Hans-Gerd Leopold

Embedding of function spaces of variable order of differentiation in function spaces of variable order of integration

Czechoslovak Mathematical Journal, Vol. 49 (1999), No. 3, 633–644

Persistent URL: `http://dml.cz/dmlcz/127515`

Czechoslovak Mathematical Journal, 49 (124) (1999), 633–644

EMBEDDING OF FUNCTION SPACES OF VARIABLE ORDER OF DIFFERENTIATION IN FUNCTION SPACES OF VARIABLE ORDER OF INTEGRATION

Hans-Gerd Leopold, Jena

(Received December 5, 1996)

Abstract. The paper deals with embeddings of function spaces of variable order of differentiation in function spaces of variable order of integration. Here the function spaces of variable order of differentiation are defined by means of pseudodifferential operators.

Keywords: function spaces, embedding theorems

MSC 2000: 46E35

1. Introduction

The main intention of this paper is to show the interaction between function spaces of variable order of differentiation $W_p^{s(x)}$ and function spaces of variable order of integration $L_{q(x)}$.

As concerns the usual spaces H_p^s and L_q, for each of them we get only non satisfactory embedding results:

$$H_p^s(\mathbb{R}^n) \hookrightarrow L_{q(x)}(\mathbb{R}^n) \quad \text{if} \quad 1 < p \leqslant \inf\{q(x)\colon\ x \in \mathbb{R}^n\} \quad \text{and} \quad s \geqslant \frac{n}{p} - \frac{n}{\sup\{q(x)\colon\ x \in \mathbb{R}^n\}} \tag{1}$$

and

$$W_p^{s(x)}(\mathbb{R}^n) \hookrightarrow L_q(\mathbb{R}^n) \quad \text{if} \quad 1 < p \leqslant q \leqslant \infty \quad \text{and} \quad \inf\{s(x)\colon\ x \in \mathbb{R}^n\} \geqslant \frac{n}{p} - \frac{n}{q}, \tag{2}$$

Supported by DFG-contract Tr 374/1.

where neither result can be improved. This comes from the fact that the spaces H^s_p and L_q are translation invariant, in contrast to the spaces $W^{s(x)}_p$ and $L_{q(x)}$, neither of which has this important property.

So it seems natural to compare these both scales. We prove the following result:

$$\begin{aligned} W^{s(x)}_p(\mathbb{R}^n) \hookrightarrow L_{q(x)}(\mathbb{R}^n) \quad \text{if} \quad & 1 < p \leqslant \inf\{q(x)\colon\ x \in \mathbb{R}^n\} \quad \text{and} \\ & \inf\{[s(x) + \frac{n}{q(x)}]\colon\ x \in \mathbb{R}^n\} > \frac{n}{p}. \end{aligned} \tag{3}$$

It contains the non-limiting cases of (1) and (2) as special cases. On the other hand, in the case that both $s(x)$ and $q(x)$ are variable, (3) is more general, because it takes into consideration the local behavior and the relation between $s(x)$ and $q(x)$, too.

Function spaces $L_{q(x)}$ have been considered since 1991 in connection with boundary problems [KR], [ER]. In the case of the interval $[0,1]$ some elements of the theory appeared in [Sh] and [Ts] in connection with some problems in the approximation theory.

Function spaces $W^{s(x)}_p$ were considered for example in [Le2] in connection with a more general theory of function spaces of variable order of differentiation, where the special case $W^{s(x)}_p$ plays a role also in connection with problems in the probability theory [JL], [Ne].

The author would like to thank J. Rákosník for valuable discussions and helpful comments.

2. Function spaces $L_{q(x)}$

We will use a special case of function spaces $L_{q(x)}(\Omega)$ defined and described in [KR] and [ER].

Let $q\colon\ \mathbb{R}^n \to [1,\infty]$ be continuous and let on the set of all measurable functions f on $\mathbb{R}^n$

$$\varrho_q(f) = \int_{\mathbb{R}^n\setminus\Omega_\infty} |f(x)|^{q(x)}\,\mathrm{d}x + \operatorname*{ess\,sup}_{\Omega_\infty} |f(x)| \tag{4}$$

where $\Omega_\infty = \{x\colon\ q(x) = \infty\}$.

The generalized Lebesgue space $L_{q(x)}(\mathbb{R}^n)$ is the class of all functions f such that $\varrho_q(\lambda f) < \infty$ for some $\lambda = \lambda(f) > 0$.

$L_{q(x)}(\mathbb{R}^n)$ is a Banach space if endowed with the norm

$$\|f\,|L_{q(x)}\| = \inf\{\lambda > 0\colon\ \varrho_q(f/\lambda) \leqslant 1\}.$$

It is obvious that for $q(x) = q = const.$ this norm coincides with the usual L_q-norm. Further properties of $L_{q(x)}(\mathbb{R}^n)$ may be found in [KR].

We need the following lemma.

Lemma 1. *Let $1 \leqslant q_* \leqslant q(x) \leqslant q^* \leqslant \infty$ for all $x \in \mathbb{R}^n$. Then*

$$L_{q_*}(\mathbb{R}^n) \cap L_{q^*}(\mathbb{R}^n) \hookrightarrow L_{q(x)}(\mathbb{R}^n)$$

and there exists a constant $c > 0$ with

$$\|f|L_{q(x)}\| \leqslant c \max(\|f|L_{q_*}\|, \|f|L_{q^*}\|)$$

for all $f \in L_{q_*}(\mathbb{R}^n) \cap L_{q^*}(\mathbb{R}^n)$.

Proof. *Step 1.* Let $q^* < \infty$ and $\lambda > 0$, then

$$\begin{aligned}
\varrho_q(f/\lambda) &= \int\limits_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{q(x)} \mathrm{d}x \\
&\leqslant \int\limits_{\{x:\,|f(x)|\leqslant\lambda\}} \left|\frac{f(x)}{\lambda}\right|^{q_*} \mathrm{d}x + \int\limits_{\{x:\,|f(x)|>\lambda\}} \left|\frac{f(x)}{\lambda}\right|^{q^*} \mathrm{d}x \\
&\leqslant \left(\frac{\|f|L_{q_*}\|}{\lambda}\right)^{q_*} + \left(\frac{\|f|L_{q^*}\|}{\lambda}\right)^{q^*}.
\end{aligned}$$

If $\lambda = 2\max(\|f|L_{q_*}\|, \|f|L_{q^*}\|)$ we get

$$\varrho_q(f/\lambda) \leqslant (1/2)^{q_*} + (1/2)^{q^*} \leqslant 1$$

and consequently

$$\|f|L_{q(x)}\| \leqslant 2\max(\|f|L_{q_*}\|, \|f|L_{q^*}\|).$$

Step 2. In the case $q^* = \infty$ we have $f \in L_\infty(\mathbb{R}^n)$, that is $\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)| = \|f|L_\infty\| < \infty$. Let again $\lambda = 2\max(\|f|L_{q_*}\|, \|f|L_\infty\|)$, then

$$\begin{aligned}
\varrho_q(f/\lambda) &= \int\limits_{\mathbb{R}^n\setminus\Omega_\infty} \left|\frac{f(x)}{\lambda}\right|^{q(x)} \mathrm{d}x + \operatorname*{ess\,sup}_{\Omega_\infty} \left|\frac{f(x)}{\lambda}\right| \\
&\leqslant \left(\frac{\|f|L_{q_*}\|}{\lambda}\right)^{q_*} + \frac{\|f|L_\infty\|}{\lambda} \\
&\leqslant (1/2)^{q_*} + 1/2 \leqslant 1
\end{aligned}$$

and this again yields

$$\|f|L_{q(x)}\| \leqslant 2\max(\|f|L_{q_*}\|, \|f|L_\infty\|).$$

$\square$

Remark 1. The same is true with Ω instead of $\mathbb{R}^n$. In case of q_* and q^* variable and $q^* < \infty$ similar results are contained in [KR]—Corollary 2.2. But they are not sufficient for our purpose. For constants q_* and q^*, $|\Omega_\infty| = 0$ and a little different definition of the norm $\|\cdot|L_{q(x)}\|$, such a result is contained also in [Sa], Lemma 3.6.

Corollary 1. *Let Ω be an open, non-empty set, $1 \leqslant q_\Omega \leqslant q(x) \leqslant q^\Omega \leqslant \infty$ for $x \in \Omega$. Then there again exists a constant $c > 0$ with*

$$\|g|L_{q(x)}\| \leqslant c\max(\|g|L_{q_\Omega}\|, \|g|L_{q^\Omega}\|)$$

for all $g \in L_{q_\Omega}(\mathbb{R}^n) \cap L_{q^\Omega}(\mathbb{R}^n)$ with $\operatorname{supp} g \subset \Omega$.

In virtue of the proof of Lemma 1, this is obvious.

Let $q_* = \inf\{q(x)\colon x \in \mathbb{R}^n\}$ and $q^* = \sup\{q(x)\colon x \in \mathbb{R}^n\}$. If additionally $1 < p \leqslant q_*$ and $s \geqslant \frac{n}{p} - \frac{n}{q^*}$, then by the classical embedding theorem we have

$$H^s_p(\mathbb{R}^n) \hookrightarrow L_{q_*}(\mathbb{R}^n) \qquad \text{and} \qquad H^s_p(\mathbb{R}^n) \hookrightarrow L_{q^*}(\mathbb{R}^n)$$

and now by Lemma 1 the embedding (1)

$$H^s_p(\mathbb{R}^n) \hookrightarrow L_{q(x)}(\mathbb{R}^n).$$

3. Function spaces $W^{s(x)}_p$

The function spaces $W^{s(x)}_p$, defined in the sequel, are a special case of function spaces of variable order of differentiation. Such spaces are defined and studied in [Le1] and [Le2]—see there for details, other examples and more references. We will use also some notation and properties from the theory of pseudodifferential operators. We refer to [Kg], [Ta] and to [Le1], [Le2] where the needed properties are also collected.

Let $\mathcal{B}(\mathbb{R}^n)$ denote the set of all C^∞-functions whose derivatives are all bounded on $\mathbb{R}^n$. Further let $s(x) \in \mathcal{B}(\mathbb{R}^n)$ be real-valued with $m = \sup\{s(x)\colon x \in \mathbb{R}^n\}$ and $m' = \inf\{s(x)\colon x \in \mathbb{R}^n\} > 0$. With respect to the embedding in Section 4, the assumption $m' > 0$ is natural.

Then the pseudodifferential operator $\Lambda^{s(x)}(D_x)$ is defined by

$$\Lambda^{s(x)}(D_x)u(x) = (2\pi)^{-n} \int e^{ix\xi}(1+|\xi|^2)^{s(x)/2}\hat{u}(\xi)\, d\xi \qquad \text{for } u \in S(\mathbb{R}^n),$$

where $S(\mathbb{R}^n)$ denotes the Schwartz class, and $\hat{u}(\xi) = \int e^{-iy\xi}u(y)\, dy$ is the Fourier transform of u.

$\Lambda^{s(x)}(D_x)$ can be extended to a continuous operator from $S'(\mathbb{R}^n)$ into $S'(\mathbb{R}^n)$, the space of all complex-valued tempered distributions on $\mathbb{R}^n$.

$\Lambda^{s(x)}(D_x)$ is a pseudodifferential operator belonging to the Hörmander class $\Psi^m_{1,\delta}$ for arbitrary $0 < \delta < 1$. Furthermore we have

$$|D^\alpha_\xi D^\beta_x \langle\xi\rangle^{s(x)}| \leqslant c_{\alpha,\beta} \langle\xi\rangle^{s(x)} \langle\xi\rangle^{-|\alpha|+\delta|\beta|}$$

and

$$\langle\xi\rangle^{s(x)} \geqslant \langle\xi\rangle^{m'},$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, $D^\beta_x = (-i)^{|\beta|}\partial^\beta_x$ and $0 < \delta < 1$. Consequently $\Lambda^{s(x)}(D_x)$ is hypoelliptic and we can construct a parametrix $Q(x, D_x) \in \Psi^{-m'}_{1,\delta}$ such that

$$\begin{aligned} \Lambda^{s(x)}(D_x)Q(x, D_x) &= I + R_R(x, D_x), \\ Q(x, D_x)\Lambda^{s(x)}(D_x) &= I + R_L(x, D_x) \end{aligned} \tag{5}$$

with $R_R, R_L \in \Psi^{-\infty}$. The class $\Psi^{-\infty} = \bigcap\limits_m \Psi^m_{1,\delta}$ is independent of $\delta < 1$. Furthermore

$$Q(x, D_x)u(x) = (2\pi)^{-n} \int e^{ix\xi} q(x, \xi)\hat{u}(\xi)\, d\xi \qquad \text{for } u \in S(\mathbb{R}^n),$$

and for the symbol $q(x, \xi)$ the inequality

$$|D^\alpha_\xi D^\beta_x q(x, \xi)| \leqslant c_{\alpha,\beta} \langle\xi\rangle^{-s(x)} \langle\xi\rangle^{-|\alpha|+\delta|\beta|} \tag{6}$$

holds for any α, β and all $x \in \mathbb{R}^n_x$, $\xi \in \mathbb{R}^n_\xi$ with $|\xi| \geqslant R_q$. See [Kg] Sect. 2, §5 or [Ta] Chapt. III, §5 for details.

For $1 < p < \infty$, the function space $W^{s(x)}_p(\mathbb{R}^n)$ is defined in a natural way by

$$W^{s(x)}_p(\mathbb{R}^n) = \{u\colon\ u \in L_p(\mathbb{R}^n) \text{ and } \|u|W^{s(x)}_p\| < \infty\},$$

$$\|u|W^{s(x)}_p\| = \|\Lambda^{s(x)}(D_x)u|L_p\| + \|u|L_p\|.$$

These spaces are a special case of function spaces of variable order $W^{1,a}_p(\mathbb{R}^n)$ defined in [Le2]. The case $s(x) \in \mathcal{B}(\mathbb{R}^n)$ was considered also separately in connection with problems in the probability theory in [KN]. From [Le2] and [Le1] we get immediately the following properties:

$W_p^{s(x)}(\mathbb{R}^n)$ is a Banach space in which $S(\mathbb{R}^n)$ is dense, the dual space and the interpolation spaces can be described explicitly and furthermore we have

$$H_p^m(\mathbb{R}^n) \hookrightarrow W_p^{s(x)}(\mathbb{R}^n) \hookrightarrow H_p^{m'}(\mathbb{R}^n).$$

The following lemma gives a justification for the name variable order of differentiation. A similar result for more complicated spaces of Besov type is contained in [Le1] Theorem 7.

Lemma 2. *Let Ω be an open subset of $\mathbb{R}^n$ and $s \leqslant s(x)$ for all $x \in \Omega$. If $1 < p < \infty$, $\sigma > 0$ and $\varphi \in C^\infty(\mathbb{R}^n)$ with $(\operatorname{supp}\varphi)_\sigma \subset \Omega$ where $(\operatorname{supp}\varphi)_\sigma = \{y\colon\ y = x + h,$ $x \in \operatorname{supp}\varphi,\ |h| < \sigma\}$, then there exists a positive constant c_σ depending on σ such that*

$$\|\varphi u | H_p^s\| \leqslant c_\sigma \|\varphi u | W_p^{s(x)}\|$$

holds for all $u \in L_p(\mathbb{R}^n)$.

Proof. Let φ and σ be as described above. Then there exists $\psi \in C^\infty(\mathbb{R}^n)$ such that $\psi(x) = 1$ on $\operatorname{supp}\varphi$, $\operatorname{supp}\psi \subset \Omega$ and $|D^\gamma\psi(x)| \leqslant c_{\gamma,\sigma}$ uniformly for all $x \in \mathbb{R}^n$ and γ.

With the notation of (5) it follows that

$$\begin{aligned}\|\Lambda^s(D_x)\varphi u | L_p\| &= \|\Lambda^s(D_x)\psi\varphi u | L_p\| \\ &\leqslant \|\Lambda^s(D_x)\psi Q(x, D_x)\Lambda^{s(x)}(D_x)\varphi u | L_p\| \\ &\quad + \|\Lambda^s(D_x)\psi R_L(x, D_x)\varphi u | L_p\|.\end{aligned}$$

If $\Lambda^s(D_x)\psi(x)Q(x, D_x)$ and $\Lambda^s(D_x)\psi(x)R_L(x, D_x)$ belong to $\Psi^0_{1,\delta}$, then by a result of [Il], see also [Le2] Theorem 2, these operators are bounded on L_p and we have the desired result. For the second operator this is obvious, because $R_L \in \Psi^{-\infty}$. Denote by $b(x, \xi)$ the symbol of the first pseudodifferential operator. Then the composition formula for pseudodifferential operators—see [Kg] Chapter 2 §3, or [Le2] Theorem 1—gives us for arbitrary natural numbers N that

$$b(x, \xi) = \sum_{|\gamma| < N} \Big[1/\gamma!\, D_\xi^\gamma(\langle\xi\rangle^s)\partial_x^\gamma(\psi(x)q(x, \xi))\Big] + r_N(x, \xi)$$

where the pseudodifferential operator $R_N(x, D_x)$ defined by the symbol $r_N(x, \xi)$ is an element of $\Psi_{1,\delta}^{s-m'-(1-\delta)N}$.

We fix N so large that $R_N(x, D_x) \in \Psi^0_{1,\delta}$ holds.

Let $0 \leqslant |\gamma| < N$ and

$$b_\gamma(x,\xi) = D_\xi^\gamma(\langle\xi\rangle^s)\partial_x^\gamma(\psi(x)q(x,\xi)).$$

By an easy calculation and the use of (6) we get

$$|D_\xi^\alpha D_x^\beta b_\gamma(x,\xi)| \leqslant c\,\langle\xi\rangle^{-|\alpha|-(1-\delta)|\gamma|+|\beta|\delta}\,\chi_\psi(x)\,\langle\xi\rangle^{s-s(x)};$$

here χ_ψ is the characteristic function of the support of ψ. The assumption $s \leqslant s(x)$ for all $x \in \Omega$ and $\operatorname{supp}\psi \subset \Omega$ now gives $B_\gamma(x,D_x) \in \Psi_{1,\delta}^{-(1-\delta)|\gamma|} \subset \Psi_{1,\delta}^0$ for the operator $B_\gamma(x,D_x)$ defined by $b_\gamma(x,\xi)$, and this completes the proof. □

The next lemma is a special case of Theorem 4 in [Le2]. That is why we give here only an outline of the proof.

Lemma 3. *Let $C(x,D_x) \in \Psi_{1,\delta}^0$ and $\delta < 1$. Then for all p with $1 < p < \infty$ there exist a constant c' and integers l, k, all independent of $C(x,D_x)$ such that*

$$\|C(x,D_x)u\big|W_p^{s(x)}\| \leqslant c'|c|_{(l,k)}^{(0)}\|u\big|W_p^{s(x)}\|$$

holds for all functions $u \in W_p^{s(x)}(\mathbb{R}^n)$.

Here

$$|c|_{(l,k)}^{(0)} = \max_{|\alpha|\leqslant l,|\beta|\leqslant k}\sup_{x,\xi}\left\{\left|D_\xi^\alpha D_x^\beta c(x,\xi)\right|\langle\xi\rangle^{|\alpha|-\delta|\beta|}\right\}$$

denotes a suitable semi-norm of the symbol $c(x,\xi)$ of $C(x,D_x)$.

P r o o f. There exists, again by (5), $Q(x,D_x) \in \Psi_{1,\delta}^{-m'}$ such that

$$Q(x,D_x)\Lambda^{s(x)}(D_x) = I + R_L(x,D_x), \quad R_L \in \Psi^{-\infty},$$

and (6) hold. If $C(x,D_x) \in \Psi_{1,\delta}^0$, then we get by the composition rule for pseudodifferential operators and by (6)

$$\Lambda^{s(x)}(D_x)C(x,D_x)Q(x,D_x) = C(x,D_x) + D(x,D_x)$$

with $D(x,D_x) \in \Psi_{1,\delta}^{-(1-\delta)}$—see also [Le2] Lemma 1.

Now we use the relations $C(x,D_x) \in \Psi_{1,\delta}^0$, $D(x,D_x) \in \Psi_{1,\delta}^{-(1-\delta)} \subset \Psi_{1,\delta}^0$, $\Lambda^{s(x)}(D_x)C(x,D_x)R_L(x,D_x) \in \Psi^{-\infty} \subset \Psi_{1,\delta}^0$ and the fact that pseudodifferential operators of order zero yield bounded operators on L_p if $1 < p < \infty$ and $\delta < 1$

(see again [Il] or [Le2] Theorem 2, respectively) to obtain

$$\begin{aligned}\|C(x,D_x)u\big|W_p^{s(x)}\| &= \|\Lambda^{s(x)}(D_x)C(x,D_x)u\big|L_p\| + \|C(x,D_x)u\big|L_p\| \\ &\leqslant \|\Lambda^{s(x)}(D_x)C(x,D_x)Q(x,D_x)\Lambda^{s(x)}(D_x)u\big|L_p\| \\ &\quad + \|\Lambda^{s(x)}(D_x)C(x,D_x)R_L(x,D_x)u\big|L_p\| + \|C(x,D_x)u\big|L_p\| \\ &\leqslant \|C(x,D_x)\Lambda^{s(x)}(D_x)u\big|L_p\| + \|D(x,D_x)\Lambda^{s(x)}(D_x)u\big|L_p\| \\ &\quad + \|\Lambda^{s(x)}(D_x)C(x,D_x)R_L(x,D_x)u\big|L_p\| + \|C(x,D_x)u\big|L_p\| \\ &\leqslant c|c|^{(0)}_{(l,k)}\|\Lambda^{s(x)}(D_x)u\big|L_p\| + c'|c|^{(0)}_{(l,k)}\|u\big|L_p\|.\end{aligned}$$

The special dependence of the constants on $C(x,D_x)$ also comes from these results. $\square$

Corollary 2. *Let $s \leqslant s(x)$ on $\mathbb{R}^n$, then we have*

$$\Lambda^s(D_x) = \Lambda^s(D_x)Q(x,D_x)\Lambda^{s(x)}(D_x) - \Lambda^s(D_x)R_L(x,D_x)$$

and $\Lambda^s(D_x)Q(x,D_x) \in \Psi^0_{1,\delta}$, $\Lambda^s(D_x)R_L(x,D_x) \in \Psi^{-\infty}$. Lemma 3 then gives

$$W_p^{s(x)}(\mathbb{R}^n) \hookrightarrow W_p^s(\mathbb{R}^n).$$

Now again by the classical embedding theorem with $s = \inf\{s(x)\colon x \in \mathbb{R}^n\}$ and $1 < p \leqslant q \leqslant \infty$ we get the embedding (2)

$$W_p^{s(x)}(\mathbb{R}^n) \hookrightarrow L_q(\mathbb{R}^n) \quad \textit{if} \quad \inf\{s(x)\colon x \in \mathbb{R}^n\} \geqslant \frac{n}{p} - \frac{n}{q}.$$

4. The general embedding

Theorem 1. *Let $1 < p \leqslant q(x) < \infty$ for all x and let $q(x)$ be uniformly continuous on $\mathbb{R}^n$. Furthermore let $s(x) \in \mathcal{B}(\mathbb{R}^n)$ and $\inf\{[s(x) + \frac{n}{q(x)}]\colon x \in \mathbb{R}^n\} > \frac{n}{p}$. Then the continuous embedding*

$$W_p^{s(x)}(\mathbb{R}^n) \hookrightarrow L_{q(x)}(\mathbb{R}^n)$$

holds.

P r o o f. *Step 1.* Let $\varepsilon = \inf\{s(x) + \frac{n}{q(x)}\} - \frac{n}{p} > 0$ by assumption and $N = [\frac{4n}{\varepsilon}] + 1$. Then there exist numbers $q_j = \frac{N}{j}$, $j = 1,\ldots,N$, $1 = q_N < q_{N-1} < \ldots < q_1 < \infty$ with

$$\frac{1}{q_{j+1}} - \frac{1}{q_j} = \frac{1}{N} < \frac{\varepsilon}{4n}.$$

We define the following sets, which are open because $q(x)$ is continuous, and cover $\mathbb{R}^n$:

$$\begin{aligned} \Omega_1 &= \{x\colon\ q_3 < q(x) < \infty\}, \\ \Omega_j &= \{x\colon\ q_{j+2} < q(x) < q_{j-1}\} \quad \text{if } j = 2, 3, \ldots, N-3, \\ \Omega_{N-2} &= \{x\colon\ 1 < q(x) < q_{N-3}\}. \end{aligned}$$

Step 2. The function $q(x)$ is uniformly continuous with values in $(1, \infty)$. Consequently there exists $\sigma > 0$ such that for arbitrary $x, x' \in \mathbb{R}^n$ with $|x - x'| < \sigma$,

$$\left| \frac{1}{q(x)} - \frac{1}{q(x')} \right| < \frac{1}{N}. \tag{7}$$

Denote

$$(\Omega_j)_\sigma = \{y\colon\ y = x + h,\ x \in \Omega_j, |h| < \sigma\}.$$

Then we have

$$(\Omega_j)_\sigma \subset \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1} \qquad \text{for } j = 1, \ldots, N-2 \tag{8}$$

with $\Omega_0 = \Omega_{N-1} = \emptyset$.

Step 3. Resolution of unity with respect to $\{\Omega_j\}_{j=1}^{N-2}$. It is clear that there exist functions $\varphi_j \in C^\infty(\Omega_j)$ with $\operatorname{supp} \varphi_j \subset \Omega_j$ and $\sum\limits_{j=1}^{N-2} \varphi_j = 1$ on $\mathbb{R}^n$. Moreover, in the following we need the estimates

$$|D^\gamma \varphi_j(x)| \leqslant c_{\gamma,j} \qquad \text{uniformly for all } x \in \mathbb{R}^n \tag{9}$$

and $j = 1, \ldots, N-2$, $\gamma \in \mathbb{N}_0$.

But by the property (8) it is always possible to find a resolution of unity with respect to Ω_1, …, Ω_{N-2} with this additional property. In the next step, we give an outline of the construction.

Step 4. We fix a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $0 \leqslant \varphi(x) \leqslant 1$, $\operatorname{supp} \varphi \subset \{x\colon\ |x| < \sqrt{n}\}$ and $1/2 \leqslant \varphi(x) \leqslant 1$ if $|x| \leqslant \sqrt{n}/2$. Furthermore we fix $t = \sigma/\sqrt{n}$ and put

$$\varphi_{k,t}(x) = \varphi(t^{-1}x - k), \qquad k \in \mathbb{Z}^n.$$

The support of such a function is contained in the ball with radius σ and center tk. Let us denote this ball by $B(tk, \sigma)$. There exists a number $c(n)$ such that every point $x \in \mathbb{R}^n$ belongs to at most $c(n)$ such balls. Consequently, the sum

$$\sum_{k \in \mathbb{Z}^n} \varphi_{k,t}(x) = \Phi(x)$$

is finite for all x and we get $\Phi \in C^\infty(\mathbb{R}^n)$ and $\Phi(x) \geqslant 1/2$ for all x. Then the functions

$$\psi_{k,t}(x) = \frac{\varphi_{k,t}(x)}{\Phi(x)}$$

form a resolution of unity with respect to these balls with radius σ, $\operatorname{supp} \psi_{k,t} \subset B(tk, \sigma)$ and, additionally, we have

$$|D^\gamma \psi_{k,t}(x)| \leqslant c'(n) c(\varphi) \sigma^{-|\gamma|}$$

uniformly for all $x \in \mathbb{R}^n$ and all $k \in \mathbb{Z}^n$.

Let

$$K_1 = \{k\colon\ k \in \mathbb{Z}^n \text{ and } q_2 \leqslant q(tk) < \infty\}.$$

Then it follows by the construction in Step 1 and property (7) that

$$\bigcup_{k \in K_1} B(tk, \sigma) \subset \Omega_1.$$

Let

$$K_j = \{k\colon\ k \in \mathbb{Z}^n \text{ and } q_{j+1} \leqslant q(tk) < q_j\} \quad \text{if } j = 2, \ldots, N-3.$$

Then

$$\bigcup_{k \in K_j} B(tk, \sigma) \subset \Omega_j,$$

because for $k \in K_j$ and for $x \in B(tk, \sigma)$ we have $q_{j+1} \leqslant q(tk) < q_j$ and $|tk - x| < \sigma$, and so by (7) $q_{j+2} < q(x) < q_{j-1}$.

By the same argument we get

$$K_{N-2} = \{k\colon\ k \in \mathbb{Z}^n \text{ and } 1 < q(tk) < q_{N-2}\}$$

and

$$\bigcup_{k \in K_{N-2}} B(tk, \sigma) \subset \Omega_{N-2}.$$

Then

$$\varphi_j(x) = \sum_{k \in K_j} \psi_{k,t}(x) \quad \text{for } j = 1, \ldots, N-2$$

are functions with the properties described in Step 3.

Step 5. Let $f \in W_p^{s(x)}(\mathbb{R}^n)$. We have $s(x) > 0$ and therefore every element of $W_p^{s(x)}(\mathbb{R}^n)$ belongs at least to $L_p(\mathbb{R}^n)$. Now

$$\|f \big| L_{q(x)}\| = \left\| \sum_{j=1}^{N-2} \varphi_j f \big| L_{q(x)} \right\| \leqslant \sum_{j=1}^{N-2} \|\varphi_j f \big| L_{q(x)}\|.$$

We have $\operatorname{supp}(\varphi_j f) \subset \Omega_j$ and $q_{j+2} < q(x) < q_{j-1}$ for $x \in \Omega_j$ where $q_0 = \infty$.

By Lemma 1 and Corollary 1 we get

$$\|\varphi_j f \big| L_{q(x)}\| \leqslant 2 \max\{\|\varphi_j f \big| L_{q_{j+2}}\|, \|\varphi_j f \big| L_{q_{j-1}}\|\}$$

where the spaces $L_{q_{j+2}}$ and $L_{q_{j-1}}$ are defined with respect to $\mathbb{R}^n$.

Now by the classical embedding theorem we have

$$\|\varphi_j f \big| L_{q(x)}\| \leqslant 2 \max\{\|\varphi_j f \big| H_p^{s_{j+2}}\|, \|\varphi_j f \big| H_p^{s_{j-1}}\|\}$$

and

$$\|f \big| L_{q(x)}\| \leqslant c \sum_{j=1}^{N-2} (\|\varphi_j f \big| H_p^{s_{j+2}}\| + \|\varphi_j f \big| H_p^{s_{j-1}}\|)$$

with $s_j = \frac{n}{p} - \frac{n}{q_j}$, $j = 0, 1, \ldots, N$.

Step 6. Because of $q_{j+2} < q_{j-1}$ and $s_{j+2} < s_{j-1}$, it is sufficient to compare only $H_p^{s_{j-1}}(\mathbb{R}^n)$ with $W_p^{s(x)}(\mathbb{R}^n)$ for functions $(\varphi_j f)$ with $\operatorname{supp}(\varphi_j f) \subset \Omega_j$.

For $x \in (\Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1})$ we have

$$\begin{aligned}
s_{j-1} &= s(x) - \left[s(x) - \frac{n}{p} + \frac{n}{q(x)}\right] + \frac{n}{q(x)} - \frac{n}{q_{j-1}} \\
&\leqslant s(x) - \inf\left\{\left[s(x) - \frac{n}{p} + \frac{n}{q(x)}\right] :\ x \in \mathbb{R}^n\right\} \\
&\qquad + \sup\left\{\frac{n}{q(x)} - \frac{n}{q_{j-1}} :\ x \in (\Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1})\right\} \\
&\leqslant s(x) - \varepsilon + \frac{4n}{N} \\
&< s(x).
\end{aligned}$$

Since $(\operatorname{supp} \varphi_j f)_\sigma \subset (\Omega_j)_\sigma \subset (\Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1})$ and $s_{j-1} \leqslant s(x)$ on $(\Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1})$, Lemma 2 implies that

$$\|\varphi_j f \big| H_p^{s_{j-1}}\| \leqslant c \|\varphi_j f \big| W_p^{s(x)}\|$$

with a constant $c > 0$, dependent on σ but not on $\varphi_j f$.

Combined with Lemma 3—here the additional property (9) of the system $\{\varphi_j\}_{j=1}^{N-2}$ is necessary—this yields

$$\begin{aligned}
\|f \big| L_{q(x)}\| &\leqslant c \sum_{j=1}^{N-2} \|\varphi_j f \big| W_p^{s(x)}\| \\
&\leqslant c \left(\sum_{j=1}^{N-2} |\varphi_j|_{(l,k)}^{(0)}\right) \|f \big| W_p^{s(x)}\|
\end{aligned}$$

and so the proof is complete. □

References

[ER] *Edmunds, D. E. and Rákosník, J.*: Density of smooth functions in $W^{k,p(x)}$. Proc. R. Soc. Lond. A *437* (1992), 229–236.

[Il] *Illner R.*: A class of L^p-bounded pseudodifferential operators. Proc. Amer. Mat. Soc. *51* (1975), 347–355.

[JL] *Jacob, N. and Leopold, H.-G.*: Pseudodifferential operators with variable order of differentiation generating Feller semigroups. Integral Equations and Operator Theory *17* (1994), 151–166.

[KN] *Kikuchi, K. and Negoro, A.*: Pseudodifferential operators and Sobolev spaces of variable order of differentiation. Rep. Fac. Liberal Arts, Shizuoka University, Sciences, *31* (1995), 19–27.

[KR] *Kováčik, O. and Rákosník, J.*: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslovak Math. J. *41* (1991), 592–618.

[Kg] *Kumano-go, H.*: Pseudo-Differential Operators. Massachusetts Institute of Technology Press, Cambridge (Massachusetts)-London, 1981.

[Le1] *Leopold, H.-G.*: On Besov spaces of variable order of differentiation. Z. Anal. Anw. *8* (1989), 69–82.

[Le2] *Leopold, H.-G.*: On function spaces of variable order of differentiation. Forum Math. *3* (1991), 1–21.

[Ne] *Negoro, A.*: Stable-like processes; construction of the transition density and the behavior of sample paths near $t = 0$. Osaka J. Math. *31* (1994), 189–214.

[Sa] *Samko, S. G.*: Differentiation and integration of variable order and the spaces $L^{p(x)}$. Preprint, 1996.

[Sh] *Sharapudinov, I. I.*: On a topology of the space $L^{p(t)}([0,1])$. Matem. Zametki *26* (1979), 613–632.

[Ta] *Taylor, M. E.*: Pseudodifferential Operators. Princeton University Press, Princeton (New Jersey), 1981.

[Tr] *Triebel, H.*: Theory of Function Spaces. Birkhäuser, Basel, 1983.

[Ts] *Tsenov, I. V.*: A generalization of the problem of the best approximation in the space L^s. Uchen. Zap. Daghestan Gos. Univ *7* (1961), 25–37.

Author's address: Mathematisches Institut, Fakultät für Mathematik und Informatik, Friedrich-Schiller-Universität Jena, D-07743 Jena, Deutschland, e-mail: `leopold@minet.uni-jena.de`.