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UNIQUE SOLVABILITY OF A LINEAR PROBLEM WITH PERTURBED PERIODIC BOUNDARY VALUES

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Abstract. We investigate the problem with perturbed periodic boundary values

$$\begin{cases} y'''(x) + a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x), \\ y^{(i)}(T) = cy^{(i)}(0), \ i = 0, 1, 2; \ 0 < c < 1 \end{cases}$$

with $a_2, a_1, a_0 \in C[0, T]$ for some arbitrary positive real number T, by transforming the problem into an integral equation with the aid of a piecewise polynomial and utilizing the Fredholm alternative theorem to obtain a condition on the uniform norms of the coefficients a_2, a_1 and a_0 which guarantees unique solvability of the problem. Besides having theoretical value, this problem has also important applications since *decay* is a phenomenon that all physical signals and quantities (amplitude, velocity, acceleration, curvature, etc.) experience.

Keywords: Ordinary differential equations, integral equations, periodic boundary value problems

MSC 2000: 34B15, 34C10

1. Transformation into an integral equation

Let L be the linear third order differential operator with continuous coefficients

$$L = \frac{\mathrm{d}^3}{\mathrm{d}x^3} + a_2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + a_0(x); \quad a_2, a_1, a_0 \in C[0, T].$$

Our aim is to investigate unique solvability, for every $f \in C[0, T]$, of the problem

$$\begin{cases} Ly(x) = f(x), \\ y^{(i)}(T) = cy^{(i)}(0), \ i = 0, 1, 2; \ 0 < c < 1 \end{cases} \tag{1.1}$$

by transforming it into an integral equation with the aid of a piecewise cubic polynomial with real coefficients

$$q(x,t)=\begin{cases}\alpha_1(x-t)^3+\beta_1(x-t)^2+\gamma_1(x-t)+\theta_1 & \text{if } 0\leqslant t\leqslant x\leqslant T,\\ \alpha_2(x-t)^3+\beta_2(x-t)^2+\gamma_2(x-t)+\theta_2 & \text{if } 0\leqslant x\leqslant t\leqslant T.\end{cases}\tag{1.2}$$

Note that (1.1) is a problem in which *periodic* boundary values are *perturbed*. Besides having theoretical value, the problem (1.1) has also important applications since all physical signals and quantities experience decay (amplitude depending on y, velocity depending on y', acceleration depending on y'', curvature depending on y' and y'', etc.).

The reason for breaking up the definition of q into two regions will be made clear as we proceed. With this choice for q we have

$$\frac{\partial q}{\partial x}(x,t)=\begin{cases}3\alpha_1(x-t)^2+2\beta_1(x-t)+\gamma_1 & \text{if } 0\leqslant t< x\leqslant T,\\ 3\alpha_2(x-t)^2+2\beta_2(x-t)+\gamma_2 & \text{if } 0\leqslant x< t\leqslant T,\end{cases}\tag{1.3}$$

$$\frac{\partial^2 q}{\partial x^2}(x,t)=\begin{cases}6\alpha_1(x-t)+2\beta_1 & \text{if } 0\leqslant t< x\leqslant T,\\ 6\alpha_2(x-t)+2\beta_2 & \text{if } 0\leqslant x< t\leqslant T,\end{cases}\tag{1.4}$$

$$\frac{\partial^3 q}{\partial x^3}(x,t)=\begin{cases}6\alpha_1 & \text{if } 0\leqslant t< x\leqslant T,\\ 6\alpha_2 & \text{if } 0\leqslant x< t\leqslant T.\end{cases}\tag{1.5}$$

We intend to select the coefficients of q in such a way that if $u\in C[0,T]$ is a solution of the integral equation

$$u(x)+\int_0^T Lq(x,t)u(t)\,\mathrm{d}t=f(x)\tag{1.6}$$

then the function y defined as

$$y(x):=\int_0^T q(x,t)u(t)\,\mathrm{d}t\tag{1.7}$$

is a solution of the problem (1.1). Now

$$y(x)=\left(\int_0^x+\int_x^T\right)q(x,t)u(t)\,\mathrm{d}t,$$

therefore

$$y'(x)=\lim_{t\to x^-}[q(x,t)u(t)]-\lim_{t\to x^+}[q(x,t)u(t)]+\int_0^T\frac{\partial q}{\partial x}(x,t)u(t)\,\mathrm{d}t$$

and we have

$$y'(x) = \int_0^T \frac{\partial q}{\partial x}(x,t)u(t)\,\mathrm{d}t \tag{1.8}$$

provided

$$\lim_{t\to x^-}[q(x,t)u(t)] = \lim_{t\to x^+}[q(x,t)u(t)]$$

or, by virtue of (1.2),

$$\theta_1 = \theta_2. \tag{1.9}$$

Starting with (1.8) we get

$$y''(x) = \lim_{t\to x^-}\left[\frac{\partial q}{\partial x}(x,t)u(t)\right] - \lim_{t\to x^+}\left[\frac{\partial q}{\partial x}(x,t)u(t)\right] + \int_0^T \frac{\partial^2 q}{\partial x^2}(x,t)u(t)\,\mathrm{d}t$$

and we have

$$y''(x) = \int_0^T \frac{\partial^2 q}{\partial x^2}(x,t)u(t)\,\mathrm{d}t \tag{1.10}$$

provided

$$\lim_{t\to x^-}\left[\frac{\partial q}{\partial x}(x,t)u(t)\right] = \lim_{t\to x^+}\left[\frac{\partial q}{\partial x}(x,t)u(t)\right]$$

or, by virtue of (1.3),

$$\gamma_1 = \gamma_2. \tag{1.11}$$

Finally, starting with (1.10) we arrive at

$$y'''(x) = \lim_{t\to x^-}\left[\frac{\partial^2 q}{\partial x^2}(x,t)u(t)\right] - \lim_{t\to x^+}\left[\frac{\partial^2 q}{\partial x^2}(x,t)u(t)\right] + \int_0^T \frac{\partial^3 q}{\partial x^3}(x,t)u(t)\,\mathrm{d}t;$$

this time we are interested in adjusting the conditions so that

$$y'''(x) = u(x) + \int_0^T \frac{\partial^3 q}{\partial x^3}(x,t)u(t)\,\mathrm{d}t, \tag{1.12}$$

which is obtained provided

$$\lim_{t\to x^-}\left[\frac{\partial^2 q}{\partial x^2}(x,t)u(t)\right] - \lim_{t\to x^+}\left[\frac{\partial^2 q}{\partial x^2}(x,t)u(t)\right] = u(x)$$

or, by virtue of (1.4),

$$\beta_1 - \beta_2 = \frac{1}{2}. \tag{1.13}$$

It is the need for this discontinuity in $\partial^2 q/\partial x^2$ over the line segment $\{x = t\}$ that inspired us to define q piecewise as we did in (1.2). From (1.7), (1.8), (1.10) and (1.12) we obtain

$$Ly(x) = u(x) + \int_0^T Lq(x,t)u(t)\,\mathrm{d}t = f(x).$$

To make y defined by (1.7) satisfy the conditions of the problem (1.1) as well, it suffices, by virtue of (1.7), (1.8) and (1.10), to place the following constraints on q:

$$\forall t \in [0,T] \qquad q(T,t) = cq(0,t), \tag{1.14}$$

$$\forall t \in [0,T] \qquad \frac{\partial q}{\partial x}(T,t) = c\frac{\partial q}{\partial x}(0,t), \tag{1.15}$$

$$\forall t \in [0,T] \qquad \frac{\partial^2 q}{\partial x^2}(T,t) = c\frac{\partial^2 q}{\partial x^2}(0,t). \tag{1.16}$$

To make q satisfy (1.14), we should have for every $t \in [0,T]$

$$\begin{gathered}(c\alpha_2 - \alpha_1)t^3 + (3T\alpha_1 + \beta_1 - c\beta_2)t^2 + (-3T^2\alpha_1 - 2T\beta_1 - \gamma_1 + c\gamma_2)t \\ + (T^3\alpha_1 + T^2\beta_1 + T\gamma_1 + \theta_1 - c\theta_2) = 0\end{gathered}$$

and hence all coefficients should be identically zero, which together with (1.9), (1.11), (1.13) and the definitions

$$\alpha := \alpha_2,\ \beta := \beta_2,\ \gamma := \gamma_2,\ \theta := \theta_2,$$

result in

$$\alpha_1 = c\alpha,\ \beta_1 = \beta + \frac{1}{2}, \qquad \gamma_1 = \gamma,\ \theta_1 = \theta$$

and

$$\begin{cases} 3cT\alpha + \frac{1}{2} + (1-c)\beta = 0, \\ 3cT^2\alpha + 2T\beta + T + (1-c)\gamma = 0, \\ cT^3\alpha + T^2\beta + T\gamma + \frac{1}{2}T^2 + (1-c)\theta = 0, \end{cases} \tag{1.17}$$

and it is easy to verify that (1.15) and (1.16) are also satisfied if conditions (1.17) hold. We have actually proved

Lemma 1.1. *Let* $q \in C([0,T] \times [0,T])$ *be the piecewise polynomial*

$$(1.18) \quad q(x,t) = \begin{cases} c\alpha(x-t)^3 + (\beta + \frac{1}{2})(x-t)^2 + \gamma(x-t) + \theta & \text{if } 0 \leqslant t \leqslant x \leqslant T, \\ \alpha(x-t)^3 + \beta(x-t)^2 + \gamma(x-t) + \theta & \text{if } 0 \leqslant x \leqslant t \leqslant T \end{cases}$$

with real coefficients α, β, γ, θ *satisfying* (1.17). *Under these conditions, for* $f \in C[0,T]$, *if* $u \in C[0,T]$ *is a solution of the integral equation* (1.6), *then* $y \in C[0,T]$ *defined by* (1.7) *is a solution of the problem* (1.1).

Conversely, we have

Lemma 1.2. *If*

$$\max_{0 \leqslant x \leqslant T} \|Lq(x,.)\|_{L^1[0,T]} < 1$$

then for any solution $y \in C[0,T]$ *of the problem* (1.1), *the function* $u \in C[0,T]$ *defined by*

$$(1.19) \qquad u(x) = f(x) + \int_0^T R(x,t) f(t) \, \mathrm{d}t$$

is a solution of the integral equation (1.6), *with* R *defined by the relation*

$$(1.20) \qquad R(x,t) + \int_0^T Lq(x,w) R(w,t) \, \mathrm{d}w = -Lq(x,t).$$

Remark 1.1. The function $R(x,t)$ is called the *resolvent* of the kernel $Lq(x,t)$ (see [3]).

Proof. For any $t \in [0,T]$, $Lq(x,t)$ considered as a function of x is piecewise continuous over $[0,T]$, and hence by a *piecewise argument* based on a reasoning similar to that used in the next section in the proof of Theorem 2.1, we can establish the existence of $R(x,t)$. We prove that (1.19) is a solution of (1.6) by inserting it into the righthand side of (1.6) and using the definition of R in (1.20):

$$
\begin{aligned}
u(x) + \int_0^T Lq(x,w)u(w)\,\mathrm{d}w &= f(x) + \int_0^T R(x,t)f(t)\,\mathrm{d}t \\
&\quad + \int_0^T Lq(x,w)\left[f(w) + \int_0^T R(w,t)f(t)\,\mathrm{d}t\right]\mathrm{d}w \\
&= f(x) + \int_0^T R(x,t)f(t)\,\mathrm{d}t + \int_0^T Lq(x,w)f(w)\,\mathrm{d}w \\
&\quad + \int_0^T \int_0^T Lq(x,w)R(w,t)f(t)\,\mathrm{d}t \\
&= f(x) + \int_0^T Lq(x,w)f(w)\,\mathrm{d}w \\
&\quad + \int_0^T \left[R(x,t) + \int_0^T Lq(x,w)R(w,t)\,\mathrm{d}w\right] f(t)\,\mathrm{d}t \\
&= f(x),
\end{aligned}
$$

which proves Lemma 1.2. □

These two lemmas yield the main result of this section:

Theorem 1.1. *With q given by* (1.18) *together with constraints* (1.17), *there is a one-to-one correspondence between the solution set of the problem* (1.1) *and the solution set of the integral equation* (1.6).

2. Unique solvability of the integral equation

Now we investigate conditions under which the integral equation

$$u(x) + \int_0^T Lq(x,t)u(t)\,\mathrm{d}t = f(x)$$

with $q \in C([0,T] \times [0,T])$ a third order piecewise polynomial, has a unique solution for every $f \in C[0,T]$. To accomplish this, we stablish conditions on the kernel $Lq(x,t)$ using the Riesz-Fredholm theory, that is, the Fredholm alternative for compact operators [1, 4]. Defining the integral operator

$$
\begin{cases}
K\colon\ C[0,T] \longrightarrow C[0,T], \\
(Ku)(x) = -\displaystyle\int_0^T Lq(x,t)u(t)\,\mathrm{d}t
\end{cases}
$$

we have

Lemma 2.1. *With $C[0,T]$ equipped with the uniform norm*

$$\|v\|_\infty = \max_{0\leqslant x\leqslant T} |v(x)|$$

the operator K is compact.

Proof. Obviously, K is linear. The kernel $Lq(x,t)$ of K is piecewise continuous on $[0,T]\times[0,T]$ since $q \in C([0,T]\times[0,T])$ and L is a linear differential operator with *continuous* coefficients. Therefore

$$\|Lq(x,.)\|_{L^1[0,T]} = \int_0^T |Lq(x,t)|d(t) < \infty. \tag{2.1}$$

By the definition of compactness of operators [4, 5], we need to show that $K(B)$, the image of the unit ball in $C[0,T]$

$$B = \{v \in C[0,T]\colon\ \|v\|_\infty < 1\}$$

under K is relatively compact in $C[0,T]$. To demonstrate this, it suffices to show that $K(B)$ is bounded and equicontinuous in $C[0,T]$. Relative compactness of $K(B)$ will then be deduced from the compactness of the interval $[0,T]$ and the Arzela-Ascoli theorem [2, 4].

Proof of the boundedness of $K(B)$. Given $v \in B$, by (2.1) we have for all $x \in [0,T]$

$$|(Kv)(x)| < \|Lq(x,.)\|_{L^1[0,T]} < \infty.$$

Taking the maximum of the lefthand side over $[0,T]$, we obtain

$$\|Kv\|_\infty < \max_{0\leqslant x\leqslant T} \|Lq(x,.)\|_{L^1[0,T]},$$

hence $K(B)$ is contained in the ball centered at the origin of $C[0,T]$ with radius $\max\limits_{0\leqslant x\leqslant T}\|Lq(x,.)\|_{L^1[0,T]}$ and is therefore bounded.

Proof of the equicontinuity of $K(B)$. The kernel $Lq(x,t)$ is continuous over each of the sets

$$S_1 := \{(x,t) \in [0,T]\times[0,T]\colon\ t < x\},$$
$$S_2 := \{(x,t) \in [0,T]\times[0,T]\colon\ x < t\},$$

but is not continuous over $[0,T]\times[0,T]$, so we need to introduce two functions

$$\begin{cases} p_1\colon\ \overline{S_1} := S_1 \cup \{(x,x)\colon\ x \in [0,T]\} \longrightarrow \mathbb{R}, \\ p_1(x,t) = \begin{cases} Lq(x,t) & \text{if } x \in S_1, \\ \lim\limits_{t\to x^+} Lq(x,t) & \text{if } x = t \end{cases} \end{cases}$$

and

$$\begin{cases} p_2\colon\ \overline{S_2} := S_2 \cup \{(x,x)\colon\ x \in [0,T]\} \longrightarrow \mathbb{R}, \\ p_2(x,t) = \begin{cases} Lq(x,t) & \text{if } x \in S_2, \\ \lim\limits_{t\to x^-} Lq(x,t) & \text{if } x = t \end{cases} \end{cases}$$

which are continuous over the *compact* sets $\overline{S_1}$ and $\overline{S_2}$, respectively, and hence are *uniformly* continuous over their respective domains. Therefore, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall (x_1,t),(x_2,t) \in \overline{S_1} \quad |x_1 - x_2| < \delta \Longrightarrow |p_1(x_1,t) - p_1(x_2,t)| < \varepsilon/(2T),$$
$$\forall (x_1,t),(x_2,t) \in \overline{S_2} \quad |x_1 - x_2| < \delta \Longrightarrow |p_2(x_1,t) - p_2(x_2,t)| < \varepsilon/(2T).$$

Without loss of generality we may assume that $x_1 < x_2$. For all $v \in B$ with $\|v\|_\infty < 1$ we conclude

$$|(Kv)(x_1) - (Kv)(x_2)| \leqslant I_1 + I_2 + I_3$$

where

$$\begin{aligned}
I_1 &:= \int_0^{x_1} |Lq(x_1,t) - Lq(x_2,t)|\,\mathrm{d}t \\
&= \int_0^{x_1} |p_1(x_1,t) - p_1(x_2,t)|\,\mathrm{d}t < x_1\varepsilon/(2T), \\
I_3 &:= \int_{x_2}^{T} |Lq(x_1,t) - Lq(x_2,t)|\,\mathrm{d}t \\
&= \int_{x_2}^{T} |p_2(x_1,t) - p_2(x_2,t)|\,\mathrm{d}t < (T - x_2)\varepsilon/(2T), \\
I_2 &= \int_{x_1}^{x_2} |Lq(x_1,t) - Lq(x_2,t)|\,\mathrm{d}t \\
&\leqslant \int_{x_1}^{x_2} |Lq(x_1,t) - Lq(t,t)|\,\mathrm{d}t + \int_{x_1}^{x_2} |Lq(t,t) - Lq(x_2,t)|\,\mathrm{d}t \\
&= \int_{x_1}^{x_2} |p_2(x_1,t) - p_2(t,t)|\,\mathrm{d}t + \int_{x_1}^{x_2} |p_1(t,t) - p_1(x_2,t)|\,\mathrm{d}t \\
&< (x_2 - x_1)\varepsilon/(2T) + (x_2 - x_1)\varepsilon/(2T) \\
&< (x_2 - x_1)\varepsilon/(2T) + \varepsilon/2
\end{aligned}$$

provided $|x_1 - x_2| < \delta$, and the equicontinuity of $K(B)$ is established, making the proof of Lemma 2.1 complete. □

Having proved the compactness of K, we immediately arrive at

Proposition 2.1. *If the corresponding homogeneous integral equation $u - Ku = 0$ has only the trivial solution $u \equiv 0$, then the main integral equation $u - Ku = f$ has a unique solution $u \in C[0,T]$ for all $f \in C[0,T]$.*

P r o o f. Direct consequence of the compactness of the integral operator K and the Fredholm alternative theorem. $\square$

So the problem of finding conditions for existence and uniqueness of the solution of the integral equation $u - Ku = f$ for all $f \in C[0,T]$ is reduced to the problem of finding conditions under which the equation $u = Ku$ has only the trivial solution. By linearity of K, $u \equiv 0$ is always a solution of $u = Ku$. Therefore by the Banach fixed point theorem we are done if we provide conditions which make K a contraction mapping.

For all $u_1, u_2 \in C[0,T]$ and for all $x \in [0,T]$

$$\begin{aligned} |(Ku_1)(x) - (Ku_2)(x)| &\leqslant \int_0^T |Lq(x,t)||u_1(x) - u_2(x)| \,\mathrm{d}t \\ &\leqslant \left(\int_0^T |Lq(x,t)| \,\mathrm{d}t\right) \|u_1 - u_2\|_\infty. \end{aligned}$$

Taking maximum on the lefthand side over $[0,T]$, we get

$$\|Ku_1 - Ku_2\|_\infty \leqslant \max_{0 \leqslant x \leqslant T} \|Lq(x,.)\|_{L^1[0,T]} \|u_1 - u_2\|_\infty,$$

This argument proves

Theorem 2.1. *If*

$$\max_{0 \leqslant x \leqslant T} \|Lq(x,.)\|_{L^1[0,T]} < 1$$

then for all $f \in C[0,T]$ the integral equation

$$u(x) + \int_0^T Lq(x,t)u(t) \,\mathrm{d}t = f(x)$$

has a unique solution $u \in C[0,T]$.

P r o o f. Under the condition stated K is a contraction mapping. $\square$

Remark 2.1. We could as well start with the mapping K with the domain equipped with the L^p*−norm* $(1 \leqslant p \leqslant \infty)$,

$$K\colon\ (C[0,T], \|.\|_{L^p[0,T]}) \longrightarrow (C[0.T], \|.\|_\infty).$$

By the Hölder inequality with $\frac{1}{p} + \frac{1}{r} = 1$ we have

$$\|Ku_1 - Ku_2\|_\infty \leqslant \max_{0 \leqslant x \leqslant T} \|Lq(x,.)\|_{L^r[0,T]} \|u_1 - u_2\|_{L^p[0,T]}$$

and the condition for K to be a contraction mapping would be

$$\max_{0 \leqslant x \leqslant T} \|Lq(x,.)\|_{L^r[0,T]} < 1.$$

Our main discussion is the special case with $p = \infty$.

3. Condition for unique solvability of the problem

With the particular q obtained in Section 1 as (1.18) we have

$$Lq(x,t)=\begin{cases} 6c\alpha+[6c\alpha(x-t)+(2\beta+1)]a_2(x) \\ +[3c\alpha(x-t)^2+(2\beta+1)(x-t)+\gamma]a_1(x) \\ +[c\alpha(x-t)^3+(\beta+\frac{1}{2})(x-t)^2+\gamma(x-t)+\theta]a_0(x) & \text{if } 0\leqslant t<x\leqslant T \\ 6\alpha+[6\alpha(x-t)+2\beta]a_2(x) \\ +[3\alpha(x-t)^2+2\beta(x-t)+\gamma]a_1(x) \\ +[\alpha(x-t)^3+\beta(x-t)^2+\gamma(x-t)+\theta]a_0(x) & \text{if } 0\leqslant x<t\leqslant T. \end{cases}$$

Taking into account the assumption $0<c<1$ we get

$$\begin{aligned}\int_0^T |Lq(x,t)|\,\mathrm{d}t &= \left(\int_0^x+\int_x^T\right)|Lq(x,t)|\,\mathrm{d}t \\ &\leqslant 6x|\alpha|+[6I_{11}|\alpha|+x(2|\beta|+1)]\|a_2\|_\infty \\ &\quad+[3I_{12}|\alpha|+I_{11}(2|\beta|+1)+x|\gamma|]\|a_1\|_\infty \\ &\quad+[I_{13}|\alpha|+I_{12}(|\beta|+\frac{1}{2})+I_{11}|\gamma|+x|\theta|]\|a_0\|_\infty \\ &\quad+6(T-x)|\alpha|+[6I_{21}|\alpha|+2(T-x)|\beta|]\|a_2\|_\infty \\ &\quad+[3I_{22}|\alpha|+2I_{21}|\beta|+(T-x)|\gamma|]\|a_1\|_\infty \\ &\quad+[I_{23}|\alpha|+I_{22}|\beta|+I_{21}|\gamma|+(T-x)|\theta|]\|a_0\|_\infty\end{aligned}$$

where

$$\begin{aligned} I_{11}&=\left|\int_0^x (x-t)\,\mathrm{d}t\right|=\frac{1}{2}x^2, & I_{21}&=\left|\int_x^T (x-t)\,\mathrm{d}t\right|=\frac{1}{2}(T-x)^2,\\ I_{12}&=\left|\int_0^x (x-t)^2\,\mathrm{d}t\right|=\frac{1}{3}x^3, & I_{22}&=\left|\int_x^T (x-t)^2\,\mathrm{d}t\right|=\frac{1}{3}(T-x)^3,\\ I_{13}&=\left|\int_0^x (x-t)^3\,\mathrm{d}t\right|=\frac{1}{4}x^4, & I_{23}&=\left|\int_x^T (x-t)^3\,\mathrm{d}t\right|=\frac{1}{4}(T-x)^4,\end{aligned}$$

so

$$\begin{aligned}\|Lq(x,.)\|_{L^1[0,T]} &\leqslant 6T|\alpha|+(3[x^2+(T-x)^2]|\alpha|+2T|\beta|+x)\|a_2\|_\infty \\ &\quad+\Big([x^3+(T-x)^3]|\alpha|+[x^2+(T-x)^2]|\beta|+T|\gamma|+\frac{1}{2}x^2\Big)\|a_1\|_\infty \\ &\quad+\Big(\frac{1}{4}[x^4+(T-x)^4]|\alpha|+\frac{1}{3}[x^3+(T-x)^3]|\beta| \\ &\quad+\frac{1}{2}[x^2+(T-x)^2]|\gamma|+T|\theta|+\frac{1}{6}x^3\Big)\|a_0\|_\infty.\end{aligned}$$

Taking the maximum of the righthand side and noting that

$$\max_{0\leqslant x\leqslant T}[x^j + (T-x)^j] = T^j,\ j = 2,3,4$$

we arrive at

$$\begin{aligned} \|Lq(x,.)\|_{L^1[0,T]} \leqslant\ & 6T|\alpha| + (3T^2|\alpha| + 2T|\beta| + T)\|a_2\|_\infty \\ & + \Big(T^3|\alpha| + T^2|\beta| + T|\gamma| + \frac{1}{2}T^2\Big)\|a_1\|_\infty \\ & \Big(\frac{1}{4}T^4|\alpha| + \frac{1}{3}T^3|\beta| + \frac{1}{2}T^2|\gamma| + T|\theta| + \frac{1}{6}T^3\Big)\|a_0\|_\infty. \end{aligned} \tag{3.1}$$

To make K a contraction mapping we use Theorem 2.1 to adjust the coefficients α, β, γ, and θ in such a manner that they satisfy (1.17) and simultaneously make the righthand side of (3.1) strictly less than 1. Selecting

$$\alpha = \frac{1}{12T}$$

we obtain from (1.17)

$$\beta = -\frac{2+c}{4(1-c)},\quad \gamma = \frac{5c+c^2}{4(1-c)^2}T,\quad \theta = -\frac{7c+10c^2+c^3}{12(1-c)^3}T^2.$$

With these values (3.1) becomes

$$\begin{aligned} \max_{0\leqslant x\leqslant T}\|Lq(x,t)\|_{L1[0,T]} \leqslant\ & \frac{1}{2} + \frac{9-3c}{4(1-c)}T\|a_2\|_\infty \\ & + \frac{13-2c+7c^2}{12(1-c)^2}T^2\|a_1\|_\infty \\ & + \frac{17+19c+43c^2-7c^3}{48(1-c)^3}T^3\|a_0\|_\infty. \end{aligned}$$

This inequality together with Theorems 1 and 2 yields in our main existence and uniqueness theorem:

Theorem 3.1. *If the uniform norms of $a_2, a_1, a_0 \in C[0,T]$ satisfy the constraint*

$$\frac{9-3c}{4(1-c)}T\|a_2\|_\infty + \frac{13-2c+7c^2}{12(1-c)^2}T^2\|a_1\|_\infty + \frac{17+19c+43c^2-7c^3}{48(1-c)^3}T^3\|a_0\|_\infty < \frac{1}{2} \tag{3.2}$$

then the problem

$$\begin{cases} y'''(x) + a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x), \\ y^{(i)}(T) = cy^{(i)}(0),\ i = 0,1,2;\ 0 < c < 1 \end{cases}$$

has a unique solution $y \in C[0,T]$.

Remark 3.1. By the same reasoning we can establish the existence of a continuous solution for the problem

$$\begin{cases} y'''(x) + a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x), \\ y^{(i)}(2T) = cy^{(i)}(T),\ i = 0, 1, 2;\ 0 < c < 1. \end{cases}$$

Proceeding in this manner, we obtain the following important by-product of our main result above:

Corollary 3.1. *If the uniform norms of the functions* $a_2, a_1, a_0 \in C([0, +\infty[)$ *satisfy the constraint* (3.2)*, then for every* $f \in C([0, +\infty[)$ *the third order differential equation*

$$y'''(x) + a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x)$$

has a solution in $C([0, +\infty[)$ *with the property that*

$$\{y(nT)\}_{n=0}^{\infty}, \qquad \{y'(nT)\}_{n=0}^{\infty}, \qquad \{y''(nT)\}_{n=0}^{\infty}$$

are geometric sequences convergent to zero, and hence the solution is stable (since it is continuous on each interval $[nT, (n+1)T]$ *and hence bounded) and has a decaying behavior toward zero (although not necessarily in a uniform manner).*

Remark 3.2. The same reasoning can, in principle, be used for the problem

$$\begin{cases} y^{(n)} + \displaystyle\sum_{k=0}^{n-1} a_k(x)y^{(k)}(x) = f(x), \\ y^{(i)}(T) = cy^{(i)}(0),\ i = 0, \dots, n-1;\ 0 < c < 1 \end{cases}$$

if we make use of an nth order piecewise polynomial, although the computations involved increase tremendously as n increases. Here we have treated the case $n = 3$.

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