} = u_j$ and $u_{j+i+1} = u_{j-i+1} = u_{j+1}$. As follows from (10),

$$u_{j+1} \ldots u_{j+i} u_{j+i+1} T x \tag{12}$$

and

$$u_{j-i} u_{j-i+1} \ldots u_j T x. \tag{13}$$

Let $yxu_{j+1} \in T$. Combining (12) with Corollary 1, we get

$$u_{j+1} \ldots u_{j+i} u_{j+i+1} T y$$

and therefore $u_j u_{j+1} y \in T$, which is a contradiction.

Let $yxu_{j+1} \notin T$. By (11), $xyu_j \in T$. Combining (13) with Corollary 1, we get

$$u_{j-i} u_{j-i+1} \ldots u_j T y$$

and therefore $u_j u_{j+1} y \in T$, which is a contradiction. Thus the lemma is proved. □

Lemma 4. *Consider $T \in \mathcal{T}_0$. Assume that T fulfils Axioms* Y_0–Y4. *Let $u_0, \ldots, u_i$, $v \in V$, where $i \geqslant 1$, and let* (5) *hold. Then* (2) *is a path in G.*

P r o o f. First, we will prove that the vertices $u_0, \ldots, u_i$ are mutually distinct. Suppose, to the contrary, that there exist g and h, $0 \leqslant g < h \leqslant i$ such that $u_g = u_h$. We have

$$u_g \ldots u_h T v.$$

Since $u_g = u_h$, it follows from Lemma 3 that

$$u_g \ldots u_h T w$$

for every $w \in V$. Therefore $u_g u_{g+1} u_g \in T$. By Y0,

$$u_{g+1} u_g u_g \in T,$$

which contradicts Y1. We have proved that $u_0, \ldots, u_i$ are mutually distinct.

Recall that T is associated with G. Therefore, (5) implies that $u_0 \ldots u_i$ is a walk in G. Since $u_0, \ldots, u_i$ are mutually distinct, (5) is a path, which completes the proof. □

Corollary 2. *Consider $T \in \mathcal{T}_0$. Assume that T fulfils Axioms* Y0–Y4. *Then every process in T is a nontrivial path in G.*

Lemma 5. *Consider $T \in \mathcal{T}_0$. Assume that T fulfils Axioms* Y0–Y5. *Let $u, v \in V$ and let $u \neq v$. There exists $\tau \in \Sigma$ such that $u\tau v$ is a process in T.*

P r o o f. Suppose, to the contrary, that the lemma does not hold. By virtue of Y5, there exists an infinite sequence

$$(u_0, u_1, \ldots)$$

of vertices in G such that

$$u_0 \ldots u_i T v \text{ for each } i = 1, 2, \ldots.$$

Since V is finite, there exist j and k such that $0 \leqslant j < k$ and $u_j = u_k$. We have

$$u_j \ldots u_k T v,$$

which contradicts Lemma 4. Hence the lemma follows. □

Remark 2. As we will see, the assumption that T fulfils Axiom Y4 cannot be removed from Lemma 5. Let $|V| = 7$,

$$\begin{aligned} &V = \{x_0, x_1, x_2, y_0, y_1, y_2, z\},\\ &E = \{\{x_0, z\}, \{y_0, z\}\} \cup \{\{x_i, x_{i+1}\}, \{y_i, y_{i+1}\};\ 1 \leqslant i \leqslant 3\} \end{aligned}$$

and

$$\begin{aligned} T = \{&x_0zz,\ zx_0x_0,\ y_0zz,\ zy_0y_0\}\\ &\cup \{x_jx_0z,\ zx_0x_j,\ y_jy_0z,\ zy_0y_j;\ 1 \leqslant j \leqslant 2\}\\ &\cup \{x_kx_{k+1}x_{k+1},\ x_{k+1}x_kx_k,\ y_ky_{k+1}y_{k+1},\ y_{k+1}y_ky_k;\ 0 \leqslant k \leqslant 2\}\\ &\cup \{x_mx_{m+1}y_n,\ y_my_{m+1}x_n;\ 0 \leqslant m \leqslant 2,\ 0 \leqslant n \leqslant 2\}, \end{aligned}$$

where $x_3 = x_0$ and $y_3 = y_0$. We see that $T \in \mathcal{T}_0$, it fulfils Axioms Y0–Y3 and Y5 but does not fulfil Axiom Y4. Moreover, we see that the conclusion of Lemma 5 does not hold for T: for example, there exists no $\tau \in \Sigma$ such that $x_0\tau y_0$ is a process in T.

2. Recall that $\mathcal{R}$ is the set of all $R \subseteq \Pi_N$ such that R fulfils Axioms X0–X5 and $\mathcal{T}$ is the set of all $T \subseteq V^3$ such that T is associated with G and fulfils Axioms Y0–Y5.

We denote by Φ the mapping from $\mathcal{R}$ into V^3 defined as follows:

Φ is the set of all $uvx \in V^3$ with the property that there exists $\xi \in R$ such that $a(\xi) = u$, $b(\xi) = v$ and $c(\xi) = x$

for each $R \in \mathcal{R}$.

Moreover, we denote by Ψ the mapping from $\mathcal{T}$ into Σ_1 such that $\Psi(T)$ is the set of all processes in T for each $T \in \mathcal{T}$.

The next theorem is the main result of the present paper:

Theorem 1. *Φ is a one-to-one mapping from $\mathcal{R}$ onto $\mathcal{T}$ and $\Psi = \Phi^{-1}$.*

Proof. (I) Consider an arbitrary $R \in \mathcal{R}$. Denote $T = \Phi(R)$. Combining the fact that $R \subseteq \Pi_N$ with X0, we see that T is associated with G. X5 implies that T fulfils Y5.

We will show that T fulfils Y0–Y4. Consider arbitrary $u, v, x, y \in V$. Let $uvx \in T$. Since $T = \Phi(R)$, there exists $\tau \in \Sigma$ such that $u\tau x \in R$ and $b(u\tau x) = v$. Since $u\tau x \in \Pi$, we have $u \neq x$.

(Verification of Y0). If $\tau = *$, then $uv \in R$. If $\tau \neq *$, then by X2, $uv \in R$, too. By X1, $vu \in R$. Since $a(vu) = v$, $b(vu) = u = c(vu)$, we have $vuu \in T$.

(Verification of Y1). Let $vuy \in T$. We wish to show that $x \neq y$. Suppose, to the contrary, that $x = y$. Then $vux \in T$. Since $u \neq x$, there exists $\beta \in \Sigma$ such that $vu\beta x \in R$. Since $u\tau x \in R$, X3 implies that $vu\tau x \in R$. Since $b(u\tau x) = v$, we see that $vu\tau x \notin \Pi_N$, which is a contradiction. Thus $x \neq y$.

(Verification of Y2 and Y3). Let $xyv \in T$. Then there exists $\pi \in \Sigma$ such that $x\pi v \in R$ and $b(x\pi v) = y$. Since $x\pi v \in \Pi_N$, $x \neq v$. Since $b(u\tau x) = v$, there exists $\alpha \in \Sigma$ such that $uv\alpha x \in R$. Recall that $x\pi v \in R$. By X1, $v\overline{\pi}x \in R$. According to X3, $uv\overline{\pi}x \in R$.

First, let $\pi = *$. Then $y = v$ and $uyx \in R$. By X2, $uy \in R$. Since $v = y$, we have $uvy \in T$. Since $uyx \in R$, X1 implies that $xyu \in R$. Hence $xyu \in T$.

Now, let $\pi \neq *$. Then there exists $\beta \in \Sigma$ such that $xy\beta v = x\pi v$. Hence $v\overline{\beta}yx = v\overline{\pi}x$. Recall that $uv\overline{\pi}x \in R$. We get $uv\overline{\beta}yx \in R$. By X1, $xy\beta vu \in R$ and therefore, $xyu \in T$. By X2, $uv\overline{\beta}y \in R$. Hence $uvy \in T$.

(Verification of Y4). Let $xyy \in T$. Then $xy \in R$. By X0, $yx \in R$.

First, let $\tau = *$. Then $v = x$. We see that $yxv \in T$.

Now, let $\tau \neq *$. Then there exists $\alpha \in \Sigma$ such that $uv\alpha x \in R$. By X1, $uv \in R$. If $u = y$, then $xu \in R$ and therefore, $xyu \in T$. If $y = v$, then $uy \in R$ and therefore, $uvy \in T$. Let $u \neq y \neq v$.

It there exists $\varrho \in \Sigma$ such that $u\varrho yx \in R$, then, by X1, $xy\overline{\varrho}u \in R$ and therefore, $xyu \in T$. If there exists $\sigma \in \Sigma$ such that $uv\sigma y \in R$, then $uvy \in T$. Assume that

$u\varrho yx \notin R$ for all $\varrho \in \Sigma$ and $uv\sigma y \notin R$ for all $\sigma \in \Sigma$. By virtue of X4, $v\alpha xy$. According to X1, $yx\overline{\alpha}v \in R$ and therefore, $yxv \in T$.

We have proved that $T \in \mathcal{T}$. This means that Φ is a mapping into $\mathcal{T}$.

(II) We will prove that

$$\text{if } R_1 \neq R_2, \text{ then } \Phi(R_1) \neq \Phi(R_2), \text{ for any } R_1, R_2 \in \mathcal{R}. \tag{14}$$

Suppose, to the contrary, that there exist distinct $R, R' \in \mathcal{R}$ such that $\Phi(R) = \Phi(R')$. Without loss of generality, let $R - R' \neq \emptyset$. Let m be the minimal $k \geqslant 1$ with the property that $(R - R') \cap V^{k+1} \neq \emptyset$. Combining the fact that $R \subseteq \Pi_N$ with X0, we see that $m \geqslant 2$. There exist $u, v, x \in V$ and $\alpha \in \Sigma$ such that

$$uv\alpha x \in (R - R') \cap V^{m+1}.$$

Hence $v \neq x$. Since $uv\alpha x \in R$, combining X1 and X2, we get $v\alpha x \in R$. By the definition of m, $v\alpha x \in R'$. Since $uv\alpha x \in R$, $uvx \in \Phi(R)$. Hence $uvx \in \Phi(R')$. Recall we have that $v \neq x$. There exists $\beta \in \Sigma$ such that $uv\beta x \in R'$. Since $v\alpha x \in R'$, X3 implies that $uv\alpha x \in R'$, which is a contradiction. Thus (14) is proved.

(III) Consider an arbitrary $T \in \mathcal{T}$. Denote $R = \Psi(T)$. By Corollary 2, $R \subseteq \Pi_N$. Since T is associated with G, Y0 implies that R fulfils X0. By Lemma 2, R fulfils X1. Combining (4) with X1, we see that R fulfils X2. By virtue of Lemma 5, R fulfils X5. We will show that R fulfils X3 and X4.

(Verification of X3). Let $\alpha u\beta x\gamma$, $u\delta x \in R$, where $u, x \in V$ and $\alpha, \beta, \gamma, \delta \in \Sigma$. By X2, $\alpha u\beta x \in R$. Since $\alpha u\beta x$ and $u\delta x$ are processes in T, we have

$$\alpha u\beta xTx \quad \text{and} \quad u\delta xTx.$$

If $\alpha = *$, then $\alpha u\delta x \in R$. Let $\alpha \neq *$. Then αuTx. By (3), $\alpha u\delta xTx$ and therefore, $\alpha u\delta x \in R$.

By X1, $\overline{\gamma}x\overline{\beta}u\overline{\alpha}$, $x\overline{\delta}u\overline{\alpha} \in R$. If $\gamma = *$, then $\overline{\gamma}x\overline{\delta}u\overline{\alpha} \in R$ and, by X1, $\alpha u\delta x\gamma \in R$. Let $\gamma \neq *$. Put $v = c(u\overline{\alpha})$. We have

$$x\overline{\delta}u\overline{\alpha}Tv \quad \text{and} \quad \overline{\gamma}xTv.$$

By virtue of (3), $\overline{\gamma}x\overline{\delta}u\overline{\alpha}Tv$. Since $v = c(u\overline{\alpha})$, $\overline{\gamma}x\overline{\delta}u\overline{\alpha} \in R$. By X1, $\alpha u\delta x\gamma \in R$.

(Verification of X4). Let $u, v, x \in V$ and $\alpha \in \Sigma$. Assume that $uv\alpha x$, $xy \in R$, $u \neq y \neq v$, $u\varrho yx \notin R$ for all $\varrho \in \Sigma$ and $uv\sigma y \notin R$ for all $\sigma \in \Sigma$. Then uvx, $xyy \in T$, $x \neq v$ and $uvy \notin T$. Moreover, by virtue of X1, we have $xy\overline{\varrho}u \notin R$ for all $\overline{\varrho} \in \Sigma$. Hence $xyu \notin T$. By X4, $yxv \in T$. Recall that $x \neq v$. According to Lemma 5, there exists $\tau \in \Sigma$ such that $x\tau v \in R$. Since $yxv \in T$, (3) implies that $yx\tau v \in R$. By

X1, $v\overline{\tau}xy \in R$. Recall that $uv\alpha x \in R$. Combining X1 and X2, we have $v\alpha x$. Since $v\overline{\tau}xy \in R$, X3 implies that $v\alpha xy \in R$.

We have proved that $R \in \mathcal{R}$. This means that Ψ is a mapping into $\mathcal{R}$.

(IV) Consider an arbitrary $T_0 \in \mathcal{T}$. It is clear that $\Phi(\Psi(T_0)) \subseteq T_0$. Applying Lemma 5 and (3), we easily get $T_0 \subseteq \Phi(\Psi(T_0))$. Hence $\Phi(\Psi(T)) = T$ for each $T \in \mathcal{T}$.

Combining the results of (I)–(IV), we obtain the statement of the theorem. Thus the proof is complete. □

Lemma 6. *Consider $T \in \mathcal{T}_0$. Assume that T fulfils Axioms* Y0, Y1 *and* Y4. *Then it fulfils Axiom* Y7 *if and only if it fulfils the following Axiom* Y7+*:*

$$\text{Y7}+ \qquad \textit{if } uvx,\ vuy,\ xyu \in T, \quad \textit{then } yxv \in T \quad (\forall u, v, x, y \in V).$$

P r o o f. Let T fulfil Y7. Consider arbitrary $u, v, x, y \in V$ and assume that $uvx, vuy, xyu \in T$. By virtue of Y0, $yxx \in T$. Y7 implies that $yxv \in T$. Thus T fulfils Y7+.

Conversely, let T fulfil Y7+. Consider arbitrary $u, v, x, y \in V$ and assume that $uvx, vuy, xyy \in T$. Since $uvx, xyy \in T$, Y4 implies that either $xyu \in T$ or $yxv \in T$ or $uvy \in T$. We will show that $xyu \in T$. Suppose that either $uvy \in T$ or $yxv \in T$. If $uvy \in T$, then, by Y1, we have $vuy \notin T$, which is a contradiction. Hence $yxv \in T$. Since $uvx, vuy \in T$, Y7+ implies that $xyu \in T$. Thus T fulfils Y7, which completes the proof. □

Theorem 2. *Let $R \in \mathcal{R}$. Denote $T = \Phi(R)$. Then R fulfils Axioms* X6 *and* X7 *if and only if T fulfils Axioms* Y6 *and* Y7.

P r o o f. (I) Let R fulfil X6 and X7. We will prove that T fulfils Y6 and Y7. Consider arbitrary $u, v, x, y \in V$. Assume that $uvx \in T$. By virtue of Y0, $uvv \in T$. Since $R = \Psi(T)$, $uv \in R$.

(Verification of Y6). Let $uvy \in T$. Assume that $y \neq v$. Then there exists $\beta \in \Sigma$ such that $uy\beta v \in R$. By X6, $uv \notin R$, which is a contradiction. Thus $y = v$. We see that T fulfils Y6.

(Verification of Y7). We first show that T fulfils Y7+. Assume that $vuy, xyu \in T$. By Y0, $yxx \in T$. If $v = x$, then $yxv \in T$. Suppose that $v \neq x$. There exists $\alpha \in \Sigma$ such that $uv\alpha x \in R$. First, let $u = y$. Then $xuu \in T$ and thus, by Y0, $uxx \in T$. By Y6, $uvx \notin T$, which is a contradiction. We have $u \neq y$. Then there exist $\beta, \gamma \in \Sigma$ such that $vu\beta y$, $xy\gamma u \in R$. Combining X1 and X2, we get $u\beta y$, $u\overline{\gamma}yx \in R$. By X3, $u\beta yx \in R$. Recall that $uv\alpha x$, $vu\beta y \in R$. By virtue of X7, $v\alpha xy \in R$. By X2,

$yx\overline{\alpha}v \in R$. Hence $yxv \in T$. We have shown that T fulfils Y7+. By Lemma 6, it fulfils Y7.

(II) Let T fulfil Y6 and Y7. We will prove that R fulfils X6 and X7. Consider arbitrary $u, v, x, y \in V$ and an arbitrary $\alpha \in \Sigma$. Assume that $uv\alpha x \in R$. Then $uvx \in T$ and $v \neq x$.

(Verification of X6). Assume that $ux \in R$. Then $uxx \in T$. Since $uvx \in T$, Y6 implies that $v = x$, which is a contradiction. Thus $ux \notin R$.

(Verification of X7). Let $vu\beta y$, $u\beta yx \in R$. Then $vuy \in T$. Since $uv\alpha x$, $u\beta yx \in R$, combining X1 and X2 we get $v\alpha x$, $yx \in R$. Hence $yxx \in T$.

Combining the fact that vuy, uvx, $yxx \in T$ with Y7, we get $yxv \in T$. Since $v \neq x$, there exists $\tau \in \Sigma$ such that $yx\tau v \in R$. By X1, $v\overline{\tau}xy \in R$. Since $v\alpha x \in R$, X3 implies that $v\alpha xy \in R$.

We have proved that R fulfils X6 and X7, which completes the proof of the theorem. □

By virtue of Theorem 2, Theorem B immediately follows from Theorem A. And similarly, by virtue of Theorem 2, Theorem A immediately follows from Theorem B.

Remark 3. Every step in G can be interpreted as a signpost showing a shortest path from a vertex to another vertex in G. Then every step uvx in G can be interpreted as the signpost located at u, "oriented" to v and signed by x.

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