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BIFURCATION OF PERIODIC SOLUTIONS TO VARIATIONAL INEQUALITIES IN $\mathbb{R}^\kappa$ BASED ON ALEXANDER-YORKE THEOREM

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Abstract. Variational inequalities

$$\begin{gathered} U(t) \in K, \\ (\dot U(t) - B_\lambda U(t) - G(\lambda, U(t)),\ Z - U(t)) \geqslant 0 \ \text{ for all } \ Z \in K, \ \text{a.a.} \ t \in [0,T) \end{gathered}$$

are studied, where K is a closed convex cone in $\mathbb{R}^\kappa$, $\kappa \geqslant 3$, B_λ is a $\kappa \times \kappa$ matrix, G is a small perturbation, λ a real parameter. The assumptions guaranteeing a Hopf bifurcation at some λ_0 for the corresponding equation are considered and it is proved that then, in some situations, also a bifurcation of periodic solutions to our inequality occurs at some $\lambda_I \neq \lambda_0$. Bifurcating solutions are obtained by the limiting process along branches of solutions to penalty problems starting at λ_0 constructed on the basis of the Alexander-Yorke theorem as global bifurcation branches of a certain enlarged system.

Keywords: bifurcation, periodic solutions, variational inequality, differential inequality, finite dimensional space, Alexander-Yorke theorem

MSC 2000: 34C23, 58F14, 34A40

0. Introduction

Let K be a closed convex cone with its vertex at the origin in $\mathbb{R}^\kappa$, $\kappa \geqslant 3$. Consider a bifurcation problem for the inequality

$$\text{(I)} \qquad \left\{ \begin{aligned} & U(t) \in K, \\ & (\dot U(t) - F(\lambda, U(t)),\ Z - U(t)) \geqslant 0 \ \text{ for all } \ Z \in K, \ \text{a.a.} \ t \in [0,T) \end{aligned} \right.$$

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with a bifurcation parameter $\lambda \in I$, I being an open interval in $\mathbb{R}$. Here $F\colon I \times \mathbb{R}^\kappa \to \mathbb{R}^\kappa$ is a smooth mapping, $F(\lambda, 0) = 0$ for all $\lambda \in I$. We will write $F(\lambda, U) = B_\lambda U + G(\lambda, U)$ where B_λ is a real matrix of the type $\kappa \times \kappa$ depending continuously on the parameter λ and $G\colon I \times \mathbb{R}^\kappa \to \mathbb{R}^\kappa$ satisfies the conditions

$$|G(\lambda, U)| = O(|U|^2), \tag{G}$$

$$\begin{cases} \text{for any } \Lambda_1, \Lambda_2 \in I,\ \Lambda_1 < \Lambda_2, R > 0 \text{ there exists } C > 0 \text{ such that} \\ |G(\lambda, U_1) - G(\lambda, U_2)| \leqslant C|U_1 - U_2| \text{ for all } \lambda \in [\Lambda_1, \Lambda_2], |U_1|, |U_2| \leqslant R. \end{cases} \tag{L}$$

By a solution of (I) on $[0, T)$ we mean an absolutely continuous function satisfying (I) for a.a. $t \in [0, T)$.

It is proved in [8], [5] that if a Hopf bifurcation of periodic solutions to the equation

$$\dot{U}(t) = F(\lambda, U(t)) \tag{E}$$

occurs at some λ_0 and certain additional assumptions are fulfilled then there exists also a bifurcation point λ_I of our inequality at which periodic solutions to (I) bifurcate from the branch of trivial solutions. The main results in [8], [5] either ensure the existence of such a bifurcation or explain in a certain sense why a bifurcation does not occur. A basic idea was to join the inequality (I) with the corresponding equation (E) by a certain homotopy and to show that the bifurcation point λ_0 of the equation is transfered to a bifurcation point of the inequality by this homotopy. The joining mentioned was given in [8] or [5] by a system of inequalities on suitable deformations of the cone K or by a system of penalty equations, respectively. This approach represents a certain nontrivial modification of the method for the investigation of bifurcations of stationary solutions to inequalities given in [6] (see also [7]). In the papers mentioned, the problem of investigation of periodic solutions was first transferred to that of the study of fixed points of a suitable mapping. These fixed points were initial conditions of periodic solutions of the inequalities on deformed cones or of the penalty problems. A certain modification of the well-known Rabinowitz global bifurcation theorem [13] was used to such stationary problem for the proof of the existence of a branch of periodic solutions of the inequalities on deformed cones or of the penalty equations representing the joining mentioned above. One of the main difficulties was the definition of a suitable mapping. (It was impossible to use directly the classical Poincaré map.) In the present paper, we use the Alexander-Yorke global bifurcation theorem [1] for the study of the branches of periodic solutions to the penalty problem. This approach seems to be more general than that from [8], [5] and essentially simpler even if it is necessary to use a more complicated penalty problem.

Of course, the corresponding linearized equation

$$\text{(LE)} \qquad \dot{U}(t) = B_\lambda U(t)$$

and the "linearized inequality"

$$\text{(LI)} \qquad \begin{cases} U(t) \in K, \\ (\dot{U}(t) - B_\lambda U(t),\ Z - U(t)) \geqslant 0 \ \text{ for all } \ Z \in K, \quad \text{a.a. } t \geqslant 0 \end{cases}$$

play an essential role. Of course, the problem (LI) is only positively homogeneous but strongly nonlinear again. One of the basic difficulties is that there is no real linearization of the problem (I).

The main results are formulated and explained in Section 1. Theorem 1.1 describes the properties of the branches of solutions to our penalty system. Theorem 1.2 gives the existence of a bifurcation point for the inequality (I) and Theorem 1.3 explains how the bifurcating solutions are obtained. In Section 2, some basic properties of the problem with the penalty are described. Proof of the main results based on the Alexander-Yorke global bifurcation theorem for equations is given in Section 3.

Notice that an elementary approach to the investigation of bifurcations of periodic solutions to inequalities (I) in the special case $\kappa = 3$ was given in [3] and it was developed for the study of the stability of bifurcating solutions in [9].

1. Main results

Set $(U, V) = \sum_{i=1}^{\kappa} u_i v_i$, $|U|^2 = (U, U)$ for $U = [u_1, \dots, u_\kappa]$, $V = [v_1, \dots, v_\kappa]$.

Basic Assumptions 1.1. We will always suppose that there is $\lambda_0 \in I$ such that

$$(\mu) \qquad \begin{cases} \text{for } \lambda = \lambda_0, \text{ there are exactly two eigenvalues } \pm \mathrm{i}\omega_0,\ \omega_0 > 0, \text{ of } B_{\lambda_0} \\ \quad \text{on the imaginary axis,} \\ \text{for } \lambda \in I \setminus \{\lambda_0\}, \text{ there is no eigenvalue of } B_\lambda \text{ on the imaginary axis,} \\ \text{the real parts of the eigenvalues of } B_\lambda \text{ near } \pm \mathrm{i}\omega_0 \text{ change sign} \\ \quad \text{as } \lambda \text{ increases past } \lambda_0. \end{cases}$$

Denote by P_K the projection on K, i.e. $P_K U$ for $U \in \mathbb{R}^\kappa$ is the unique point from K satisfying

$$|P_K U - U| = \min_{V \in K} |V - U|.$$

Set $\beta = I - P_K$. For any $\varrho > 0$, we will consider a system of penalty equations

$$\text{(PS)} \qquad \begin{cases} \dot U(t) &= B_\lambda U(t) + G(\lambda, U(t)) - \varepsilon(t)\beta U(t), \\ \dot\varepsilon(t) &= -\varrho^2 \frac{\varepsilon(t)}{1+|\varepsilon(t)|} + |U(t)|^2 \end{cases}$$

and the "linearized" penalty equation

$$\text{(LPE)} \qquad \dot U(t) = B_\lambda U(t) - \tau\beta U(t)$$

where τ is an additional real parameter. Notice that the second equation in (PS) can be written with an emphasis on the linear term as

$$\dot\varepsilon(t) = -\varrho^2\varepsilon(t) + \varrho^2 \frac{\varepsilon(t)|\varepsilon(t)|}{1+|\varepsilon(t)|} + |U(t)|^2.$$

Remark 1.1. We obtain (E) and (I) in a certain sense from the first equation in (PS) for $\varepsilon \equiv 0$ and $\varepsilon \to +\infty$, respectively (precisely see Lemma 2.4). Hence, the penalty system (PS) can be understood in a certain sense as a homotopy joining our inequality with the corresponding equation. Cf. also [5], Remark 1.2, Theorem 2.3, where only the first equation from (PS) with a constant $\varepsilon \equiv \tau$ was considered.

Notation 1.1.
K^0, ∂K—the interior and the boundary of K,
$U^\tau_{\varrho,\lambda}(\cdot, V)$, $\varepsilon^\tau_{\varrho,\lambda}(\cdot, V)$—the solution of (PS) satisfying the initial condition $U(0) = V$, $\varepsilon(0) = \tau$,
$U^\tau_{0,\lambda}(\cdot, V)$—the solution of (LPE) satisfying the initial condition $U(0) = V$,
$U^\infty_\lambda(\cdot, V)$, $U^\infty_{0,\lambda}(\cdot, V)$—the solution of (I) and (LI), respectively, satisfying the initial condition $U(0) = V$.
Set

$$\begin{aligned} \mathcal{B} &= \Big\{\Big[\frac{2k\pi}{\omega_0}, 0, 0, \lambda_0\Big] \in (0,+\infty) \times \mathbb{R}^\kappa \times \mathbb{R} \times I;\ k \text{ positive integer}\Big\}, \\ \mathcal{L}_\varrho &= \{[T, V, \tau, \lambda] \in [0,+\infty) \times \mathbb{R}^\kappa \times \mathbb{R} \times I;\ U^\tau_{\varrho,\lambda}(T, V) = V,\ \varepsilon^\tau_{\varrho,\lambda}(T, V) = \tau\}, \\ \mathcal{C}_\varrho &= \big(\mathcal{L}_\varrho \setminus (0,+\infty) \times \{0\} \times \{0\} \times I\big) \cup \mathcal{B} \end{aligned}$$

and denote by $\mathcal{C}^0_\varrho$ the component of $\mathcal{C}_\varrho$ containing $[\frac{2\pi}{\omega_0}, 0, 0, \lambda_0]$.

We will see in Observation 2.2 that if $[T, V, \tau, \lambda] \in \mathcal{C}_\varrho$, $T > 0$ then either $|V| = \tau = 0$ or $|V| > 0$, $\tau > 0$. Particularly, the only points in $\mathcal{C}_\varrho$ with $T > 0$, $\tau = 0$ are $[\frac{2k\pi}{\omega_0}, 0, 0, \lambda_0]$, k positive integer. Now, if $[T, 0, 0, \lambda]$ lies in the closure of $\mathcal{L}_\varrho \setminus (0,+\infty) \times \{0\} \times \{0\} \times I$, $T > 0$ then there are $[T_n, V_n, \tau_n, \lambda_n] \in \mathcal{L}_\varrho$ such that $[T_n, V_n, \tau_n, \lambda_n] \to [T, 0, 0, \lambda]$, $|V_n| > 0$, $\tau_n > 0$. We have $U^{\tau_n}_{\varrho,\lambda_n}(T_n, V_n) = V_n$ and if

$\frac{V_n}{|V_n|} \to W$ then $U^0_{0,\lambda}(T, W) = W$ (see Lemma 2.5 below). Under the assumption (μ), this can occur only for $\lambda = \lambda_0$, $T = \frac{2k\pi}{\omega_0}$ with k positive integer. It follows that $\mathcal{C}_\varrho$ is closed and contains the closure of $\mathcal{L}_\varrho \setminus (0, +\infty) \times \{0\} \times \{0\} \times I$.

Theorem 1.1. *Let the assumptions* (μ), (G), (L) *be fulfilled. Then for any* $\varrho > 0$, $\mathcal{C}^0_\varrho$ *is a closed connected and unbounded set in* $[0, +\infty) \times \mathbb{R}^\kappa \times [0, +\infty) \times \mathbb{R}$ *either containing a point of the type* $[0, V, \tau, \lambda]$ *or having the following property:*

$$(1.1)\quad \begin{cases} \textit{if } [T, V, \tau, \lambda] \in \mathcal{C}^0_\varrho, \tau \neq 0 \textit{ then } T > 0,\ \tau > 0,\ U^\tau_{\varrho,\lambda}(\cdot, V),\ \varepsilon^\tau_{\varrho,\lambda}(\cdot, V) \textit{ are} \\ \textit{(nonstationary) } T\textit{-periodic and} \int_0^T |U^\tau_{\varrho,\lambda}(t, V)|^2 \,\mathrm{d}t = \int_0^T \varrho^2 \frac{\varepsilon^\tau_{\varrho,\lambda}(t,V)}{1+|\varepsilon^\tau_{\varrho,\lambda}(t,V)|} \,\mathrm{d}t. \end{cases}$$

Remark 1.2. If U, ε is a solution of (PS) and U is constant then ε cannot be nonconstant periodic. Hence, if $[T, V, \tau, \lambda] \in \mathcal{C}^0_\varrho$, $T > 0$, $U^\tau_{\varrho,\lambda}(\cdot, V)$ is constant then also $\varepsilon^\tau_{\varrho,\lambda}(\cdot, V)$ must be constant. If at the same time $|V| > 0$, $\tau > 0$ then we get $[t, V, \tau, \lambda] \in \mathcal{C}^0_\varrho$ for all $t \geqslant 0$. In particular, if $\mathcal{C}^0_\varrho$ contains no point of the type $[0, V, \tau, \lambda]$ then $U^\tau_{\varrho,\lambda}(\cdot, V)$, $\varepsilon^\tau_{\varrho,\lambda}(\cdot, V)$ are T-periodic nonstationary for any $[T, V, \tau, \lambda] \in \mathcal{C}^0_\varrho$, $|V| > 0$, $\tau > 0$.

Basic Assumptions 1.2. We are interested in situations when the set $\mathcal{C}^0_\varrho$ in Theorem 1.1 is unbounded in τ and the condition (1.1) holds for ϱ small enough. Then small periodic solutions of (I) can be obtained by the limiting process $\tau \to +\infty$ along the branches $\mathcal{C}^0_\varrho$ (see Theorem 1.3). Therefore we will suppose that there exist $\varrho_0 > 0$, $\gamma > 0$, $t_M > 0$, $\Lambda_1, \Lambda_2 \in I$ such that

$$(1.2)\qquad \text{if } [T, V, \tau, \lambda] \in \mathcal{C}^0_\varrho,\ \varrho \in (0, \varrho_0) \text{ then } \gamma < T < t_M,\ \lambda \in (\Lambda_1, \Lambda_2).$$

To exclude the existence of stationary solutions of (LI) and simultaneously the existence of a bifurcation of stationary solutions to (I), we will assume that

$$(1.3)\qquad \begin{cases} \text{for any } U \in \partial K, |U| > 0,\ \lambda \in [\Lambda_1, \Lambda_2] \\ \text{there is } Z \in K \text{ such that } (B_\lambda U, Z - U) > 0. \end{cases}$$

(The condition (1.3) means that for any $U \in \partial K, |U| > 0$, $B_\lambda U$ does not lie in the normal cone to K at U.) We will explain on Model Example 1.1 below how the assumptions (1.2), (1.3) can be verified. In fact, this example can be treated by a more elementary approach (see [3], [9]). However, our aim is not to solve complicated examples where main ideas are hidden in technical computation. More general examples in $\mathbb{R}^\kappa$, $\kappa > 3$ will be contained in a forthcoming paper.

Remark 1.3. If we supose (in addition to (1.3)) that

$$B_\lambda U - \tau \beta U \neq 0 \quad \text{for all } |U| \neq 0,\ \tau > 0,\ \lambda \in [\Lambda_1, \Lambda_2], \tag{1.4}$$

that means the linearized penalty equation has no nontrivial stationary solution for the parameters under consideration, then we can omit the lower estimate $T > \gamma$ in the assumption (1.2) because it is fulfilled automatically. See Appendix for details. Note that the inequality in (1.4) for $U \in K$ or $\tau = 0$ follows directly from the assumption (μ).

Theorem 1.2. *Let* (μ), (G), (L) *and* (1.2), (1.3) *be fulfilled. Then there exists a bifurcation point* $\lambda_I \in [\Lambda_1, \Lambda_2]$ *of (I) at which periodic (nonstationary) solutions of* (I) *bifurcate from the branch of trivial solutions. Precisely, there exist* $T_n \in (\gamma, t_M)$, $\lambda_n \in [\Lambda_1, \Lambda_2]$, $V_n \in \mathbb{R}^\kappa$ *such that* $|V_n| \to 0$, $\lambda_n \to \lambda_I$ *and* $U^\infty_{\lambda_n}(\cdot, V_n)$ *are* T_n*-periodic (nonstationary) solutions of* (I). *If* $T_n \to T$, $\frac{V_n}{|V_n|} \to W$ *then* $U^\infty_{0,\lambda_I}(\cdot, W)$ *is (nonstationary)* T*-periodic.*

P r o o f of Theorem 1.2 follows directly from the following Theorem 1.3 explaining how bifurcating solutions of (I) can be obtained by using branches of solutions of the penalty problem. □

Theorem 1.3. *Let* (μ), (G), (L) *and* (1.2), (1.3) *be fulfilled. Then for any* $\varrho \in (0, \varrho_0)$ *there exists at least one sequence* $\{[T_n, V_n, \tau_n, \lambda_n]\} \subset \mathcal{C}^0_\varrho$ *such that* $T_n \to T_\varrho, V_n \to V_\varrho, \tau_n \to +\infty, \lambda_n \to \lambda_\varrho$. *For any such sequence we have* $T_\varrho \in [\gamma, t_M]$, $\lambda_\varrho \in [\Lambda_1, \Lambda_2]$, $|V_\varrho| > 0$,

$$U^{\tau_n}_{\varrho,\lambda_n}(\cdot, V_n) \to U^\infty_{\lambda_\varrho}(\cdot, V_\varrho) \text{ in } C([0, t_M]) \text{ and weakly in } W^1_2(0, t_M), \tag{1.5}$$

$$\int_0^{T_\varrho} |U^\infty_{\lambda_\varrho}(t, V_\varrho)|^2 \,\mathrm{d}t = \varrho^2 T_\varrho \tag{1.6}$$

where $U^\infty_{\lambda_\varrho}(\cdot, V_\varrho)$ *is a* T_ϱ*-periodic (nonstationary) solution of* (I).

If $\varrho_n \to 0$ *and* $V_{\varrho_n}, \lambda_{\varrho_n}$ *are obtained by this procedure then* $|V_{\varrho_n}| \to 0$. *Particularly, any accumulation point* λ_I *of* λ_{ϱ_n} *is a bifurcation point of* (I) *announced in Theorem 1.2. If* $\varrho_n \to 0$, $T_{\varrho_n} \to T$, $\frac{V_{\varrho_n}}{|V_{\varrho_n}|} \to W$ *then* $U^\infty_{0,\lambda_I}(\cdot, W)$ *is (nonstationary)* T*-periodic.*

Model Example 1.1. Set $\kappa = 3$, $K = \{U = [u_1, u_2, u_3] \in \mathbb{R}^3;\ u_3 \geqslant 0, u_3 \geqslant u_1\}$,

$$B_\lambda = \begin{pmatrix} \lambda, & 1, & 0 \\ -1, & \lambda, & 0 \\ 0, & 0, & -1 \end{pmatrix},$$

$I = \mathbb{R}$. Let $G\colon \mathbb{R}^4 \to \mathbb{R}^3$ be a mapping satisfying (G), (L), $G = [g_1, g_2, g_3]$. For simplicity, let us suppose that

$$g_3(\lambda, U) \geqslant 0 \text{ for all } \lambda \in \mathbb{R},\ U = [u_1, u_2, u_3],\ u_3 \leqslant 0. \tag{1.7}$$

We will show that the assumptions of Theorem 1.2 are fulfilled and that the existence of a bifurcation point $\lambda_I \in (0,1)$ of the inequality (I) follows.

The eigenvalues and the corresponding eigenvectors of B_λ are $\mu_{1,2}(\lambda) = \lambda \pm \mathrm{i}$, $\mu_3(\lambda) = -1$ and $W_{1,2}(\lambda) = U_1 \pm \mathrm{i} U_2$, $W_3(\lambda) = U_3$ with $U_1 = [1,0,0]$, $U_2 = [0,1,0]$, $U_3 = [0,0,1]$ for all $\lambda \in \mathbb{R}$. Set

$$P_L U = [u_1, u_2, 0],\ P_L^* U = [-u_2, u_1, 0] \text{ for } U = [u_1, u_2, u_3].$$

The linearized equation (LE) has the form

$$\begin{cases} \dot u_1(t) = \lambda u_1(t) + u_2(t), \\ \dot u_2(t) = -u_1(t) + \lambda u_2(t), \\ \dot u_3(t) = -u_3(t). \end{cases} \tag{1.8}$$

Its solutions (with the exception of those starting on $\operatorname{Lin}\{U_3\}$) circulate around the U_3-axis and tend to the plane $\operatorname{Lin}\{U_1, U_2\}$.

The assumption (1.3) is fulfilled for any Λ_1, Λ_2.

In accordance with the comment after the definition of $\mathcal{C}_\varrho$, we will consider only $\tau \geqslant 0$ in the sequel (see also Observation 2.1 below). Taking into account the assumption (1.7) and the direction of βU, it is easy to see that

$$\begin{cases} \text{if } \varrho > 0,\ V = [v_1, v_2, v_3],\ v_3 \geqslant 0,\ \tau \geqslant 0,\ \lambda \in \mathbb{R}, \\ U(t) = [u_1(t), u_2(t), u_3(t)] = U^\tau_{\varrho,\lambda}(t, V) \\ \text{then } u_3(t) \geqslant 0 \text{ for all } t \text{ where } U(t) \text{ is defined ;} \\ \text{all periodic solutions of (PS) and (LPE) with any } \tau \geqslant 0,\ \lambda \in \mathbb{R} \\ \text{lie in the halfspace } \{U \in \mathbb{R}^3\,;\ u_3 \geqslant 0\}. \end{cases} \tag{1.9}$$

Hence, we need not deal with points U, $u_3 < 0$ in our considerations.

We have $P_K U = U$ for $U \in K$,

$$P_K U = \Big[\frac{u_1 + u_3}{2}, u_2, \frac{u_1 + u_3}{2}\Big] \text{ for all } U = [u_1, u_2, u_3], u_1 \geqslant u_3 \geqslant 0,$$
$$\beta U = \Big[\frac{(u_3 - u_1)^-}{2}, 0, -\frac{(u_3 - u_1)^-}{2}\Big] \text{ for all } U = [u_1, u_2, u_3], u_3 \geqslant 0.$$

In the halfspace $\{U \in \mathbb{R}^3 ;\ u_3 \geqslant 0\}$ the first equation in (PS) reads

$$(1.10)\qquad \begin{cases} \dot{u}_1(t) = \lambda u_1(t) + u_2(t) + g_1(\lambda, U(t)) - \varepsilon(t)\frac{(u_3(t)-u_1(t))^-}{2}, \\ \dot{u}_2(t) = -u_1(t) + \lambda u_2(t) + g_2(\lambda, U(t)), \\ \dot{u}_3(t) = -u_3(t) + g_3(\lambda, U(t)) + \varepsilon(t)\frac{(u_3(t)-u_1(t))^-}{2}. \end{cases}$$

In what follows, we will explain only the main ideas. Precise proofs of the assertions (1.11)–(1.16) using some general facts from Section 2 will be given in Appendix.

Denote by $r^\tau_{\varrho,\lambda}(t,V)$, $\varphi^\tau_{\varrho,\lambda}(t,V)$ and $r^\tau_{0,\lambda}(t,V)$, $\varphi^\tau_{0,\lambda}(t,V)$ the polar coordinates of $P_L P_K U^\tau_{\varrho,\lambda}(t,V)$ and of $P_L P_K U^\tau_{0,\lambda}(t,V)$, respectively, in the plane $\mathrm{Lin}\{U_1, U_2\}$ with the angle measured from $P_L P_K V$. In other words, we have $\varphi^\tau_{\varrho,\lambda}(0,V) = 0$,

$$P_L P_K U^\tau_{\varrho,\lambda}(t,V) = r^\tau_{\varrho,\lambda}(t,V)\big[\cos\big(\varphi^\tau_{\varrho,\lambda}(t,V) + \varphi_V\big)\cdot U_1 + \sin\big(\varphi^\tau_{\varrho,\lambda}(t,V) + \varphi_V\big)\cdot U_2\big]$$

for $t \in [0, t_0)$ if $|P_L P_K U^\tau_{\varrho,\lambda}(t,V)| \neq 0$ on $[0,t_0)$ where

$$P_L P_K V = r^\tau_{\varrho,\lambda}(0,V)\big(\cos\varphi_V \cdot U_1 + \sin\varphi_V \cdot U_2\big),$$

and analogously for $r^\tau_{0,\lambda}(t,V)$, $\varphi^\tau_{0,\lambda}(t,V)$. For $V \notin \mathrm{Lin}\{U_3\}$ set

$$t^\tau_{\varrho,\lambda}(V) = \inf\{t_0 ;\ r^\tau_{\varrho,\lambda}(t,V) > 0 \text{ for } t \in [0,t_0],\ \varphi^\tau_{\varrho,\lambda}(t_0,V) = -2\pi\}.$$

Hence, $t^\tau_{\varrho,\lambda}(V)$ is the time of one circuit of $P_L P_K U^\tau_{\varrho,\lambda}(\cdot,V)$ (and simultaneously of $U^\tau_{\varrho,\lambda}(\cdot,V)$) around the axis U_3 if it is finite. Analogously we define $t^\tau_{0,\lambda}(V)$.

Writing

$$\dot{\varphi}^\tau_{\varrho,\lambda}(t,V) = \frac{(\frac{\mathrm{d}}{\mathrm{d}t}(P_L P_K U^\tau_{\varrho,\lambda}(t,V)), P_L^* P_K U^\tau_{\varrho,\lambda}(t,V))}{|P_L^* P_K U^\tau_{\varrho,\lambda}(t,V))|^2}$$

and analogously for $\dot{\varphi}^\tau_{0,\lambda}(t,V)$, we can prove by using (1.10), (1.9) that there are $\eta \in (0,1)$, $\varrho_1 > 0$ such that

$$(1.11)\qquad \begin{cases} \text{if } \tau \in [0,+\infty],\ \lambda \leqslant 1,\ W = [w_1,w_2,w_3] \notin \mathrm{Lin}\{U_3\},\ w_3 \geqslant 0, \\ \ \text{then } \dot{\varphi}^\tau_{0,\lambda}(t,W) < -\eta \text{ for all } t \geqslant 0; \\ \text{if } \varrho > 0,\ \tau \in [0,+\infty),\ \lambda \leqslant 1,\ V = [v_1,v_2,v_3],\ v_3 \geqslant 0, \\ U^\tau_{\varrho,\lambda}(t,V) \notin \mathrm{Lin}\{U_3\},\ |U^\tau_{\varrho,\lambda}(t,V)| < \varrho_1 \text{ for } t \in [0,\tilde{t}] \\ \ \text{then } \dot{\varphi}^\tau_{\varrho,\lambda}(t,V) < -\eta \text{ for all } t \in [0,\tilde{t}], \end{cases}$$

which means $P_K U^\tau_{0,\lambda}(t,W)$ and $P_K U^\tau_{\varrho,\lambda}(t,V)$ circulate around the U_3-axis with the velocity greater than η under the assumptions considered. We will choose a fixed $\eta \in (0,1)$ such that (1.11) holds, $\eta \neq \frac{1}{k}$, $k = 1,2,\ldots$, and set

$$t_M = \frac{2\pi}{\eta} \quad (t_M > 2\pi,\ \neq 2k\pi,\ k = 1,2,\ldots).$$

A simple calculus of the expressions $\frac{\mathrm{d}}{\mathrm{d}t}(|U^{\tau}_{0,\lambda}(t,W)|^2)$, $\frac{\mathrm{d}}{\mathrm{d}t}(|P_K U^{\tau}_{0,\lambda}(t,W)|^2)$ shows that if $U^{\tau}_{0,\lambda}(\cdot,W)$ were periodic and $\tau \in (0,+\infty]$, $\lambda \leqslant 0$ or $\lambda \geqslant 1$ then $|U^{\tau}_{0,\lambda}(t,W)|$ would decrease or $|P_K U^{\tau}_{0,\lambda}(t,W)|$ would increase, respectively, during one period. This is impossible and therefore

$$\text{(1.12)}\quad \text{if } W \in \mathbb{R}^3,\ |W| > 0,\ \tau \in (0,+\infty],\ U^{\tau}_{0,\lambda}(T,W) = W,\ T > 0 \text{ then } \lambda \in (0,1).$$

Further, let us consider fixed $\gamma \in (0,\pi)$, $\xi > 0$. Set

$$\mathcal{C}^M_{\varrho} \ = \ \text{the component of } \{[T,V,\tau,\lambda] \in \mathcal{C}^0_{\varrho};\ \gamma \leqslant T \leqslant t_M\} \text{ containing } [2\pi,0,0,0].$$

(We will see latter that in fact $\mathcal{C}^M_{\varrho} = \mathcal{C}^0_{\varrho}$.) We will show that there is $\varrho_0 > 0$ such that

$$\text{(1.13)}\qquad \text{if } \varrho \in (0,\varrho_0),\ [T,V,\tau,\lambda] \in \mathcal{C}^M_{\varrho} \text{ then } \lambda \in (-\xi,1).$$

Idea of the proof: $\mathcal{C}^M_{\varrho}$ is connected and starts at $\lambda = \lambda_0 = 0$. If (1.13) were true for no ϱ_0 then for arbitrarily small ϱ, $\mathcal{C}^M_{\varrho}$ would contain a point $[T,V,\tau,\lambda]$ with $|V| > 0$ and $\lambda = 1$ or $\lambda = -\xi$. (This need not be true with $\xi = 0$). We have $U^{\tau}_{\varrho,\lambda}(T,V) = V$ and the limiting proces for $\varrho \to 0$ (using the fact that small solutions of (I) and (PS) with small ϱ behave similarly to those of (LI) and (LPE), respectively, by virtue of the assumption (G)) would give $U^{\tau}_{0,\lambda}(T,W) = W$ with some $T \in [\gamma,t_M]$, $|W| = 1$, $\tau \in [0,\infty]$ and $\lambda = -\xi$ or $\lambda = 1$. This would contradict (1.12) or, in the case $\tau = 0$, the fact that (1.8) has periodic solutions only for $\lambda = 0$.

It is possible to show that the assumptions of the second part of (1.11) are automatically fulfilled if $[2k\pi,0,0,0] \neq [T,V,\tau,\lambda] \in \mathcal{C}^M_{\varrho}$ with ϱ small enough. Hence,

$$\text{(1.14)}\quad \begin{cases} \text{if } \varrho \in (0,\varrho_0),\ [T,V,\tau,\lambda] \in \mathcal{C}^M_{\varrho},\ [T,V,\tau,\lambda] \neq [2k\pi,0,0,0],\ k = 1,2,\ldots, \\ \text{then } U^{\tau}_{\varrho,\lambda}(t,V) \notin \mathrm{Lin}\{U_3\},\ \dot{\varphi}^{\tau}_{\varrho,\lambda}(t,V) < -\eta \text{ for all } t \geqslant 0, \\ \qquad \gamma < t^{\tau}_{\varrho,\lambda}(V) < t_M,\ T \geqslant t^{\tau}_{\varrho,\lambda}(V). \end{cases}$$

Idea of the proof: It follows from the second equation in (PS) (which plays a role of a norm condition—see Observation 2.3) and the T-periodicity that if $[T,V,\tau,\lambda] \in \mathcal{C}^M_{\varrho}$, ϱ is small enough then $|U^{\tau}_{\varrho,\lambda}(t,V)| < \varrho_1$ for $t \geqslant 0$ (ϱ_1 is from (1.11)). If there were $\varrho_n \to 0$, $[2k\pi,0,0,0] \neq [T_n,V_n,\tau_n,\lambda_n] \in \mathcal{C}^M_{\varrho_n}$, $t_n \geqslant 0$, $U^{\tau_n}_{\varrho_n,\lambda_n}(t_n,V_n) \in \mathrm{Lin}\{U_3\}$ then we would obtain the existence of a nontrivial periodic or stationary solution to (1.8) in $\mathrm{Lin}\{U_3\}$, which is impossible. Hence, $U^{\tau}_{\varrho,\lambda}(t,V) \notin \mathrm{Lin}\{U_3\}$ for $t \geqslant 0$ and we can use (1.11). Thus, any trajectory under consideration circulates around the U_3-axis. Particularly, more than one half of such a trajectory lies in K^0 where it coincides with that of (1.8) and consequently $\gamma < t^{\tau}_{\varrho,\lambda}(V)$. The other estimates follow from (1.11).

The branch $\mathcal{C}^0_{\varrho}$ is connected and for $[T, V, \tau, \lambda]$ near $[2\pi, 0, 0, 0]$, the period T as well as $t^{\tau}_{\varrho,\lambda}(V)$ are close to 2π. This together with (1.14) (guaranteeing a uniform circulation around the U_3-axis) leads to the conclusion that

$$\tag{1.15} \begin{cases} \text{for any } \varrho \in (0, \varrho_0) \text{ there exists } \delta > 0 \text{ such that if } [T, V, \tau, \lambda] \in \mathcal{C}^M_{\varrho}, \\ [T, V, \tau, \lambda] \neq [2\pi, 0, 0, 0],\ |T - 2\pi| \leqslant \delta,\ |V| \leqslant \delta,\ \tau \leqslant \delta,\ |\lambda| \leqslant \delta \\ \ \text{then } V \notin \operatorname{Lin}\{U_3\},\ T = t^{\tau}_{\varrho,\lambda}(V). \end{cases}$$

It follows from (1.15), the connectedness of $\mathcal{C}^M_{\varrho}$ and the continuous dependence of $t^{\tau}_{\varrho,\lambda}(V)$ on parameters that

$$\tag{1.16} \begin{aligned} &\text{if } \varrho \in (0, \varrho_0),\ [2\pi, 0, 0, 0] \neq [T, V, \tau, \lambda] \in \mathcal{C}^M_{\varrho}, \\ &\ \text{then } V \notin \operatorname{Lin}\{U_3\},\ T = t^{\tau}_{\varrho,\lambda}(V). \end{aligned}$$

Now, we can conclude that

$$\tag{1.17} \text{if } [T, V, \tau, \lambda] \in \mathcal{C}^0_{\varrho} \text{ then } \gamma < T < t_M \text{ (in particular, } \mathcal{C}^M_{\varrho} = \mathcal{C}^0_{\varrho}).$$

Indeed, if this were not true then it would follow from the connectedness that there exists $[T, V, \tau, \lambda] \in \mathcal{C}^M_{\varrho}$ with $T = \gamma$ or $T = t_M$. It would be $T = t^{\tau}_{\varrho,\lambda}(V)$ by (1.16) but this would contradict (1.14).

It follows from 1.13) and (1.17) that the condition (1.2) holds with $\Lambda_1 = -\xi$, $\Lambda_2 = 1$. However, $\xi > 0$ was arbitrary and therefore we obtain the existence of a bifurcation point $\lambda_I \in [0, 1]$ from Theorem 1.2. The cases $\lambda_I = 0$, $\lambda_I = 1$ are excluded by the last assertion of Theorem 1.2 and by (1.12) for $\tau = +\infty$. This means $\lambda_I \in (0, 1)$.

2. Properties of the penalty system and of the inequality (I)

Remark 2.1. The solution of (PS) and of (I) (for fixed $\varrho > 0$, $\lambda \in I$, $\tau \in \mathbb{R}$, $V \in \mathbb{R}^{\kappa}$ and $V \in K$, respectively) is unique and exists at least on some interval $[0, T_0)$, $T_0 > 0$. Further, if $T > 0$ and a solution of (PS) or of (I) is bounded on any subinterval of [0,T) on which it is defined then it exists on $[0, T)$. For (PS), this follows from the standard theory of ODE's (see e.g. [10]). For the inequality (I) see [2]. Particularly, $U^{\tau}_{0,\lambda}(\cdot, V)$ and $U^{\infty}_{0,\lambda}(\cdot, V)$ always exist on $[0, +\infty)$ for all $\lambda \in I$, $\tau \in \mathbb{R}$, $V \in \mathbb{R}^{\kappa}$ and $V \in K$, respectively. For τ finite, the boundedness on any finite interval follows from estimates analogous to those from the proof of Lemma 2.1 which becomes simpler in the case $G = 0$, $\varepsilon(t) \equiv \tau$. For $\tau = +\infty$ cf. [8], Lemma 2.1.

Remark 2.2. The operators P_K, $\beta = I - P_K$ are lipschitzian and

$$\text{(P)} \qquad (\beta U, U) > 0 \text{ for all } U \notin K,\ \beta U = 0 \text{ if and only if } U \in K,$$

$$\text{(H)} \qquad \beta(tU) = t\beta U \text{ for all } t > 0,\ U \in \mathbb{R}^\kappa \ (\beta \text{ is positively homogeneous}),$$

$$\text{(M)} \qquad (\beta U - \beta V, U - V) \geqslant 0 \text{ for all } U, V \in \mathbb{R}^\kappa \ (\text{i.e. } \beta \text{ is monotone}),$$

$$\text{(Pt)} \qquad \beta W = \frac{1}{2}\operatorname{grad}|\beta W|^2 \ (\text{i.e. } \beta \text{ is potential})$$

(see [14]).

Lemma 2.1. *Let $\Lambda_1, \Lambda_2 \in I$, $\Lambda_1 < \Lambda_2$, $t_M > 0$. Then there exist $\varrho_0 > 0$, $r > 0$, $C > 0$ and for any $\tau^* \geqslant 0$ there is C_{τ^*} such that if $\varrho \in (0, \varrho_0)$, $V \in \mathbb{R}^\kappa$, $|V| \leqslant \varrho_0$, $\lambda \in [\Lambda_1, \Lambda_2]$ then*

$$\text{(2.1)} \qquad \begin{cases} U^\tau_{\varrho,\lambda}(\cdot, V),\ \varepsilon^\tau_{\varrho,\lambda}(\cdot, V) \text{ exist on } [0, t_M + 1) \text{ for any } \tau \geqslant 0, \\ |U^\tau_{\varrho,\lambda}(t, V)|^2 \leqslant |V|^2 \mathrm{e}^{rt} \text{ for all } \tau \geqslant 0,\ t \in [0, t_M + 1), \\ |\dot U^\tau_{\varrho,\lambda}(t, V)|^2 \leqslant C_{\tau^*} |V|^2 \mathrm{e}^{rt} \text{ for all } \tau \in [0, \tau^*],\ t \in [0, t_M + 1). \end{cases}$$

P r o o f. Choose $\tilde\varrho > 0$. It follows from (G), (L) that there exists $C > 0$ such that

$$\frac{|G(\lambda, U)|}{|U|} \leqslant C \text{ for all } |U| \leqslant \tilde\varrho,\ \lambda \in [\Lambda_1, \Lambda_2].$$

Consider a fixed $\varrho > 0$, $V \in \mathbb{R}^\kappa$, $|V| \leqslant \tilde\varrho$, $\tau \geqslant 0$, $\lambda \in [\Lambda_1, \Lambda_2]$ and set $U(t) = U^\tau_{\varrho,\lambda}(t, V)$, $\varepsilon(t) = \varepsilon^\tau_{\varrho,\lambda}(t, V)$. It follows directly from the second equation in (PS) that if $\tau \geqslant 0$ then $\varepsilon(t) \geqslant 0$ for all t for which our solution exists. Multiplying the first equation in (PS) by $U(t)$ and using (P) (see Remark 2.2) we obtain

$$\begin{gathered} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|U(t)|^2) = \big(\dot U(t), U(t)\big) = \big(F(\lambda, U(t)) - \varepsilon(t)\beta U(t), U(t)\big) \\ \leqslant \big(B_\lambda U(t) + G(\lambda, U(t)), U(t)\big) \leqslant C_1 |U(t)|^2 \text{ for } t \text{ such that } |U(t)| \leqslant \tilde\varrho \end{gathered}$$

with some $C_1 > 0$. The Gronwall lemma implies

$$|U(t)|^2 \leqslant |V|^2 \mathrm{e}^{rt} \text{ for all } t \in [0, t_0) \text{ if } |U(t)| \leqslant \tilde\varrho \text{ on } [0, t_0)$$

with $r = 2C_1$. If $\varrho_0^2 = \tilde\varrho^2 \mathrm{e}^{-r(t_M+1)}$ then we obtain

$$|U(t)|^2 \leqslant \tilde\varrho^2 \text{ for } |V| \leqslant \varrho_0, t \in [0, t_M + 1)$$

and the first estimate in (2.1) follows. Further, suppose that $\varrho \in (0, \varrho_0)$. We obtain from (PS) that $-\varrho^2 \leqslant \dot\varepsilon(t) \leqslant |U(t)|^2$ for $t \in [0, t_M + 1]$. Hence, for any $\tau^* \in [0, +\infty)$

there is C_{τ^*} such that $|\varepsilon(t)| \leqslant C_{\tau^*}$ for all $t \in [0, t_M + 1)$ if $\tau \in [0, \tau^*]$, $\varepsilon(0) = \tau$. Particularly, Remark 2.1 ensures the existence of our solution on $[0, t_M + 1)$.

Further, it follows from (PS) and the above estimates that

$$\begin{aligned}(\dot{U}(t), \dot{U}(t)) &\leqslant |(B_\lambda U(t) + G(\lambda, U(t)) - \varepsilon(t)\beta U(t), \dot{U}(t))| \\ &\leqslant C_1 |U(t)||\dot{U}(t)| + C_{\tau^*}|U(t)||\dot{U}(t)| \text{ for } t \in [0, t_M + 1)\end{aligned}$$

if $|V| \leqslant \varrho$, $\tau \in [0, \tau^*]$, and therefore the second estimate in (2.1) is a consequence of the first. □

Observation 2.1. If $\varrho > 0$, $\lambda \in I$, U, ε is a nontrivial periodic solution of (PS) then $\varepsilon(t) > 0$ for all t. Indeed, if $\varepsilon(t) \leqslant 0$ for some t then we obtain from (PS) that $\dot{\varepsilon}(t) > 0$. (We use the fact that $\varepsilon(t) = |U(t)| = 0$ can occur only in the case of the trivial solution.) Hence, if $\varepsilon(t_0) \leqslant 0$ for some t_0 then $\varepsilon(t) > \varepsilon(t_0)$ for all $t > t_0$ and therefore ε cannot be periodic. Particularly, we have $\tau \geqslant 0$ for any $[T, V, \tau, \lambda] \in \mathcal{C}_\varrho$.

Observation 2.2. If $[T, V, \tau, \lambda] \in \mathcal{C}_\varrho$, $T > 0$, $\lambda \in I$, $|V| + |\tau| > 0$ then $|V| > 0$, $\tau > 0$. Indeed, $\tau > 0$ follows directly from Observation 2.1. If $|V| = 0$, $\tau > 0$ then $U^\tau_{\varrho,\lambda}(t, 0) \equiv 0$ and $\varepsilon^\tau_{\varrho,\lambda}(t, V)$ is the solution of $\dot{\varepsilon} = -\varrho^2 \frac{\varepsilon}{1+|\varepsilon|}$, $\varepsilon(0) = \tau$ which is not periodic, that means $[T, V, \tau, \lambda] \notin \mathcal{C}_\varrho$ for $T > 0$.

Observation 2.3. If $[T, V, \tau, \lambda] \in \mathcal{C}_\varrho$, $T > 0$, $U(t) = U^\tau_{\varrho,\lambda}(t, V)$, $\varepsilon(t) = \varepsilon^\tau_{\varrho,\lambda}(t, V)$ then there exists t_0 such that $|U(t_0)| \leqslant \varrho$. Indeed, otherwise we would have

$$\dot{\varepsilon}(t) = -\varrho^2 \frac{\varepsilon(t)}{1 + |\varepsilon(t)|} + |U(t)|^2 > -\varrho^2 + \varrho^2 = 0 \text{ for all } t \geqslant 0,$$

which contradicts the condition $\varepsilon(T) = \tau$ $(= \varepsilon(0))$.

Remark 2.3. Any solution of (PS) fulfils $\dot{\varepsilon}(t) \geqslant -\varrho^2$. Particularly, if ϱ_0 is from Lemma 2.1, $\varrho_n \in (0, \varrho_0)$, $|V_n| \leqslant \varrho_0$, $\lambda_n \in [\Lambda_1, \Lambda_2]$, $\tau_n \to +\infty$, then

$$\lim_{n\to\infty} \inf_{t \in [0, t_M]} \varepsilon^{\tau_n}_{\varrho_n,\lambda_n}(t, V_n) = +\infty. \tag{2.2}$$

Lemma 2.2. *Let $t_M > 0$, $\Lambda_1, \Lambda_2 \in I$, $\Lambda_1 < \Lambda_2$, let $\varrho_0 > 0$ be from Lemma 2.1. Then there exists $C > 0$ such that if $U^\tau_{\varrho,\lambda}(\cdot, V)$, $\varepsilon^\tau_{\varrho,\lambda}(\cdot, V)$ are T-periodic, $\varrho \in (0, \varrho_0)$, $0 < T \leqslant t_M$, $|V| \leqslant \varrho_0$, $\lambda \in [\Lambda_1, \Lambda_2]$, $\tau \geqslant 0$ then*

$$-\int_0^T \varepsilon^\tau_{\varrho,\lambda}(t, V)(\beta U^\tau_{\varrho,\lambda}(t, V), \dot{U}^\tau_{\varrho,\lambda}(t, V))\,\mathrm{d}t \leqslant CT|V|^2.$$

P r o o f. Let $U(t) = U^{\tau}_{\varrho,\lambda}(t, V)$, $\varepsilon(t) = \varepsilon^{\tau}_{\varrho,\lambda}(t, V)$ be from the assumptions. Then we obtain by using the periodicity and (Pt) (Remark 2.2) that

$$0 = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\big(\varepsilon(t)|\beta U(t)|^2\big)\,\mathrm{d}t = \int_0^T \dot{\varepsilon}(t)|\beta U(t)|^2 + 2\varepsilon(t)\big(\beta U(t), \dot{U}(t)\big)\,\mathrm{d}t.$$

If $T \leqslant t_M$ then Lemma 2.1 gives $|U(t)|^2 \leqslant C_1|V|^2$ for all $t \in [0, T]$. We can suppose $\varrho_0 \leqslant 1$, that means $|V|^4 \leqslant |V|^2$. Hence, we obtain from the last equality and Observation 2.1 that

$$\begin{aligned} &-2\int_0^T \varepsilon(t)\big(\beta U(t), \dot{U}(t)\big)\,\mathrm{d}t = \int_0^T \dot{\varepsilon}(t)|\beta U(t)|^2\,\mathrm{d}t \\ &= \int_0^T \Big[-\varrho^2\frac{\varepsilon(t)}{1+|\varepsilon(t)|}|\beta U(t)|^2 + |U(t)|^2|\beta U(t)|^2\Big]\,\mathrm{d}t \leqslant CT|V|^4 \leqslant CT|V|^2. \end{aligned}$$

□

Lemma 2.3. *Let $t_M > 0$, $\Lambda_1, \Lambda_2 \in I$, $\Lambda_1 < \Lambda_2$, let ϱ_0 be from Lemma 2.1. Then there exists $C > 0$ such that if $U^{\tau}_{\varrho,\lambda}(\cdot, V)$, $\varepsilon^{\tau}_{\varrho,\lambda}(\cdot, V)$ are T-periodic, $\varrho \in (0, \varrho_0)$, $0 < T \leqslant t_M$, $|V| \leqslant \varrho_0$, $\tau \geqslant 0$, $\lambda \in [\Lambda_1, \Lambda_2]$ then*

$$\int_0^{t_M} |\dot{U}^{\tau}_{\varrho,\lambda}(t, V)|^2\,\mathrm{d}t \leqslant C|V|^2.$$

P r o o f. It follows from Lemma 2.1 and the assumption (L) that

$$|U^{\tau}_{\varrho,\lambda}(t, V)| \leqslant C_1|V|,\ \ |F(\lambda, U^{\tau}_{\varrho,\lambda}(t, V))| \leqslant C_2|V|$$

for all solutions under consideration, $t \in [0, t_M]$. Multiplying the first equation in (PS) by $\dot{U}^{\tau}_{\varrho,\lambda}(t, V)$ and using Lemma 2.2 (and the inequality $ab \leqslant \frac{a^2}{2\delta} + \frac{\delta b^2}{2}$) we obtain

$$\begin{aligned} &\int_0^T |\dot{U}^{\tau}_{\varrho,\lambda}(t, V)|^2\,\mathrm{d}t = \int_0^T \big((F(\lambda, U^{\tau}_{\varrho,\lambda}(t, V)) - \varepsilon^{\tau}_{\varrho,\lambda}(t, V)\beta U^{\tau}_{\varrho,\lambda}(t, V), \dot{U}^{\tau}_{\varrho,\lambda}(t, V)\big)\,\mathrm{d}t \\ &\leqslant C_2\int_0^T |V||\dot{U}^{\tau}_{\varrho,\lambda}(t, V)|\,\mathrm{d}t + C_3T|V|^2 \leqslant \frac{C_2T|V|^2}{2\delta} + \frac{C_2\delta}{2}\int_0^T |\dot{U}^{\tau}_{\varrho,\lambda}(t, V)|^2\,\mathrm{d}t + C_3T|V|^2 \end{aligned}$$

for any $\delta > 0$. Choosing δ small enough, we obtain

$$\int_0^T |\dot{U}^{\tau}_{\varrho,\lambda}(t, V)|^2\,\mathrm{d}t \leqslant C_4T|V|^2.$$

Setting $k_m = \max\{k;\ k \text{ is positive integer};\ kT \leqslant t_M\}$ and using the periodicity we get

$$\int_0^{t_M} |\dot{U}^{\tau}_{\varrho,\lambda}(t, V)|^2\,\mathrm{d}t \leqslant (k_m+1)TC_4|V|^2 \leqslant 2t_MC_4|V|^2 = C|V|^2.$$

□

Lemma 2.4. *Let $\Lambda_1, \Lambda_2 \in I$, $\Lambda_1 < \Lambda_2$, $t_M > 0$ and suppose that ϱ_0 is from Lemma 2.1, $\varrho_n \in (0, \varrho_0)$, $0 < T_n \leqslant t_M$, $\lambda_n \in [\Lambda_1, \Lambda_2]$, $V_n \in \mathbb{R}^\kappa$, $|V_n| \leqslant \varrho_0$, $\tau_n \in [0, +\infty)$, $U^{\tau_n}_{\varrho_n,\lambda_n}(T_n, V_n) = V_n$, $\varepsilon^{\tau_n}_{\varrho_n,\lambda_n}(T_n, V_n) = \tau_n$, $\tau_n \to +\infty$, $T_n \to T$, $\lambda_n \to \lambda$, $V_n \to V$. Then*

$$U^{\tau_n}_{\varrho_n,\lambda_n}(\cdot, V_n) \to U^\infty_\lambda(\cdot, V) \text{ in } C([0, t_M]) \text{ and weakly in } W^1_2(0, t_M),$$
$$U^\infty_\lambda(T, V) = V.$$

In the case $T = 0$, $U^\infty_\lambda(\cdot, V)$ is stationary.

If, moreover, $V = 0$, $W_n = \frac{V_n}{|V_n|} \to W$ then

$$\frac{U^{\tau_n}_{\varrho_n,\lambda_n}(\cdot, V_n)}{|V_n|} \to U^\infty_{0,\lambda}(\cdot, W) \text{ in } C([0, t_M]) \text{ and weakly in } W^1_2(0, t_M),$$
$$U^\infty_{0,\lambda}(T, W) = W.$$

In the case $T = 0$, $U^\infty_{0,\lambda}(\cdot, W)$ is stationary.

Proof. Set $U_n(t) = U^{\tau_n}_{\varrho_n,\lambda_n}(t, V_n)$, $\varepsilon_n(t) = \varepsilon^{\tau_n}_{\varrho_n,\lambda_n}(t, V_n)$. The conditions (P), (M) from Remark 2.2, the nonnegativeness of ε_n and the first equation in (PS) give

$$\int_0^{t_M} (\dot{U}_n - F(\lambda_n, U_n), Z - U_n)\,\mathrm{d}t = \int_0^{t_M} (\varepsilon_n \beta Z - \varepsilon_n \beta U_n, Z - U_n)\,\mathrm{d}t \geqslant 0$$
$$\text{for all } Z \in L^2(0, t_M) \text{ such that } Z(t) \in K \text{ for } t \in [0, t_M].$$

It follows from the first estimate in Lemma 2.1 and from Lemma 2.3 that $\{U_n\}$ is bounded in $W^1_2(0, t_M)$. Suppose that $U_n \to U$ weakly in $W^1_2(0, t_M)$. Then $U_n \to U$ in $C([0, t_M])$ by virtue of the compactness of the imbedding and the limiting process in the last inequality gives

$$\text{(2.3)} \qquad \begin{cases} \int_0^{t_M} \big(\dot{U} - F(\lambda, U), Z - U\big)\,\mathrm{d}t \geqslant 0 \\ \text{for all } Z \in L^2(0, t_M) \text{ such that } Z(t) \in K \text{ for a.a. } t \in [0, t_M]. \end{cases}$$

We claim to show that

$$\text{(2.4)} \qquad \begin{cases} U(t) \in K \text{ for all } t \in [0, t_M], \\ \big(\dot{U}(t) - F(\lambda, U(t)), Z - U(t)\big) \geqslant 0 \text{ for all } Z \in K, \text{ a.a. } t \in [0, t_M]. \end{cases}$$

We have

$$\left|\int_0^{t_M} (\dot{U}_n(t), U_n(t))\,\mathrm{d}t\right| = \frac{1}{2}\left|\int_0^{t_M} \frac{\mathrm{d}}{\mathrm{d}t}\big(|U_n(t)|\big)^2\,\mathrm{d}t\right| = \frac{1}{2}\big||U_n(t_M)|^2 - |U_n(0)|^2\big| \leqslant C_5.$$

Hence, it follows from Observation 2.1, (P) and (PS) that there exists $C_6 > 0$ such that

$$0 \leqslant \inf_{s\in[0,t_M]} \varepsilon_n(s) \int_0^{t_M} (\beta U_n(t), U_n(t))\,\mathrm{d}t \leqslant \int_0^{t_M} \varepsilon_n(t)(\beta U_n(t), U_n(t))\,\mathrm{d}t$$
$$= -\int_0^{t_M} (\dot{U}_n(t) - F(\lambda_n, U_n(t)), U_n(t))\,\mathrm{d}t \leqslant C_6.$$

It follows from Remark 2.3 that $\inf\limits_{t\in[0,t_M]} \varepsilon_n(t) \to +\infty$ and we obtain by using (P) that

$$(\beta U(t), U(t)) = \lim(\beta U_n(t), U_n(t)) = 0 \ \text{ for } t \in [0, t_M].$$

Hence, (P) implies the first line in (2.4). Suppose that the second line in (2.4) does not hold. Let $E \subset [0, t_M]$, $Z_0 \in K$ be such that $meas(E) > 0$ and

$$(\dot{U}(t) - F(\lambda, U(t)), Z_0 - U(t)) < 0 \text{ for all } t \in E.$$

Set

$$Z(t) = Z_0 \text{ for } t \in E,\ Z(t) = U(t) \text{ for } t \notin E.$$

Then $Z(t) \in K$ for $t \in [0, t_M]$ and $Z \in L^2(0, t_M)$. Hence

$$\int_0^{t_M} \bigl(\dot{U} - F(\lambda, U), Z - U\bigr)\,\mathrm{d}t = \int_E \bigl(\dot{U} - F(\lambda, U), Z_0 - U\bigr)\,\mathrm{d}t < 0,$$

which contradicts (2.3) and (2.4) is proved. Hence, $U(t) = U_\lambda^\infty(t, V)$. All these considerations could be done for an arbitrary subsequence of U_n weakly convergent in $W_2^1(0, t_M)$ and it follows that $U_n \to U = U_\lambda^\infty(\cdot, V)$ in $C([0, t_M])$ and weakly in $W_2^1([0, t_M])$.

Now, consider the case $T_n \to T = 0$. We have

$$0 = U_n(T_n) - V_n = \int_0^{T_n} \dot{U}_n(t)\,\mathrm{d}t = \int_0^{T_n} F(\lambda_n, U_n(t)) - \varepsilon_n(t)\beta U_n(t)\,\mathrm{d}t.$$

Multiply this equation by a fixed $Z \in K$. We obtain

$$\int_0^{T_n} (F(\lambda_n, U_n(t)), Z)\,\mathrm{d}t = \int_0^{T_n} \varepsilon_n(t)(\beta U_n(t), Z)\,\mathrm{d}t. \tag{2.5}$$

Further,

$$0 = |U_n(T_n)|^2 - |V_n|^2 = 2\int_0^{T_n} (\dot{U}_n(t), U_n(t))\,\mathrm{d}t$$
$$= 2\int_0^{T_n} (F(\lambda_n, U_n(t)) - \varepsilon_n(t)\beta U_n(t), U_n(t))\,\mathrm{d}t$$

and therefore

$$\int_0^{T_n} (F(\lambda_n, U_n(t)), U_n(t))\,\mathrm{d}t = \int_0^{T_n} (\varepsilon_n(t)\beta U_n(t), U_n(t))\,\mathrm{d}t.$$

This together with (2.5), (P) and (M) implies that for any $Z \in K$ we have

$$\frac{1}{T_n}\int_0^{T_n} (F(\lambda_n, U_n(t)), Z - U_n(t))\,\mathrm{d}t = \frac{1}{T_n}\int_0^{T_n} \varepsilon_n(t)(\beta U_n(t) - \beta Z, Z - U_n(t))\,\mathrm{d}t \leqslant 0.$$

The limiting proces (by using the fact that $T = 0$ is a Lebesgue point) gives

$$\frac{1}{T_n}\int_0^{T_n} (F(\lambda_n, U_n(t)), Z - U_n(t))\,\mathrm{d}t \to (F(\lambda, V), Z - V) \leqslant 0.$$

That means $U_\lambda^\infty(t, V) = V$ is a stationary solution of (I).

The case $V = 0$ can be treated similarly by dividing all expressions by $|V_n|$ and using the condition (G). $\square$

Lemma 2.5. *Let $\varrho_n \in (0, \varrho_0)$, $V_n \in \mathbb{R}^\kappa$, $\tau_n \geqslant 0$, $\lambda_n \in I$, $|V_n| \to 0$, $\tau_n \to \tau \in [0, +\infty)$, $\lambda_n \to \lambda$, $\frac{V_n}{|V_n|} \to W$ and either $\varrho_n \to 0$ or $\tau = 0$. Then*

$$\varepsilon_{\varrho_n,\lambda_n}^{\tau_n}(\cdot, V_n) \to \varepsilon \equiv \tau \text{ in } C^1([0,T]) \text{ for any } T > 0, \tag{2.6}$$

$$\frac{U_{\varrho_n,\lambda_n}^{\tau_n}(\cdot, V_n)}{|V_n|} \to U_{0,\lambda}^\tau(\cdot, W) \text{ in } C^1([0,T]) \text{ for any } T > 0. \tag{2.7}$$

If, moreover, $U_{\varrho_n,\lambda_n}^{\tau_n}(\cdot, V_n)$ are T_n-periodic, $T_n \to T$ then $U_{0,\lambda}^\tau(T, W) = W$. If $T = 0$ then $U_\lambda^\infty(\cdot, W)$ is stationary.

Proof. Set $U_n(t) = U_{\varrho_n,\lambda_n}^{\tau_n}(t, V_n)$, $\varepsilon_n(t) = \varepsilon_{\varrho_n,\lambda_n}^{\tau_n}(t, V_n)$. Lemma 2.1 implies that for any $T > 0$ there is n_0 such that (2.1) with t_M and $U_{\varrho,\lambda}^\tau$ replaced by T and U_n holds for all $n \geqslant n_0$ (i.e. for $|V_n|$ small enough). In particular, we get $|U_n| \to 0$ in $C([0,T])$ for any $T > 0$. We have

$$\dot{\varepsilon}_n(t) = -\varrho_n^2 \frac{\varepsilon_n(t)}{1 + |\varepsilon_n(t)|} + |U_n(t)|^2.$$

In the case $\varrho_n \to 0$, it follows that $\dot{\varepsilon}_n \to 0$ in $C([0,T])$, $\varepsilon_n \to \varepsilon \equiv \tau$ in $C([0,T])$ for any $T > 0$. Consider the case $\tau = 0$ (and not $\varrho_n \to 0$). Since $\tau_n \geqslant 0$, we obtain from the last equation successively that $\varepsilon_n(t) \geqslant 0$ for all t, $\dot{\varepsilon}_n(t) \leqslant |U_n(t)|^2$, and again $\varepsilon_n \to 0$ in $C([0,T])$ for any $T > 0$, which means that (2.6) holds.

Further, it follows from Lemma 2.1 that $\frac{U_n}{|V_n|}$ is bounded in $C^1([0,T])$ and therefore there exists a subsequence convergent in $C([0,T])$. It is sufficient to show that any such subsequence converges in $C^1([0,T])$ to $U^{\tau}_{0,\lambda}(\cdot, W)$. We will suppose without loss of generality that $\frac{U_n}{|V_n|} \to U_0$ in $C([0,T])$ and prove that $\frac{U_n}{|V_n|} \to U_0$ in $C^1([0,T])$, $U_0(t) = U^{\tau}_{0,\lambda}(t, W)$. We have

$$\frac{\dot{U}_n(t)}{|V_n|} = \frac{B_{\lambda_n} U_n(t)}{|V_n|} + \frac{G(\lambda_n, U_n(t))}{|V_n|} - \frac{\varepsilon_n(t)\beta U_n(t)}{|V_n|}.$$

It follows by using (G), (H) (see Remark 2.2) and (2.6) that

$$\frac{\dot{U}_n(t)}{|V_n|} \to B_\lambda U_0(t) - \tau\beta U_0(t) \text{ in } \; C([0,T]).$$

Hence, $\frac{U_n(t)}{|V_n|}$ is convergent in $C^1([0,T])$, $\dot{U}_0(t) = B_\lambda U_0(t) - \tau\beta U_0(t)$, i.e. $U_0(t) = U^{\tau}_{0,\lambda}(t, W)$, and (2.7) is proved.

It remains to show that if $T = 0$ then $U^{\tau}_{0,\lambda}(t, W)$ is stationary. The periodicity of U_n together with the assumption (G) (and the fact that $T = 0$ is a Lebesgue point) implies

$$0 = U_n(T_n) - V_n = \int_0^{T_n} \dot{U}_n(t)\,\mathrm{d}t = \int_0^{T_n} F(\lambda_n, U_n(t)) - \varepsilon_n(t)\beta U_n(t)\,\mathrm{d}t,$$

$$0 = \frac{1}{T_n}\int_0^{T_n} \frac{B_{\lambda_n} U_n(t)}{|V_n|} + \frac{G(\lambda_n, U_n(t))}{|V_n|} - \varepsilon_n(t)\frac{\beta U_n(t)}{|V_n|}\,\mathrm{d}t \to B_\lambda W - \tau\beta W.$$

This means $U^{\tau}_{0,\lambda}(t, W) \equiv W$ and our assertion is proved. □

3. Alexander-Yorke Theorem and Proof of Main Results

Consider an ordinary differential equation

$$\dot{X}(t) = H(\lambda, X(t)) \tag{3.1}$$

with a real parameter $\lambda \in I$, I being an open interval in $\mathbb{R}$. Suppose that $H\colon I \times \mathbb{R}^s \to \mathbb{R}^s$ is a smooth mapping, $H(\lambda, 0) = 0$ for all $\lambda \in I$. Hence, we have $H(\lambda, X) = A_\lambda X + N(\lambda, X)$, A_λ is an $s \times s$ matrix, $N\colon I \times \mathbb{R}^s \longmapsto \mathbb{R}^s$,

$$N(\lambda, X) = O(|X|^2). \tag{3.2}$$

We will denote by $X_\lambda(t, Y)$ the solution of (3.1) with the initial condition Y at $t = 0$.

Suppose that there is a set of isolated parameters $\mathcal{P} = \{\lambda_1, \lambda_2, \ldots\} \subset I$ such that

(3.3) for $\lambda = \lambda_j \in \mathcal{P}$, there are exactly two eigenvalues $\pm\, \mathrm{i}\omega_j, \omega_j > 0$ of A_{λ_j} on the imaginary axis,

(3.4) for $\lambda \in I \setminus \mathcal{P}$, no eigenvalue of A_λ lies on the imaginary axis,

(3.5) the real parts of the eigenvalues of A_λ near $\pm\, \mathrm{i}\omega_j$ change sign as λ increases past λ_j.

It follows that the linearized equation

$$\dot{X}(t) = A_\lambda X(t) \tag{3.6}$$

has periodic solutions only for $\lambda = \lambda_j \in \mathcal{P}$, the corresponding smallest periods being $T_j = \frac{2\pi}{\omega_j}$. A bifurcation of periodic solutions of (3.1) from the branch of trivial solutions can occur only at $\lambda = \lambda_j \in \mathcal{P}$, $T = \frac{2k\pi}{\omega_j}$, $k = 1, 2, \ldots$.

Set

$$\begin{aligned}
&\mathcal{B} = \Big\{[T, 0, \lambda] \in (0, +\infty) \times \mathbb{R}^s \times I;\ T = \frac{2\pi k}{\omega_j}, \lambda = \lambda_j \in \mathcal{P}, k \text{ positive integer}\Big\},\\
&\mathcal{L} = \{[T, Y, \lambda] \in [0, +\infty) \times \mathbb{R}^s \times I;\ X_\lambda(T, Y) = Y\},\\
&\mathcal{C} = \big(\mathcal{L} \setminus (0, +\infty) \times \{0\} \times I\big) \cup \mathcal{B}.
\end{aligned}$$

Then $\mathcal{C}$ is closed and contains the closure of $\mathcal{L} \setminus (0, +\infty) \times \{0\} \times I$.

We will use the following version of the Alexander-Yorke Theorem given in [4].

Theorem 3.1. (Alexander and Yorke). *Let the assumptions* (3.2)–(3.5) *be fulfilled. Consider a fixed $\lambda_j \in \mathcal{P}$ and a positive integer k. Denote by $\mathcal{C}_0$ the component of $\mathcal{C}$ containing $\big[\frac{2\pi k}{\omega_j}, 0, \lambda_j\big]$. Then at least one of the following conditions is fulfilled:*

(i) *$\mathcal{C}_0$ contains a point $\big[\frac{2\pi m}{\omega_l}, 0, \lambda_l\big]$ with some $\lambda_l \in \mathcal{P}$, $\lambda_l \neq \lambda_j$, m positive integer;*

(ii) *$\mathcal{C}_0$ is unbounded in the sense that it contains points $[T, Y, \lambda]$ for which T is arbitrarily large or $|Y|$ is arbitrarily large or λ lies outside any given compact subset of I.*

P r o o f see [4]. □

P r o o f of Theorem 1.1: Let $\varrho > 0$ be fixed. Set $s = \kappa + 1$,

$$N(\lambda, X) = N(\lambda, U, \varepsilon) = \Big[G(\lambda, U) - \varepsilon\beta U, -\varrho^2 \frac{\varepsilon^2}{1 + |\varepsilon|} + |U|^2\Big]$$

$$\text{for } \lambda \in I, X = [U, \varepsilon] \in \mathbb{R}^{\kappa+1},$$

$$A_\lambda = \begin{pmatrix} B_\lambda, & 0 \\ 0, & -\varrho^2 \end{pmatrix},$$

i.e., $a^\lambda_{j,l} = b^\lambda_{j,l}$ for $j, l = 1, 2, \ldots, \kappa$, $a^\lambda_{j,\kappa+1} = a^\lambda_{\kappa+1,j} = 0$ for $j = 1, \ldots, \kappa$, $a^\lambda_{\kappa+1,\kappa+1} = -\varrho^2$, where $A_\lambda = (a^\lambda_{j,l})^{\kappa+1}_{j,l=1}$, $B_\lambda = (b^\lambda_{j,l})^{\kappa}_{j,l=1}$. Then (3.1) with $X(t) = [U(t), \varepsilon(t)]$ is equivalent to the system (PS) and (3.2) is fulfilled under the assumption (G). If $\mu \neq -\varrho^2$ then μ is an eigenvalue of B_λ (for some $\lambda \in I$) if and only if μ is an eigenvalue of A_λ. It follows that (3.3)–(3.5) are fulfilled under the assumption (μ). For our special choice of A_λ, N and $k = 1$, the sets $\mathcal{P}$, $\mathcal{B}$, $\mathcal{L}$, $\mathcal{C}$ and $\mathcal{C}_0$ from Theorem 3.1 coincide with $\{\lambda_0\}$, $\mathcal{B}, \mathcal{L}_\varrho$, $\mathcal{C}_\varrho$ and $\mathcal{C}^0_\varrho$, respectively, introduced in Section 1. Let $\varrho > 0$ be fixed. Since $\mathcal{P} = \{\lambda_0\}$, the condition (i) in Theorem 3.1 is excluded and $\mathcal{C}^0_\varrho$ must be unbounded. The closedness of $\mathcal{C}^0_\varrho$ was explained after Notation 1.1. If $\mathcal{C}^0_\varrho$ contains no point of the type $[0, V, \tau, \lambda]$ then the condition (1.1) follows from Remark 1.2, Observations 2.1, 2.2 and by integrating the second equation in (PS). $\square$

P r o o f of Theorem 1.3. Let ϱ_0, γ, Λ_1, Λ_2, t_M be from the assumption (1.2). We will suppose that ϱ_0 is simultaneously such that (2.1) from Lemma 2.1 holds. Let $\varrho \in (0, \varrho_0)$ be fixed. The set $\mathcal{C}^0_\varrho$ contains no point of the type $[0, V, \tau, \lambda]$ by the assumption (1.2). Hence, (1.1) holds by Theorem 1.1.

It follows from Observation 2.3 and (1.2) that for any $[T, V, \tau, \lambda] \in \mathcal{C}^0_\varrho$ there exists t_0 such that $|U^\tau_{\varrho,\lambda}(t_0, V)| < \varrho$. Lemma 2.1 together with the T-periodicity and (1.2) implies the existence of $C(\varrho) > 0$ such that

(3.7) $$|V| \leqslant C(\varrho) \quad \text{for all } [T, V, \tau, \lambda] \in \mathcal{C}^0_\varrho,\ C(\varrho) \to 0 \text{ as } \varrho \to 0.$$

The set $\mathcal{C}^0_\varrho$ is unbounded by Theorem 1.1 and it follows from (1.2), (3.7) that it must be unbounded in τ. According to Observation 2.1, this means that there exists a sequence $\{[T_n, V_n, \tau_n, \lambda_n]\} \subset \mathcal{C}^0_\varrho$ such that $T_n \to T_\varrho \in [\gamma, t_M]$, $V_n \to V_\varrho, \lambda_n \to \lambda_\varrho \in [\Lambda_1, \Lambda_2]$, $\tau_n \to +\infty$. Lemma 2.4 implies that (1.5) holds and $U^\infty_{\lambda_\varrho}(T_\varrho, V_\varrho) = V_\varrho$. We have

$$\int_0^{T_n} |U^{\tau_n}_{\varrho,\lambda_n}(t, V_n)|^2 \,\mathrm{d}t = \int_0^{T_n} \varrho^2 \frac{\varepsilon^{\tau_n}_{\varrho,\lambda_n}(t, V_n)}{1 + |\varepsilon^{\tau_n}_{\varrho,\lambda_n}(t, V_n)|} \,\mathrm{d}t$$

by (1.1). The limiting process in this equation by using Remark 2.3 gives (1.6). Particularly, it follows that $|V_\varrho| > 0$ because $U^\infty_{\lambda_\varrho}(t, 0) = 0$ for all $t \geqslant 0$.

We get $|V_\varrho| \leqslant C(\varrho)$ as a consequence of the above considerations. Hence, if $\varrho_n \to 0$ then $|V_{\varrho_n}| \to 0$ for V_{ϱ_n} obtained by the procedure described above by (3.7). Let us prove that the solutions $U^\infty_{\lambda_\varrho}(\cdot, V_\varrho)$ obtained are nonstationary for all $\varrho \in (0, \varrho_0)$ if ϱ_0 is small enough. Suppose by way of contradiction that there are $\varrho_n \to 0$ such that $U^\infty_{\lambda_{\varrho_n}}(\cdot, V_{\varrho_n})$ are stationary, that means

(3.8) $$V_{\varrho_n} \in K,\ \bigl(-B_{\lambda_{\varrho_n}} V_{\varrho_n} - G(\lambda_{\varrho_n}, V_{\varrho_n}), Z - V_{\varrho_n}\bigr) \geqslant 0 \text{ for all } Z \in K,$$

$|V_{\varrho_n}| \to 0$. Suppose that $\frac{V_{\varrho_n}}{|V_{\varrho_n}|} \to W$, $\lambda_{\varrho_n} \to \lambda_I$. Dividing (3.8) by $|V_{\varrho_n}|$ (and writting $|V_{\varrho_n}|Z$ instead of Z) we obtain by the limiting process by using (G) that

$$W \in K,\ \big(-B_{\lambda_I}W, Z-W\big) \geqslant 0 \text{ for all } Z \in K. \tag{3.9}$$

If $W \in \partial K$ then this is a contradiction with the assumption (1.3). If $W \in K^0$ then the last inequality is equivalent to $B_{\lambda_I}W = 0$ and this contradicts the assumption (μ).

Suppose that $\varrho_n \to 0$ and T_{ϱ_n}, V_{ϱ_n}, λ_{ϱ_n} are obtained by the procedure described above, $T_{\varrho_n} \to T \in [\gamma, t_M]$, $|V_{\varrho_n}| \to 0$, $\frac{V_{\varrho_n}}{|V_{\varrho_n}|} \to W$, $\lambda_{\varrho_n} \to \lambda_I$. We shall prove that $U^{\infty}_{0,\lambda_I}(\cdot, W)$ is nonstationary T-periodic. For any n fixed there exist $[T^k_{\varrho_n}, V^k_{\varrho_n}, \tau^k_{\varrho_n}\lambda^k_{\varrho_n}] \in \mathcal{C}^0_{\varrho_n}$, $k = 1, 2, \dots$, such that $T^k_{\varrho_n} \to T_{\varrho_n}$, $V^k_{\varrho_n} \to V_{\varrho_n}$, $\tau^k_{\varrho_n} \to +\infty$, $\lambda^k_{\varrho_n} \to \lambda_{\varrho_n}$ as $k \to +\infty$. We can find k_n such that, setting $T_n = T^{k_n}_{\varrho_n}$, $V_n = V^{k_n}_{\varrho_n}$, $\tau_n = \tau^{k_n}_{\varrho_n}$, $\lambda_n = \lambda^{k_n}_{\varrho_n}$, we obtain $T_n \to T$, $|V_n| \to 0$, $\frac{V_n}{|V_n|} \to W$, $\tau_n \to +\infty$, $\lambda_n \to \lambda_I$. It follows from Lemma 2.4 that $\frac{U^{\tau_n}_{\varrho_n,\lambda_n}(\cdot, V_n)}{|V_n|} \to U^{\infty}_{0,\lambda_I}(\cdot, W)$ in $C([0, t_M])$. We have $U^{\tau_n}_{\varrho_n,\lambda_n}(T_n, V_n) = V_n$ and this yields $U^{\infty}_{0,\lambda_I}(T, W) = W$. Analogously as above, (3.9) is impossible by the assumptions (1.3), (μ), which means that $U^{\infty}_{0,\lambda_I}(\cdot, W)$ is not stationary. $\square$

4. Appendix

For the sake of completeness we give here technical proofs of some assertions used in the previous text.

P r o o f o f t h e a s s e r t i o n o f R e m a r k 1.3. We have to show that if (1.3), (1.4) and

$$\text{if } [T, V, \tau, \lambda] \in \mathcal{C}^0_{\varrho},\ \varrho \in (0, \varrho_0) \text{ then } T < t_M,\ \lambda \in (\Lambda_1, \Lambda_2) \tag{1.2$'$}$$

hold then there exists $\gamma > 0$ such that

$$\text{if } [T, V, \tau, \lambda] \in \mathcal{C}^0_{\varrho},\ \varrho \in (0, \varrho_0) \text{ then } T > \gamma.$$

Suppose by contradiction that there are $\varrho_n \to 0$, $[T_n, V_n, \tau_n, \lambda_n] \in \mathcal{C}^0_{\varrho_n}$ such that $T_n \to 0$. By the fact that $\mathcal{C}^0_{\varrho}$ is connected, $\big[\frac{2\pi}{\omega_0}, 0, 0, \lambda_0\big] \in \mathcal{C}^0_{\varrho}$ for any ϱ, we can suppose without loss of generality that $T_n > 0$. Observation 2.3 together with the T_n-periodicity of $U^{\tau_n}_{\varrho_n,\lambda_n}(\cdot, V_n)$ implies that $|V_n| \to 0$ and we can suppose $\frac{V_n}{|V_n|} \to W$, $\tau_n \to \tau \in [0, +\infty]$, $\lambda_n \to \lambda \in [\Lambda_1, \Lambda_2]$ by (1.2'). It follows from Lemma 2.4 or 2.5 (if τ is infinite or finite, respectively) that $U^{\tau}_{0,\lambda}(\cdot, W)$ is stationary. If $\tau = +\infty$ then this contradicts (1.3) or (μ) in the case $U \in \partial K$ or $U \in K^0$, respectively. (In the latter case $U^{\tau}_{0,\lambda}(\cdot, W)$ would coincide with a stationary solution of (LE).) If $\tau \in (0, +\infty)$ or $\tau = 0$ then we get a contradiction with (1.4) or (μ), respectively. $\square$

Remark 4.1. Let $t_M > 0$, $\Lambda_1, \Lambda_2 \in I$, $\Lambda_1 < \Lambda_2$ be given and let ϱ_0 be from Lemma 2.1, $\varrho_n \in (0, \varrho_0)$, $\varrho_n \to \varrho$, $V_n \to V$, $\tau_n \to \tau \in [0, +\infty]$, $\lambda_n \in [\Lambda_1, \Lambda_2]$, $\lambda_n \to \lambda$. Then the following implications hold:

$$\begin{gathered}
\text{if } t^{\tau}_{\varrho,\lambda}(V) < t_M,\ \dot{\varphi}^{\tau}_{\varrho,\lambda}(t^{\tau}_{\varrho,\lambda}(V), V) < 0 \text{ then } t^{\tau_n}_{\varrho_n,\lambda_n}(V_n) \to t^{\tau}_{\varrho,\lambda}(V);\\
\text{if } |V| = 0,\ \frac{V_n}{|V_n|} \to W,\ t^{\tau}_{0,\lambda}(W) < +\infty,\ \dot{\varphi}^{\tau}_{0,\lambda}(t^{\tau}_{0,\lambda}(W), W) < 0\\
\text{then } t^{\tau_n}_{\varrho_n,\lambda_n}(V_n) \to t^{\tau}_{0,\lambda}(W).
\end{gathered}$$

The proof can be performed in the same elementary way as that of Theorem 2.2 in [8].

P r o o f o f (1.11). We will prove only the second assertion. The proof of the first for $\tau \in [0, +\infty)$ is simpler and for $\tau = +\infty$ is the same as in [9], Model Situation.

Consider a fixed solution $U(t) = U^{\tau}_{\varrho,\lambda}(t, V)$ with $v_3 \geqslant 0$, $\tau \in [0, +\infty)$, $\lambda \leqslant 1$ such that $U(t) \notin \operatorname{Lin}\{U_3\}$, $|U(t)| < \varrho_1$ for $t \in [0, \tilde{t}]$, and set $\varphi(t) = \varphi^{\tau}_{\varrho,\lambda}(t, V)$.

A simple geometrical idea or a calculus yield that

$$\dot{\varphi}(t) = \frac{\big(\frac{\mathrm{d}}{\mathrm{d}t}\big(P_L P_K U(t)\big), P_L^* P_K U(t)\big)}{|P_L P_K U(t)|^2}. \tag{4.1}$$

If t is fixed then

$$\text{either } \frac{\mathrm{d}}{\mathrm{d}t}\big(P_K U(t)\big) = \dot{U}(t) \text{ or } \frac{\mathrm{d}}{\mathrm{d}t}\big(P_K U(t)\big) = \Big[\frac{\dot{u}_1(t) + \dot{u}_3(t)}{2}, \dot{u}_2(t), \frac{\dot{u}_1(t) + \dot{u}_3(t)}{2}\Big].$$

The former case occurs always if $U(t) \in K^0$, the latter always if $U(t) \notin K$. If $U(t) \in \partial K$ then the first or the second equality holds if the trajectory of the equation (E) at t is directed into K or outside of K, precisely if $F(\lambda, U(t)) \in K_{U(t)}$ or $F(\lambda, U(t)) \notin K_{U(t)}$, respectively, where $K_{U(t)}$ is the contingent cone to K at $U(t)$. In the first case, by the fact that $u_3^2(t) \leqslant C_1[u_1^2(t) + u_2^2(t)]$ for $t \in [0, \tilde{t}]$ (which follows from the assumption $U(t) \notin \operatorname{Lin}\{U_3\}$, $|U(t)| < \varrho_1$ for $t \in [0, \tilde{t}]$) and by the assumption (G) we get

$$\begin{aligned}
\dot{\varphi}(t) &= \frac{-\big(\lambda u_1(t) + u_2(t) + g_1(\lambda, U(t))\big)u_2(t) + \big(-u_1(t) + \lambda u_2(t) + g_2(\lambda, U(t))\big)u_1(t)}{u_1^2(t) + u_2^2(t)}\\
&\leqslant -1 + C\frac{|G(\lambda, U(t))|}{|U(t)|} \leqslant -\frac{1}{2}
\end{aligned}$$

if ϱ_1 is small enough. In the other case we have $u_1(t) \geqslant u_3(t) \geqslant 0$ and (4.1) gives

$$\begin{aligned}
&\dot\varphi(t)\\
&= \frac{-\frac12[\lambda u_1 + u_2 - u_3 + g_1(\lambda,U) + g_3(\lambda,U)]u_2 + \frac12[-u_1 + \lambda u_2 + g_2(\lambda,U)][u_1+u_3]}{(\frac{u_1+u_3}{2})^2 + u_2^2}\\
&\leqslant \frac{-u_1^2 - u_2^2 + (\lambda+1)u_2u_3 - u_1u_3}{\frac12(u_1+u_3)^2 + 2u_2^2} + C\frac{|G(\lambda,U)|}{|U|}\\
&\leqslant \frac{-u_1^2 - u_2^2 + 2\big|u_2|u_3 - u_1u_3}{\frac12(u_1+u_3)^2 + 2u_2^2} + C\frac{|G(\lambda,U)|}{|U|}.
\end{aligned}$$

If $\frac12 u_1(t) \geqslant u_3(t)$ then

$$-u_1^2 - u_2^2 + 2|u_2|u_3 - u_1u_3 \leqslant -u_1^2 - u_2^2 + |u_2|u_1 - u_1u_3 \leqslant -\frac12(u_1^2+u_2^2)$$

and we get

$$\dot\varphi(t) \leqslant \frac{-\frac12(u_1^2+u_2^2)}{2(u_1^2+u_2^2)} + C\frac{|G(\lambda,U)|}{|U|} \leqslant -\frac14 + C\frac{|G(\lambda,U)|}{|U|} \leqslant -\frac18$$

if ϱ_1 is small enough. If $\frac12 u_1(t) \leqslant u_3(t) \leqslant u_1(t)$ then

$$\begin{aligned}
-u_1^2 - u_2^2 + 2|u_2|u_3 - u_1u_3 &\leqslant -\frac32 u_1^2 - u_2^2 + 2\big|u_2|u_1\\
&\leqslant -\frac32 u_1^2 - u_2^2 + \frac43 u_1^2 + \frac34 u_2^2 \leqslant -\frac16(u_1^2+u_2^2)
\end{aligned}$$

and for ϱ_1 small enough we get

$$\dot\varphi(t) \leqslant \frac{-\frac16(u_1^2+u_2^2)}{2(u_1^2+u_2^2)} + C\frac{|G(\lambda,U)|}{|U|} \leqslant -\frac1{12} + C\frac{|G(\lambda,U)|}{|U|} \leqslant -\frac1{24}.$$

□

P r o o f o f (1.12). Consider an arbitrary fixed T-periodic solution $U(t) = U^\tau_{0,\lambda}(t,W)$ with some $|W| > 0$, $\lambda \geqslant 1$ or $\lambda \leqslant 0$, $\tau \in (0,\infty]$. It is sufficient to show that

$$\begin{cases}
\text{if } \lambda \geqslant 1 \text{ then } \frac{\mathrm d}{\mathrm dt}\big(|P_K U(t)|^2\big) > 0 \text{ for a.a. } t \geqslant 0,\\
\text{if } \lambda \leqslant 0, \text{ then } \frac{\mathrm d}{\mathrm dt}\big(|U(t)|^2\big) \leqslant 0 \text{ for a.a. } t \geqslant 0 \text{ and}\\
\frac{\mathrm d}{\mathrm dt}\big(|U(t)|^2\big) < 0 \text{ in a nonempty subinterval of } [0,T],
\end{cases} \tag{4.2}$$

which contradicts the periodicity. Recall that $u_3(t) \geqslant 0$ for all t (see (1.9)) and clearly $|U(t)| > 0$ for all $t \geqslant 0$. First, let $\lambda \geqslant 1$. It follows from (1.8) that $u_1^2(t) + u_2^2(t)$

increases and $u_3(t)$ does not increase on any interval of t where $u_3(t) > u_1(t)$ because $U(t)$ coincides with the solution of (1.8) on such an interval. Since there is no periodic solution of (1.8) with $u_3(t) > u_1(t)$ for all t, there exists t_0 such that $u_3(t_0) \leqslant u_1(t_0)$ and it follows by using the periodicity

$$u_3^2(t) < u_1^2(t) + u_2^2(t) \quad \text{on any interval where } u_3(t) > u_1(t). \tag{4.3}$$

Hence, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(|P_K U(t)|^2\big) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(|U(t)|^2\big) = \big(\dot{U}(t), U(t)\big) = \lambda[u_1^2(t)+u_2^2(t)]-u_3^2(t) > 0 \tag{4.4}$$

on any such interval if $\lambda \geqslant 1$ by (4.3). Let $0 \leqslant u_3(t) \leqslant u_1(t)$. If $\tau = +\infty$ then we obtain from (LI) (setting successively $Z = 0$ and $Z = 2U(t)$) that $(\dot{U}(t) - B_\lambda U(t), U(t)) = 0$. We get the same expression as in (4.4) but it is equal to zero for $\lambda = 1$, $u_2(t) = 0$, $u_3(t) = u_1(t)$ (and remains positive for $\lambda \geqslant 1$, $u_2(t) \neq 0$). However, the points t for which $u_2(t) = 0$ are isolated by virtue of (1.11) if $\lambda = 1$. If $\tau \in (0, +\infty)$ then we obtain by using the formula for $P_K U$ and (1.10) with $g_j \equiv 0$ and $\varepsilon(t) \equiv \tau$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(|P_K U(t)|^2\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big[\Big(\frac{u_1(t)+u_3(t)}{2}\Big)^2 + (u_2(t))^2 + \Big(\frac{u_1(t)+u_3(t)}{2}\Big)^2\Big] \\ &= \lambda u_1^2 - u_3^2 + 2\lambda u_2^2 + (\lambda-1)u_1u_3 + u_2u_3 - u_2u_1 \\ &\geqslant (\lambda-1)u_1^2 + 2(\lambda-1)u_2^2 + (\lambda-1)u_1u_3 + [u_1+u_3-u_2][u_1-u_3] + 2u_2^2.\end{aligned}$$

(In fact, (4.4) holds if $u_3(t) = u_1(t)$ and the trajectory of (E) at t is not directed outside of K, i.e. if $F(\lambda, U(t)) \in K_{U(t)}$—cf. the considerations in the proof of (1.11). However, the equality in (4.4) is equivalent to the last one for $u_3(t) = u_1(t)$.) The last expression is positive if $u_2 < u_1 + u_3$ and either $u_2 \neq 0$ or $\lambda > 1$. For $\lambda = 1$, the times t for which $u_2(t) = 0$ are isolated by (1.11) again. For $u_2 \geqslant u_1 + u_3$ (which implies $u_1 - u_3 \leqslant u_2$), the last expression is not less than

$$-u_2[u_1-u_3] + 2u_2^2 \geqslant u_2^2 > 0$$

and the first implication in (4.2) is proved. Further, let $\lambda \leqslant 0$. If $\tau = +\infty$ then we have the same expression as in (4.4). This time it is always negative if $\lambda < 0$ and is always nonpositive and strictly negative at least on some interval if $\lambda = 0$ because $u_3(t)$ cannot be zero for all t according to (1.11) and the form of K. If $\tau \in (0, +\infty)$ then we obtain by using (1.9), (1.10) with $g_j \equiv 0$, $\varepsilon(t) \equiv \tau$ that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(|U|^2\big) &= \Big(\lambda u_1 + u_2 - \tau\frac{(u_3-u_1)^-}{2}\Big)u_1 - (u_1 - \lambda u_2)u_2 - \Big(u_3 - \tau\frac{(u_3-u_1)^-}{2}\Big)u_3 \\ &= \lambda\big(u_1^2+u_2^2\big) - u_3^2 - \frac{\tau}{2}(u_3-u_1)^-(u_1-u_3).\end{aligned}$$

The last expression is always nonpositive and is negative at least for t such that $u_3(t) < u_1(t)$. This is true on a nonempty subinterval because there is no periodic solution of (1.8) with $u_3(t) \geqslant u_1(t)$ for all t. $\square$

P r o o f o f (1.13). Suppose by contradiction that this is not true. Since $[2\pi, 0, 0, 0] \in \mathcal{C}_\varrho^M$ for any $\varrho > 0$, it follows from the connectedness of $\mathcal{C}_\varrho^M$ that there are $\varrho_n \to 0$, $[T_n, V_n, \tau_n, \lambda] \in \mathcal{C}_{\varrho_n}^M$, $T_n \in [\gamma, t_M]$, $T_n \to T$, $\tau_n \to \tau \in [0, +\infty]$ and either $\lambda = -\xi$ or $\lambda = 1$. It follows from Observation 2.3, Lemma 2.1 and the T_n-periodicity of $U_{\varrho_n,\lambda_n}^{\tau_n}(t, V_n)$, $T_n \leqslant t_M$ that $|V_n| \to 0$. We can suppose $\frac{V_n}{|V_n|} \to W$. We have $U_{\varrho_n,\lambda_n}^{\tau_n}(T_n, V_n) = V_n$ and Lemma 2.4 or 2.5 implies that $U_{0,\lambda}^{\tau}(\cdot, W)$ is periodic or stationary. This contradicts (1.12) or the fact that for $\tau = 0$, (LPE) is equivalent to (1.8) which has a periodic solution only if $\lambda = 0$. $\square$

P r o o f o f (1.14). First, suppose that there are $\varrho_n \to 0$, $[T_n, V_n, \tau_n, \lambda_n] \in \mathcal{C}_{\varrho_n}^M$, $[T_n, V_n, \tau_n, \lambda_n] \neq [2k\pi, 0, 0, 0]$, $k = 1, 2, \ldots$, $T_n \to T$, $\tau_n \to \tau \in [0, +\infty]$, $\lambda_n \to \lambda \in [-\xi, 1]$ (see (1.13)), $t_n \to t_0 \in [0, t_M]$, $U_{\varrho_n,\lambda_n}^{\tau_n}(t_n, V_n) \in \operatorname{Lin}\{U_3\}$. Set $Z_n = U_{\varrho_n,\lambda_n}^{\tau_n}(t_n, V_n)$. We have $|V_n| > 0$ by Observation 2.2, therefore also $|Z_n| > 0$. It follows from Observation 2.3, Lemma 2.1 and the T_n-periodicity that $|Z_n| \to 0$ and we can suppose $\frac{Z_n}{|Z_n|} \to Z = [z_1, z_2, z_3]$. We have $U_{\varrho_n,\lambda_n}^{\tau_n}(T_n, Z_n) = Z_n$ and it follows by using Lemma 2.4 or 2.5 that $U_{0,\lambda}^{\tau}(T, Z) = Z \in \operatorname{Lin}\{U_3\}$, $U_{0,\lambda}^{\tau}(\cdot, Z)$ is stationary if $T = 0$. Simultaneously $U_{0,\lambda}^{0}(t, Z) = [0, 0, e^{-t} z_3] \in K^0$ for all $t \geqslant 0$ and therefore this solution coincides with $U_{0,\lambda}^{\tau}(t, Z)$. This contradiction proves that if ϱ_0 is small enough and $[T, V, \tau, \lambda]$ is from (1.14) then $U_{\varrho,\lambda}^{\tau}(t, V) \notin \operatorname{Lin}\{U_3\}$ for $t \in [0, t_M]$, which means also for all $t \geqslant 0$ by virtue of the T-periodicity, $T \leqslant t_M$.

Let ϱ_1 be from (1.11). It follows from Lemma 2.1 that we can choose $\varrho_0 > 0$ such that $|U_{\varrho,\lambda}^{\tau}(t, V)| < \varrho_1$ for all $t \in [0, t_M]$ if $|V| < \varrho_0$, $\tau \in [0, +\infty)$, $\lambda \in [-\xi, 1]$. If $[T, V, \tau, \lambda]$ satisfies the assumptions of (1.14) then Observation 2.3 together with the T-periodicity, $T \leqslant t_M$, the choice of ϱ_0 and (1.13) imply that $|U_{\varrho,\lambda}^{\tau}(t, V)| < \varrho_1$ for all $t \geqslant 0$. Now, the estimate of $\dot{\varphi}_{\varrho,\lambda}^{\tau}(t, V)$ is a consequence of (1.11) and (1.9). The estimates $t_{\varrho,\lambda}^{\tau}(V) < t_M$ and $T \geqslant t_{\varrho,\lambda}^{\tau}(V)$ follow. Thanks to the form of K, more than one half of a circuit of any trajectory under consideration lies in K^0 where it coincides with the trajectory of the equation (1.8). The time of a half circuit of any solution of (1.8) not starting on $\operatorname{Lin}\{U_3\}$ is π, which means $t_{\varrho,\lambda}^{\tau}(V) > \pi > \gamma$. $\square$

P r o o f o f (1.15). Let $\varrho \in (0, \varrho_0)$ be fixed and suppose that there is no δ with the properties from (1.15). Then there are $[T_n, V_n, \tau_n, \lambda_n] \in \mathcal{C}_\varrho^0$, $[T_n, V_n, \tau_n, \lambda_n] \neq [2\pi, 0, 0, 0]$, $[T_n, V_n, \tau_n, \lambda_n] \to [2\pi, 0, 0, 0]$ such that either $V_n \in \operatorname{Lin}\{U_3\}$ or $T_n \neq t_{\varrho,\lambda_n}^{\tau_n}(V_n)$. In fact, we necessarily have $V_n \notin \operatorname{Lin}\{U_3\}$, $T_n > t_{\varrho,\lambda_n}^{\tau_n}(V_n)$ by (1.14). Simultaneously $\dot{\varphi}_{\varrho,\lambda_n}^{\tau_n}(t, V_n) < -\eta$ for all $t \geqslant 0$ by (1.14) and it follows that $\varphi_{\varrho,\lambda_n}^{\tau_n}(T_n, V_n) = -2k_n\pi$ with some $k_n > 1$. We can suppose $\frac{V_n}{|V_n|} \to W$. Lemma 2.5 ensures $U_{0,0}^{0}(2\pi, W) = W$ and therefore $W \notin \operatorname{Lin}\{U_3\}$, $\dot{\varphi}_{0,0}^{0}(t_{\varrho,\lambda}^{\tau}(W), W) < 0$

according to the form of (1.8). We have $t^{\tau_n}_{\varrho,\lambda_n}(V_n) \to t^0_{0,0}(W) = 2\pi$ by Remark 4.1. Hence, we obtain $t^{\tau_n}_{\varrho,\lambda_n}(Z_n) \leqslant T_n - t^{\tau_n}_{\varrho,\lambda_n}(V_n) \to 0$ with $Z_n = U^{\tau_n}_{\varrho,\lambda_n}(t^{\tau_n}_{\varrho,\lambda_n}(V_n), V_n)$. However, $[T_n, Z_n, \tau_n, \lambda_n] \in \mathcal{C}^0_\varrho$ and we should have $t^{\tau_n}_{\varrho,\lambda_n}(Z_n) > \gamma$ by (1.14), which is the contradiction. $\square$

P r o o f o f (1.16). Consider the sets

$$\mathcal{C}^1_\varrho = \{[T, V, \tau, \lambda] \in \mathcal{C}^M_\varrho;\ \tau > 0,\ T = t^\tau_{\varrho,\lambda}(V)\} \cup \{[2\pi, 0, 0, 0]\},$$
$$\mathcal{C}^2_\varrho = \{[T, V, \tau, \lambda] \in \mathcal{C}^M_\varrho;\ \tau > 0,\ T > t^\tau_{\varrho,\lambda}(V)\} \cup \Big\{[2k\pi, 0, 0, 0];\ 2 \leqslant k \leqslant \frac{t_M}{2\pi}\Big\}.$$

According to (1.14) and the comment after the definition of $\mathcal{C}_\varrho$ (where $\lambda_0 = 0$), we have $\mathcal{C}^M_\varrho = \mathcal{C}^1_\varrho \cup \mathcal{C}^2_\varrho$. Suppose that (1.16) is not true. Then it follows from (1.14), (1.15) that $\mathcal{C}^1_\varrho$, $\mathcal{C}^2_\varrho$ are nonempty. They are not separated because of the connectedness of $\mathcal{C}^M_\varrho$. Hence, there exist $[T_{jn}, V_{jn}, \tau_{jn}, \lambda_{jn}] \in \mathcal{C}^M_\varrho$, $|V_{jn}| > 0$, $j = 1, 2$, $n = 1, 2, \ldots$ such that $\gamma \leqslant T_{jn} \leqslant t_M$, $T_{1n} = t^{\tau_{1n}}_{\varrho,\lambda_{1n}}(V_{1n})$, $T_{2n} > t^{\tau_{2n}}_{\varrho,\lambda_{2n}}(V_{2n})$, $[T_{jn}, V_{jn}, \tau_{jn}, \lambda_{jn}] \to [T, V, \tau, \lambda] \in \mathcal{C}^M_\varrho$ for $n \to \infty$, $j = 1, 2$. It follows from (1.15) that $[2\pi, 0, 0, 0] \in \operatorname{int} \mathcal{C}^1_\varrho$ where *int* is the interior with respect to $\mathcal{C}^M_\varrho$. Suppose that we also know $[2k\pi, 0, 0, 0] \in int\ \mathcal{C}^2_\varrho$ for $2 \leqslant k \leqslant \frac{t_M}{2\pi}$. Then $|V| > 0$, $\tau > 0$ (see also Observation 2.2). We have $t^\tau_{\varrho,\lambda}(V) < t_M$, $\dot\varphi^\tau_{\varrho,\lambda}(t^\tau_{\varrho,\lambda}(V), V) < 0$ by (1.14). Remark 4.1 implies that $t^{\tau_{jn}}_{\varrho,\lambda_{jn}}(V_{jn}) \to t^\tau_{\varrho,\lambda}(V)$ for $j = 1, 2$, $n \to +\infty$. In particular, $T = t^\tau_{\varrho,\lambda}(V)$. Analogously as in the proof of (1.15), we obtain from (1.14) that $\varphi^{\tau_{2n}}_{\varrho,\lambda_{2n}}(T_{2n}, V_{2n}) = -2k_n\pi$ with some k_n and we have $k_n > 1$ because $T_{2n} > t^{\tau_{2n}}_{\varrho,\lambda_{2n}}(V_{2n})$. It follows that $t^{\tau_{2n}}_{\varrho,\lambda_{2n}}(Z_n) \to 0$ with $Z_n = U^{\tau_{2n}}_{\varrho,\lambda_{2n}}(t^{\tau_{2n}}_{\varrho,\lambda_{2n}}(V_{2n}), V_{2n})$. Clearly $[T_{2n}, Z_n, \tau_{2n}, \lambda_{2n}] \in \mathcal{C}^0_\varrho$, $|Z_n| \neq 0$, and we obtain a contradiction with (1.14).

It remains to prove that $[2k\pi, 0, 0, 0] \in \operatorname{int} \mathcal{C}^2_\varrho$ for $2 \leqslant k \leqslant \frac{t_M}{2\pi}$. If this were not true then $[T_n, V_n, \tau_n, \lambda_n] \in \mathcal{C}^1_\varrho$ would exist such that $[T_n, V_n, \tau_n, \lambda_n] \to [2k\pi, 0, 0, 0]$ for some $k > 1$, $|V_n| > 0$, $T_n = t^{\tau_n}_{\varrho,\lambda_n}(V_n)$, $\frac{V_n}{|V_n|} \to W$. We would have $W \notin \operatorname{Lin}\{U_3\}$ because $U^0_{0,0}(2k\pi, W) = W$ by Lemma 2.5 and there are no periodic solutions of (1.8) starting in $\operatorname{Lin}\{U_3\}$. We would get $T_n = t^{\tau_n}_{\varrho,\lambda_n}(V_n) \to t^0_{0,0}(W) = 2\pi < 2k\pi$ (see Remark 4.1), a contradiction. $\square$

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