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A DANIELL INTEGRAL APPROACH TO NONSTANDARD KURZWEIL-HENSTOCK INTEGRAL

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Abstract. A workable nonstandard definition of the Kurzweil-Henstock integral is given via a Daniell integral approach. This allows us to study the HL class of functions from [9]. The theory is recovered together with a few new results.

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1. Introduction

The usefulness of nonstandard analysis lies mainly in the simplification of some limit arguments in classical analysis. This fact is nicely shown in the treatment of stochastic calculus (see [1]). We attempt here to treat the Henstock-Kurzweil integral using this framework, obtaining some simplifications of the proofs of the classical results and finding new ones. Let us start giving an informal summary of nonstandard analysis. For a more specific treatment, see [1], [7], [3]. For other approaches to Henstock-Kurzweil integral, see [2], [5] and [4].

1.1. A sketch of nonstandard analysis. We assume that we can construct an extension ${}^*\mathbb{R}$ of $\mathbb{R}$, containing infinitesimal and infinitely big elements with respect to the standard reals $\mathbb{R}$. See [7] for an explicit construction using the technique of ultrafilters and ultraproducts. This construction endows ${}^*\mathbb{R}$ with the same field structure as $\mathbb{R}$. In ${}^*\mathbb{R}$ we have the (equivalence) relation $x \approx y$ (which reads x is infinitely close to y), if they differ by an infinitesimal element.

We take the opportunity to stress that there does not exist *the* set of the hyperreals (as opposed to the set of reals which is the unique complete ordered field up

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to isomorphisms); instead, depending upon the choice of the ultrafilter, we may get different objects, and they may even have different cardinalities. The point of nonstandard analysis is not the uniqueness of the nonstandard objects (or "entities" in the jargon of the area) but rather the invariance of the set of sentences that are "simultaneously" true in both worlds (standard and nonstandard). We put quotes around word "simultaneously" because since we are talking about different worlds, the names and predicates that appear in the sentences cannot be the same; nonetheless the nonstandard world is aiming at being an extension of the original one, hence it contains sorts of "replicas" of the standard entities.

So we will work with two *universes*, a standard one and a nonstandard one. These will consist of two collections of sets V_{st} and V_{ns}, containing the standard reals $\mathbb{R}$ and its nonstandard counterpart ${}^*\mathbb{R}$, respectively. Each of these collections (which will be both denoted by V) must satisfy some requirements: if $A, B \in V$ then the cartesian product $A \times B$ is also in V; if $A \in V$ and $B \subset A$, then $B \in V$, and also the powerset of A, $P(A) \in V$; if $A, B \in V$, the set of all functions from A to B, $\mathrm{Fun}(A, B)$, is in V.

These universes are related by a map ${}^*\colon V_{st} \to V_{ns}$ which sends sets A to sets ${}^*A$, functions $f\colon A \to B$ to functions ${}^*f\colon {}^*A \to {}^*B$ and also satisfies ${}^*(f(a)) = {}^*f({}^*a)$, for all $a \in A$; ${}^*\mathrm{id}_A = \mathrm{id}_{{}^*A}$ (i.e. * sends the identity function of A to the identity function of ${}^*A$) and * respects composition, i.e. ${}^*(g \circ f) = {}^*g \circ {}^*f$ (that is, * is a *functor*); ${}^*(A_1 \times A_2) = {}^*A_1 \times {}^*A_2$, and if $p_i\colon A_1 \times A_2 \longrightarrow A_i$ is the projection on the i^{th} coordinate then ${}^*p_i\colon {}^*A_1 \times {}^*A_2 \longrightarrow {}^*A_i$ is also the projection on the i^{th} coordinate. It must also satisfy the *Transfer Principle* which we explain now. A mathematical assertion about elements of one of these worlds will be obtained by using only two kinds of quantification: "there exists" and "for all". These will range over the elements of some set in V. A sentence can then be construed using boolean combinations of assertions, quantification over assertions, the simplest kind of assertion being equations or relations on the elements of some set in the universe.

When a nonstandard sentence contains only those replicas of standard entities we call it the *-transform of the corresponding standard sentence, that is, the sentence which has the same structure as the nonstandard one and has standard names and predicates substituted for the replicas they came from. The invariance of the set of sentences simultaneously true in both worlds is the essence of the *transfer principle* which we now informally state:

The Transfer Principle: If a standard assertion is true then its *-transform is also true in the nonstandard world; and conversely, if a true nonstandard sentence is the *-transform of a standard assertion then the latter is true as well.

We caution the reader that not all nonstandard sentences are *-transforms of standard ones. In fact, nonstandard methods owe their richness to this fact.

There is another important property of nonstandard superstructures we will employ many times in the sequel. Basically it allows us to embed any standard entity in a nonstandard one that is *hyper-finite*, that is, is in a one-to-one onto "good" correspondence with a set like $\{1, \ldots, \omega\} \subseteq {}^*\mathbb{N}$ for some $\omega \in {}^*\mathbb{N}$, where ${}^*\mathbb{N}$ is the replica of the usual natural numbers. We point out the possibility of the standard entity being infinite. The quotes for "good" mean that the correspondence must itself be an element of a replica of a standard entity; these kinds of "good" objects are usually called *internal* entities. These are entities $A \in V_{ns}$ for which there is a set $B \in V_{st}$ such that $A \in {}^*B$. A theorem (due to H. J. Keisler; see [7]) gives a useful characterization of these sets. It says that a subset of an internal set defined by a statement involving only internal entities is also an internal set. Naturally there are many nonstandard objects that are not internal. They are called *external* entities. For example, the set $\mathbb{N}$ of the usual natural numbers when viewed from inside the nonstandard world is an external entity. The property we have been talking about is called *enlargement* and all of our nonstandard superstructures will be enlargements of the standard ones. Also this property is closely connected with the ultrafilter employed in the construction of the nonstandard superstructure. Let us state the enlargement property:

The Enlargement Property: Given any superstructure it is always possible to build up its nonstandard analog in such a way that any object of the standard world can be embedded in a hyper-finite nonstandard internal entity. This is equivalent to saying that if P is a standard binary relation on $A \times B$ such that for all $a_1, \ldots, a_n \in A$ there is some $b \in B$ such that $(a_j, b) \in P$ for all $j = 1, \ldots, n$.

1.2. A sketch of the Henstock-Kurzweil integral. This integral is a generalized form of the usual Riemann integral as a limit of Riemann sums, the difference being in the net of partitions used to take this limit. Let us make this more precise. (See [9].)

A *gauge* in $[a, b]$ is a (strictly) positive function $\delta\colon [a, b] \longrightarrow \mathbb{R}$. A *tagged interval* is a pair $T = (\tau, J)$ where $J = [c, d]$ and $\tau \in J$. A *partition* of $[a, b]$ is a set $P = \{T_i\colon 1 \leqslant i \leqslant n\}$ of tagged intervals $T_i = (\tau_i, J_i)$ with $J_i = [x_{i-1}, x_i]$, $x_0 = a$ and $x_n = b$. Let $\mathcal{P}$ denote the set of partitions of $[a, b]$. Given a gauge δ, a partition D is called *δ-fine* if for all $T = (\tau, J) \in D$, $J \subset [\tau - \delta(\tau), \tau + \delta(\tau)]$. A function $f\colon [a, b] \longrightarrow \mathbb{R}$ is said to be *integrable* (in the sense of Henstock and Kurzweil) if there is a real number A (which we denote by $\int_a^b f$) such that for all $\varepsilon > 0$

there is a gauge δ such that for all δ-fine partitions P, $|S(f,P) - A| < \varepsilon$, where $S(f,P) = \sum_{i=1}^{n} f(\tau_i)(x_i - x_{i-1})$.

Now we are going to transfer these concepts to the nonstandard setting. We assume that we are working with an enlargement V_{ns}. The set of gauges $\mathcal{G}$ can be partially ordered by the relation $\delta \geqslant \eta$ if for all $x \in [a,b]$, $\delta(x) \geqslant \eta(x)$. This relation is clearly concurrent on $\mathcal{G}$. So, by the enlargement property, there is some $\delta \in {}^*\mathcal{G}$ such that $\delta \leqslant {}^*\eta$, for all $\eta \in \mathcal{G}$. We call such δ a *special $*$-gauge*. An element of ${}^*\mathcal{P}$ will be called an internal partition and the notion of a δ-fine partition or division is the transfered of the standard one. The sums $S(f,P)$ can be viewed as functions $S\colon \mathrm{Fun}([a,b],\mathbb{R}) \times \mathcal{P} \longrightarrow \mathbb{R}$, so we have the starred version ${}^*S\colon {}^*\mathrm{Fun}([a,b],\mathbb{R}) \times {}^*\mathcal{P} \longrightarrow {}^*\mathbb{R}$.

Theorem 1.1. (See [4].) *The following statements are equivalent:*

1. *the function $f\colon [a,b] \longrightarrow \mathbb{R}$ is integrable, with integral $\int_a^b f = A$;*
2. *there is a $*$-gauge $\delta \in {}^*\mathcal{G}$ such that for all δ-fine internal partition P, ${}^*S({}^*f,P) \approx A$;*
3. *there is a special $*$-gauge δ such that for all δ-fine internal partition P, ${}^*S({}^*f,P) \approx A$;*
4. *for all special $*$-gauges δ, for all δ-fine internal partitions P, ${}^*S({}^*f,P) \approx A$.*

There is a curious property shared by all the δ-fine internal partitions for a special $*$-gauge δ.

Proposition 1.1. (See [4].) *If δ is special and P is an internal δ-fine partition, then for each standard $c \in [a,b]$ there is $(\tau,J) \in P$ such that $c = \tau$.*

2. The integration structures

In this section we imitate the Daniell integral approach to nonstandard integration, due to P. A. Loeb (see [7]). But we include a small modification in order to allow nonabsolutely integrable functions.

We start with an internal δ-fine partition Δ for a special $*$-gauge δ and we let P to be the set of all τ such that for some J, $(\tau,J) \in \Delta$. Let L be the set of all internal functions $f\colon P \longrightarrow {}^*\mathbb{R}$ and $I\colon L \longrightarrow {}^*\mathbb{R}$ the (internal) function $I(f) = {}^*S(f,P)$. Let L_0 be the set of all functions $g\colon P \longrightarrow \mathbb{R}$ such that for all standard $\varepsilon > 0$, there is some $\psi \in L$ such that $|g(t)| < \psi(t)$ for all $t \in P$, and $I(\psi) < \varepsilon$. Let $\hat{L}$ be the set of all functions $f\colon P \longrightarrow \mathbb{R}$ (notice that f assumes only standard values) such that there are $\varphi \in L$ and $g \in L_0$, $f = \varphi + g$, and $I(\varphi)$ is a bounded nonstandard real (i.e. there is some $r \in \mathbb{R}$, $r \approx I(\varphi)$). Notice that in [7] it is required that $I(|\varphi|)$ is bounded. For $f \in \hat{L}$ define $\hat{I}(f) = {}^\circ I(\varphi)$, if $f = \varphi + g$, $\varphi \in L$ and $g \in L_0$. This is well defined:

Proposition 2.1. *If $f \in \hat{L}$ is decomposed as $f = \varphi_1 + g_1 = \varphi_2 + g_2$, $g_1, g_2 \in L_0$ and $\varphi_1, \varphi_2 \in L$, then ${}^\circ I(\varphi_1) = {}^\circ I(\varphi_2)$.*

Proof. It is immediate from the fact that $\varphi_1 - \varphi_2 \in L_0$. But this follows from the fact that $g_2 - g_1 \in L_0$. □

Let us work towards proving a monotone convergence theorem.

Proposition 2.2. *Assume that $\alpha \leqslant \beta$ are elements of $\hat{L}$, then it is possible to choose decompositions $\alpha = \varphi_\alpha + h_\alpha$, $\beta = \varphi_\beta + h_\beta$ where $\varphi_\alpha, \varphi_\beta \in L$, $h_\alpha, h_\beta \in L_0$ with $\varphi_\alpha \leqslant \varphi_\beta$.*

Proof. Let $\alpha = \varphi_a + h_a$, $\beta = \varphi_b + h_b$ be decompositions of α and β. Then $\alpha \wedge \beta - \varphi_a \wedge \varphi_b \in L_0$,[1], hence there is a decomposition of α of the form $\alpha = \psi + h$ with $\psi \leqslant \varphi_b$. □

Proposition 2.3. *Let $\alpha, \beta, f, g \in \hat{L}$ with $\alpha \leqslant f$, $g \leqslant \beta$, then $f \wedge g$, $f \vee g \in \hat{L}$.*

Proof. If $f = \varphi_f + h_f$, $g = \varphi_g + h_g$ are decompositions then $f \wedge g - \varphi_f \wedge \varphi_g \in L_0$. Hence it suffices to show that ${}^o I(\varphi_f \wedge \varphi_g) \in \mathbb{R}$. But we may assume that $\varphi_\alpha \leqslant \varphi_f$, $\varphi_g \leqslant \varphi_\beta$ where $\alpha = \varphi_\alpha + h_\alpha$ and $\beta = \varphi_\beta + h_\beta$ are decompositions. The result then follows. □

Proposition 2.4. *If $\alpha \leqslant f \leqslant \beta$ with $\alpha, \beta \in \hat{L}$ and $f = \varphi + h$ with $\varphi \in L$ and $h \in L_0$, then $f \in \hat{L}$.*

Proof. It suffices to observe that we may assume that $\varphi_\alpha \leqslant \varphi \leqslant \varphi_\beta$ where $\alpha = \varphi_\alpha + h_\alpha$ and $\beta = \varphi_\beta + h_\beta$ are decompositions. □

Now, an easy modification of the proof of Theorem 1.15 in Chapter *IV* of [7] gives the following theorem.

Theorem 2.1. (Monotone Convergence) *Suppose that $(f_n \in \hat{L}\colon n \in \mathbb{N})$ is a monotone increasing sequence converging pointwise to some f and such that $\hat{I}(f_n)$ converge to some real number A. Then $f \in \hat{L}$ and $\hat{I}(f) = A$.* □

[1] For a proof of this, see [7] Chapter IV, Lemma 1.8, page 169.

And from this, the usual standard argument (as for instance in [9], Theorem 4.3) yields the following form of the Dominated Convergence Theorem.

Theorem 2.2. (Dominated Convergence) *Suppose that* $(f_n \in \hat{L}\colon\ n \in \mathbb{N})$ *is a sequence converging pointwise to some* f *and such that* $g \leqslant f_n \leqslant h$ *for all* $n \in \mathbb{N}$ *and for some* $g, h \in \hat{L}$. *Then* $f \in \hat{L}$ *and* $\hat{I}(f) = \lim n \to \infty \hat{I}(f_n)$. □

Now let us work towards a characterization of a subclass of the Henstock-Kurzweil integrable functions for which their integrals coincide with the limit of certain truncations of these functions. This class is called HL in [9], Section 18.

Definition 2.1. Let $\hat{\mathcal{L}} \triangleq \{A \subseteq {}^*[0,1]\colon\ 1_A \in \hat{L}\}$. This is the class of *measurable sets* with respect to $\hat{L}$.

In the following we let $\hat{L}^+$ denote the set of all nonnegative $f \in \hat{L}$.

Proposition 2.5. *If* $f \in \hat{L}^+$ *and* $A = \{f > \alpha\}$ *where* $\alpha > 0$, *then* $A \in \hat{\mathcal{L}}$.

P r o o f. It suffices to consider the case $\alpha = 1$ (think of f/α). Since $0 \leqslant 1 \wedge f \leqslant f$ and $1 \wedge f - \varphi_1 \wedge \varphi_f \in L_0$, where $1 = \varphi_1 + h_1$ and $f = \varphi_f + h_f$ are decompositions, we see that $1 \wedge f \in \hat{L}^+$. Hence if $g = f - f \wedge 1$ and $B = \{g > 0\}$ we have $g \in \hat{L}^+$ and $A = B$. In other words it suffices to show that for any $f \in \hat{L}^+$ the set $\{f > 0\}$ is in $\hat{\mathcal{L}}$. So consider $1 \wedge nf$, $n \in \mathbb{N}$, then these functions are in $\hat{L}^+$ and $\hat{I}(1 \wedge nf) \leqslant \hat{I}(1 \wedge f) \leqslant \hat{I}(f)$. Since $1 \wedge nf \uparrow 1_B$ we get $1_B \in \hat{L}$ by completeness. □

Since for non-negative functions the definitions of integral agree we have the same classes $\hat{L}^+$ and $\hat{\mathcal{L}}$ for both integration structures.

Definition 2.2. $\hat{M}^+ \triangleq \{h \in \overline{\mathbb{R}}_+^{*[0,1]}\colon\ \forall n \in \mathbb{N}\ \forall A \in \hat{\mathcal{L}}\ \ h \wedge n1_A \in \hat{L}^+\}$.

As commented above the class $\hat{M}^+$ is the same for both schemes, so the same holds for the class $\hat{M}$ formed by differences between members of $\hat{M}^+$.

Definition 2.3. Given $f \in \hat{M}$ and $n \in \mathbb{N}$ we set $f_+^n = f_+ 1_{\{f_+ \leqslant n\}}$, $f_-^n = f_- 1_{\{f_- \leqslant n\}}$, $f^n = f_+^n - f_-^n$ and define $\hat{J}(f) \triangleq \lim\limits_{n \longrightarrow \infty} \hat{I}(f_+^n - f_-^n) = \lim\limits_{n \longrightarrow \infty} \hat{I}(f_n)$ whenever this last limit exists and is finite. Among the members of $\hat{L}$ we select those f for which it is true that $\hat{I}(f) = \lim\limits_{n \longrightarrow \infty} \hat{I}(f_n)$, collecting them in the class $\hat{L}_{\mathrm{ac}}$ (for accessible). The class of all functions of $\hat{M}$ for which $\hat{J}$ is defined will be denoted by $\hat{L}_1$.

We recall from [7] the definition of $\hat{L}$, which we will denote here by $\hat{L}_{\mathrm{old}}$. This is the set of all real valued functions f such that $f = \varphi + g$, where $\varphi \in L$ and $I|\varphi|$ is bounded by a standard real number. Note that $\hat{L}_{\mathrm{old}} \subseteq \hat{L}_{\mathrm{ac}} \subseteq \hat{L}_1$.

Definition 2.4. We say that $\varphi \in L$ is SHKL-integrable if ${}^{o}\varphi \in \hat{L}_1$ and $\hat{J}({}^{o}\varphi) = {}^{o}I(\varphi)$.

Theorem 2.3. *A function f is HKL-integrable iff ${}^{o*}f$ is SHKL-integrable.*

P r o o f. Suppose that f is HKL-integrable. Then $\int f = \lim\limits_{n\to\infty} \int f^n$, that is ${}^{o}I({}^*f) = \lim\limits_{n\to\infty} {}^{o}I^*f^n$. Hence ${}^{o*}f \in \hat{L}_1$ and $\hat{J}({}^*f) = {}^{o}I({}^*f)$.

The converse is obvious. □

Theorem 2.4. *A function $\varphi \in L$ is SHKL-integrable iff there exists an $\eta \in {}^*\mathbb{N}_\infty$ such that for all $\omega \in {}^*\mathbb{N}_\infty$ with $\omega \leqslant \eta$ we have $I(\varphi - \varphi^\omega) \approx 0$.*

P r o o f. Suppose that $\varphi \in L$ is SHKL-integrable. Then ${}^{o}I(\varphi) = \lim\limits_{n\to\infty, n\in\mathbb{N}} {}^{o}I(\varphi^n)$, hence it is true that the set $\{n \in {}^*\mathbb{N} \colon |I(\varphi - \varphi^n)| < 1/k\}$ where $k \in \mathbb{N}$ contains an interval of the form $[N_k, \eta_k]$ where $N_k \in \mathbb{N}$ and $\eta_k \in {}^*\mathbb{N}_\infty$. By saturation the intersection of all these intervals is non-empty containing an element, say η. It is easy to see that this element works.

Conversely, given an arbitrary standard positive ε we must have $|I(\varphi - \varphi^n)| < \varepsilon$ for all sufficiently large standard natural n (the set $\{n \in {}^*\mathbb{N} \colon |I(\varphi - \varphi^n)| < \varepsilon \vee n > \eta\}$ contains all non-standard natural numbers), consequently, since $I(\varphi^n)$ is finite, $I(\varphi)$ is finite and $\lim\limits_{n\to\infty, n\in\mathbb{N}} {}^{o}I(\varphi - \varphi^n) = 0$. It follows then that φ is SHKL-integrable. □

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