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LATTICES OF QUASIORDERS ON UNIVERSAL ALGEBRAS

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Lattices of quasiorders were studied mainly by G. Czédli and A. Lenkehegyi [2] and by A. G. Pinus and I. Chajda [9]. These investigations were done both for universal algebras and algebras of special sorts: lattices, semilattices etc. In some cases, the lattice of all quasiorders of an algebra $\mathscr{A}$ has similar properties as the congruence lattice $\operatorname{Con}\mathscr{A}$, however, there are also essential distinctions. One of the traditional questions concerning congruence lattices is a characterization of congruence lattices satisfying given identities. It was partly solved for quasiorder lattices and for varieties of algebras in [2], [8], [9]. An abstract algebraic characterization of quasiorder lattices was settled in [1], [8]. The aim of this paper is to characterize concrete quasiorder lattices and to represent these lattices by quasiorder lattices of algebras of restricted similarity types.

By a *quasiorder* on an algebra $\mathscr{A} = (A, F)$ we mean a reflexive and transitive binary relation on A which has the substitution property with respect to all operations of F, i.e. for all pairs $\langle a_i, b_i\rangle$ of this relation ($i = 1, \ldots, n$) and each n-ary $f \in F$ also the pair $\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)\rangle$ is its member. Hence, quasiorders on $\mathscr{A}$ are reflexive and transitive subalgebras of $\mathscr{A}^2$. The set $\operatorname{Quord}\mathscr{A}$ of all quasiorders on $\mathscr{A}$ forms an algebraic lattice with respect to set inclusion. Of course, $\operatorname{Con}\mathscr{A}$ is a sublattice of $\operatorname{Quord}\mathscr{A}$ with the same least and greatest elements.

§ 1.

As was shown in [1], [8], every algebraic lattice is isomorphic to $\operatorname{Quord}\mathscr{A}$ for some algebra $\mathscr{A}$. This raises the question on a concrete characterization of $\operatorname{Quord}\mathscr{A}$, i.e. a question whether a lattice L of reflexive and transitive binary relations on a set A is isomorphic to $\operatorname{Quord}\mathscr{A}$ for some algebra $\mathscr{A} = (A, F)$. For equivalences and congruences, an analogous problem was solved by B. Jónsson [4].

Let φ be a mapping of A^2 into the set of all reflexive and transitive binary relations on A and let $a, b \in A$. Denote by $\mathrm{St}_{a,b}(\varphi)$ the set of all pairs $\langle f(a), f(b)\rangle$, where f runs over the set of all mappings $A \to A$ satisfying

$$\langle f(c), f(d)\rangle \in \varphi(\langle c, d\rangle).$$

Denote by $Q_{a,b}(\varphi)$ the reflexive and transitive relation on A generated by $\mathrm{St}_{a,b}(\varphi)$. Denote by Δ_A the diagonal of A^2, i.e. $\Delta_A = \{\langle a, a\rangle;\ a \in A\}$.

A set S of subsets of a given set C is called an *algebraic closure system* if S is closed under arbitrary intersections and is up-directed with respect to inclusion. Evidently, the set of all quasiorders on an algebra $\mathscr{A}$ is an algebraic closure system.

Theorem 1. *Let* $\mathbf{Q}$ *be an algebraic closure system of some reflexive and transitive binary relations on a set* A*, let* $\Delta_A \in \mathbf{Q}$ *and let* $a, b \in A$*,* $a \neq b$*. The following conditions are equivalent:*

(1) *there exists an algebra* $\mathscr{A} = (A, F)$ *with* $\mathbf{Q} = \mathrm{Quord}\,\mathscr{A}$*;*

(2) *for every mapping* $\varphi\colon A^2 \to \mathbf{Q}$*,* $Q_{a,b}(\varphi) \in \mathbf{Q}$*.*

P r o o f. Suppose $\mathbf{Q} = \mathrm{Quord}\,\mathscr{A}$ for some algebra $\mathscr{A} = (A, F)$. Denote by $q_{c,d}(\mathscr{A})$ the least quasiorder on $\mathscr{A}$ containing the pair $\langle c, d\rangle$, the so called principal quasiorder generated by $\langle c, d\rangle$. Taking into account the definition of $Q_{a,b}(\varphi)$, we need only to prove that for every $\varphi\colon A^2 \to \mathbf{Q}$, the relation $\mathrm{St}_{a,b}(\varphi)$ is compatible with all operations of F. With respect to reflexivity and transitivity, we need only to show compatibility with respect to all unary polynomials over $\mathscr{A}$. Let $\langle c, d\rangle \in \mathrm{St}_{a,b}(\varphi)$ and let $g(x)$ be a unary polynomial over $\mathscr{A}$. By the definition of $\mathrm{St}_{a,b}(\varphi)$, there exists a mapping $f\colon A \to A$ with $\langle c, d\rangle = \langle f(a), f(b)\rangle$ and for each $u, v \in A$ we have $\langle f(u), f(v)\rangle \in \varphi(\langle u, v\rangle)$, i.e. $q_{f(u),f(v)}(\mathscr{A}) \subseteq \varphi(\langle u, v\rangle)$. Evidently, gf is a mapping of A into itself with

$$\langle g\left(f(u)\right), g\left(f(v)\right)\rangle \in g_{f(u),f(v)}(\mathscr{A}) \subseteq \varphi(\langle u, v\rangle),$$

i.e.

$$\langle g(c), g(d)\rangle = \langle g\left(f(a)\right), g\left(f(b)\right)\rangle \in \mathrm{St}_{a,b}(\varphi).$$

By the foregoing remark, we conclude that $Q_{a,b}(\varphi)$ is a quasiorder of the algebra $\mathscr{A}$, i.e. $Q_{a,b}(\varphi) \in \mathbf{Q}$. This completes the proof of (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1): Let $\mathbf{Q}$ satisfy (2). Evidently, for each $c, d \in A$ and every $\varphi\colon A^2 \to \mathbf{Q}$ we have $Q_{c,d}(\varphi) \in \mathbf{Q}$. Denote $p(c, d) = \bigcap\{r \in \mathbf{Q};\ \langle c, d\rangle \in r\}$. Hence $p\colon A^2 \to \mathbf{Q}$. Denote by G the set all mappings $A \to A$ preserving $p(c, d)$ for every $c, d \in A$. Let

$\mathscr{A} = (A, G)$. We are going to show that $\mathbf{Q} = \operatorname{Quord}\mathscr{A}$. The inclusion $\mathbf{Q} \subseteq \operatorname{Quord}\mathscr{A}$ is clear. To prove the converse inclusion we need to show that $q_{c,d}(\mathscr{A}) = p(c,d)$ for every c, d of A. The inclusion $q_{c,d}(\mathscr{A}) \subseteq p(c,d)$ follows by $\mathbf{Q} \subseteq \operatorname{Quord}\mathscr{A}$. We prove $p(c,d) \subseteq q_{c,d}(\mathscr{A})$. By definition, $\operatorname{St}_{c,d}(p) = \{\langle f(c), f(d)\rangle;\ f \in G\}$. Hence $\operatorname{St}_{c,d}(p) \subseteq q_{c,d}(\mathscr{A})$, i.e. $Q_{c,d}(p) \subseteq q_{c,d}(\mathscr{A})$. However, $Q_{c,d}(p) \in \mathbf{Q}$ and $p(c,d) \subseteq Q_{c,d}(p) \subseteq q_{c,d}(\mathscr{A})$. Together, $p(c,d) = q_{c,d}(\mathscr{A})$, which yields $\mathbf{Q} = \operatorname{Quord}\mathscr{A}$. □

§ 2.

It is known that for an algebra $\mathscr{A} = (A, F)$ there exist algebras $\mathscr{B}$ with restricted similarity types such that $\operatorname{Con}\mathscr{A} \cong \operatorname{Con}\mathscr{B}$. These results were settled by R. Freese, W. Lampe, W. Taylor [3], [6], [7], B. Jónsson [4] and S. R. Kogalovskij and V. V. Soldatova [5]. We are now going to prove similar results for lattices $\operatorname{Quord}\mathscr{A}$ instead of $\operatorname{Con}\mathscr{A}$ by heavily using the methods for congruence lattices in the quoted papers.

Theorem 2. *For any finite algebra $\mathscr{A}$ there exists a finite algebra $\mathscr{B}$ with only 4 unary operations such that $\operatorname{Quord}\mathscr{A} \cong \operatorname{Quord}\mathscr{B}$.*

P r o o f. Since $\mathscr{A}$ is finite, we may assume that $\mathscr{A}$ is of a finite similarity type F. Let $f \in F$ be n-ary, let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be elements of $\mathscr{A}$ and let Q be a reflexive and transitive relation on $\mathscr{A}$. Put $u_i(x) = f(b_1, \ldots, b_{i-1}, x, a_{i+1}, \ldots, a_n)$. Evidently, $\langle u_i(a_i), u_i(b_i)\rangle \in Q$ for $i = 1, \ldots, n$ imply also

$$\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)\rangle \in Q$$

because of reflexivity and transitivity of Q. Hence, $\mathscr{A}$ can be considered to be unary. Let $f_1, \ldots, f_n$ be all unary operations of $\mathscr{A}$ and let $\{a_1, \ldots, a_m\}$ be the support of $\mathscr{A}$. Put

$$B = \{a_1, \ldots, a_m\}^{m+n+1}$$

and $\mathscr{B} = (B; \{g_1, g_2, g_3, g_4\})$, where g_1, g_2, g_3, g_4 are unary operations on B defined as follows: for $x = (x_1, x_2, \ldots, x_{m+n+1})$ let

$$\begin{aligned}
g_1(x) &= (a_1, \ldots, a_m, f_1(x_1), f_2(x_1), \ldots, f_n(x_1), x_1),\\
g_2(x) &= (x_2, x_2, x_3, \ldots, x_{m+n+1}),\\
g_3(x) &= (x_{m+n+1}, x_1, x_2, \ldots, x_{m+n}),\\
g_4(x) &= (x_2, x_1, x_3, x_4, \ldots, x_{m+n+1}).
\end{aligned}$$

It is an easy excercise to show that for any mapping π of $\{1, 2, \ldots, m+n+1\}$ into itself the mapping $H_\pi\colon B \to B$ given by

$$H_\pi(x) = (x_{\pi(1)}, \ldots, x_{\pi(m+n+1)})$$

is a term operation of B.

Let $R \subseteq A \times A$ be a binary relation. Define $\overline{R} \subseteq B \times B$ as follows:

$$\langle x, y \rangle \in \overline{R} \quad \text{iff} \quad \langle x_k, y_k \rangle \in R \quad \text{for} \quad k = 1, 2, \ldots, m+n+1,$$

where $x = (x_1, x_2, \ldots, x_{m+n+1})$, $y = (y_1, y_2, \ldots, y_{m+n+1})$. Evidently, $R \subseteq S$ if and only if $\overline{R} \subseteq \overline{S}$, and hence the mapping of the system of all subsets of $A \times A$ into the system of all subsets of $B \times B$ defined by $R \mapsto \overline{R}$ is an injection. It is also obvious that if R is reflexive and transitive then also $\overline{R}$ has these properties. By virtue of the definition of g_1, g_2, g_3, g_4, R has the substitution property with respect to $f_1, \ldots, f_n$ if and only if $\overline{R}$ has the substitution property with respect to g_1, g_2, g_3, g_4. So $Q \in \operatorname{Quord} \mathscr{A}$ if and only if $\overline{Q} \in \operatorname{Quord} \mathscr{B}$. It remains to show that the mapping $Q \mapsto \overline{Q}$ is a surjection of $\operatorname{Quord} \mathscr{A}$ onto $\operatorname{Quord} \mathscr{B}$.

Let $S \in \operatorname{Quord} \mathscr{B}$. Introduce $Q \subseteq A \times A$ as follows:

$$Q = \{\langle u, v \rangle \in A \times A; \langle (u, u, \ldots, u), (v, v, \ldots, v) \rangle \in S\}.$$

Clearly Q is reflexive and transitive. By using the term operations H_π (with π as a constant map) we conclude that

$$\langle x, y \rangle \in S \Rightarrow \langle x_k, y_k \rangle \in Q \quad \text{for} \quad k = 1, \ldots, m+n+1.$$

We prove the converse implication. If $\langle x, y \rangle \in S$ and $r \leqslant m+n+1$ and $x', y' \in B$ are such that

$$x'_r = x_r, \quad y'_r = y_r \quad \text{and} \quad x'_k = x_k \quad \text{for} \quad r \neq k$$

then also $\langle x', y' \rangle \in S$. (Indeed, we can assume $r = 0$ and x', y' are obtained from x, y by first applying g_1 and then, since all elements of A occur among the first m coordinates, applying a suitable term H_π; hence $\langle x', y' \rangle \in S$).

Now, let

$$z^{(k)} = (y_1, \ldots, y_k, x_{k+1}, \ldots, x_{m+n+1}).$$

If $\langle x_k, y_k \rangle \in Q$ then $\langle x^{(k)}, z^{(k+1)} \rangle \in S$. Since S is reflexive and transitive and $x = z^{(1)}$, $y = z^{(m+n+1)}$, we conclude

$$\langle x_k, y_k \rangle \in Q \quad \text{for} \quad k = 1, \ldots, m+n+1 \Rightarrow \langle x, y \rangle \in S.$$

Hence $\overline{Q} = S$. It remains to show the substitution property of Q. Suppose $\langle u, v \rangle \in Q$ and put $x = (u, u, \ldots, u)$, $y = (v, v, \ldots, v)$. Then $\langle x, y \rangle \in S$, but $S \in \operatorname{Quord} \mathscr{B}$ implies

$$\langle g_1(x), g_1(y) \rangle \in S,$$

thus also $\langle g_1(x)_k, g_1(y)_k \rangle \in Q$ for $k = 1, \ldots, m+n+1$. Since $f_i(u)$ or $f_i(v)$ occurs as the first m coordinates in $g_1(x)$ or $g_1(y)$, respectively, clearly also $\langle f_i(u), f_i(v) \rangle \in Q$ for $i = 1, \ldots, n$ completing the proof. $\square$

Theorem 3. *For every finite algebra $\mathscr{A}$ of finite similarity type there exists a finite algebra $\mathscr{B}$ of type* $(2,1,1)$ *such that* $\operatorname{Quord}\mathscr{A} \cong \operatorname{Quord}\mathscr{B}$.

Proof. For $\mathscr{A} = (A, F)$ suppose $F = \{f_1, \dots, f_n\}$ where each f_i is considered to be n-ary. Let $C = A^n$ and introduce one binary and two unary operations of C as follows: for $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$

$$x \bullet y = (x_1, y_1, y_2, \dots, y_{n-1}),$$
$$g(x) = (f_1(x), f_2(x), \dots, f_n(x)),$$
$$h(x) = (x_2, x_3, \dots, x_n, x_1).$$

Then $\mathscr{C} = (C; \{\bullet, g, h\})$ is a finite algebra of type $(2,1,1)$. For $x^{(1)}, x^{(2)}, \dots, x^{(k)} \in C$ $(k \geqslant 2)$ we put

$$(*) \qquad x^{(1)} \bullet x^{(2)} \bullet \dots \bullet x^{(k)} = x^{(1)} \bullet \bigl(x^{(2)} \bullet \bigl(\dots x^{(k)}\bigr) \dots\bigr).$$

Define the mapping $\varphi\colon \operatorname{Quord}\mathscr{A} \to \operatorname{Quord}\mathscr{C}$ as follows:

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle \in \varphi(R) \quad \text{iff} \quad \langle x_i, y_i \rangle \in R$$

for $i = 1, 2, \dots, n$ and $R \in \operatorname{Quord}\mathscr{A}$. Clearly, $\varphi(R)$ is reflexive and transitive binary relation on C and, by the definition of operations $\bullet, g, h, \varphi(R) \in \operatorname{Quord}\mathscr{C}$. Evidently, for $R, S \in \operatorname{Quord}\mathscr{A}$ we have $R \subseteq S$ if and only if $\varphi(R) \subseteq \varphi(S)$, i.e. φ is an injection. It remains to prove that φ is a surjection.

For $x = (x_1, x_2, \dots, x_n) \in C$ we put $I(x) = x_1$. Let $R \in \operatorname{Quord}\mathscr{C}$. Let $T \subseteq A \times A$ be such that $\langle u, v \rangle \in T$ if and only if there exist $x, y \in C$ with $\langle x, y \rangle \in R$ and $I(x) = u, I(y) = v$. Evidently, T is reflexive. Suppose $\langle u, v \rangle \in T$ and $\langle v, w \rangle \in T$. Hence, there exist $x, y^{(1)}, y^{(2)}, z \in C$ with $\langle x, y^{(1)} \rangle \in R, \langle y^{(2)}, z \rangle \in R$ and $I(x) = u, I(y^{(1)}) = v = I(y^{(2)})$, $I(z) = w$. By $(*)$ and the definition of $\bullet$ we have $x^n = x \bullet x \bullet \dots \bullet x = (u, u, \dots, u)$. Analogously,

$$(y^{(1)})^n = (v, v, \dots, v) = (y^{(2)})^n, \qquad z^n = (w, w, \dots, w).$$

Hence $\langle x^n, (y^{(1)})^n \rangle \in R, \langle (y^{(2)})^n, z^n \rangle \in R$ and, by the transitivity of R, also $\langle x^n, z^n \rangle \in R$. Thus $I(x^n) = u, I(z^n) = w$ give $\langle u, w \rangle \in T$ proving transitivity of T.

Now we show that $\langle (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \rangle \in R$ whenever $\langle x_i, y_i \rangle \in T$ for all $i = 1, 2, \dots, n$. Assume $\langle x_i, y_i \rangle \in T$. Then there exist $x^{(i)}, y^{(i)} \in C$ such that $\langle x^{(i)}, y^{(i)} \rangle \in R$ and $I(x^{(i)}) = x_i, I(y^{(i)}) = y_i$. However,

$$x = (x_1, \dots, x_n) = x^{(1)} \bullet x^{(2)} \bullet \dots \bullet x^{(n)},$$
$$y = (y_1, \dots, y_n) = y^{(1)} \bullet y^{(2)} \bullet \dots \bullet y^{(n)},$$

so $\langle x, y\rangle \in R$.

It remains to show that $T \in \operatorname{Quord}\mathscr{A}$. Let $\langle x_i, y_i\rangle \in T$ for $i = 1, 2, \ldots, n$. Then $\langle x, y\rangle \in R$ for $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$. Hence $\langle g(x), g(y)\rangle \in R$ and so $\langle f_1(x), f_1(y)\rangle \in T$. Analogously, for $k = 1, 2, \ldots, n-1$ we have $\langle h^k g(x), h^k g(y)\rangle \in R$, so $\langle f_i(x), f_i(y)\rangle \in T$ for $i = 2, 3, \ldots, n$. Thus $T \in \operatorname{Quord}\mathscr{A}$.

Finally we show that $R = \varphi(T)$. Suppose $\langle x, y\rangle = \langle (x_1, \ldots, x_n), (y_1, \ldots y_n)\rangle \in R$. Then $\langle h^k(x), h^k(y)\rangle \in R$ for $k = 1, 2 \ldots, n-1$, i.e. $\langle x_i, y_i\rangle \in T$ for $i = 1, \ldots, n$. This gives $\langle x, y\rangle \in \varphi(T)$, i.e. $R \subseteq \varphi(T)$. Assume $\langle x, y\rangle \in \varphi(T)$. Then $\langle x_i, y_i\rangle \in T$ for $i = 1, \ldots, n$, thus also $\langle x, y\rangle \in R$, i.e. $\varphi(T) \subseteq R$. □

The foregoing construction can be generalized also for algebras which need not be finite:

Theorem 4. *For every algebra $\mathscr{A}$ of finite similarity type there exists an algebra $\mathscr{B}$ of type $(2, 1, 1)$ such that $\operatorname{Quord}\mathscr{A} \cong \operatorname{Quord}\mathscr{B}$.*

P r o o f. Let $\mathscr{A} = (A; \{f_n, \ldots, f_m\})$. Without loss of generality suppose that all f_i are n-ary. Let B be the set of all (infinite) sequences

$$u = (a_1, a_2, a_3, \ldots) \quad \text{of elements} \quad a_i \in A \quad \text{such that}$$
$$\text{for some} \quad n_0 \in \mathbb{N}, a_j = a_k \quad \text{for} \quad j, k \geqslant n_0.$$

Introduce one binary and two unary operations on B as follows: for $u = (x_1, x_2, \ldots)$, $v = (y_1, y_2, \ldots)$

$$\begin{aligned} d(u, v) &= \big(f_1(y_1, \ldots, y_n), \ldots, f_m(y_1, \ldots, y_n), x_1, y_1, y_2, y_3, \ldots\big) \\ g_1(u) &= (x_1, x_1, x_1, \ldots) \\ g_2(u) &= (x_2, x_3, x_4, \ldots). \end{aligned}$$

Put $\mathscr{B} = (B; \{d, g_1, g_2\})$. For each $p \in \mathbb{N}$ we put

$$h_p(u^{(1)}, \ldots, u^{(p+1)}, v) = (x_1^{(1)}, x_1^{(2)}, \ldots x_1^{(p+1)}, y_1, y_2, \ldots),$$

where $u^{(s)} = (u_1^{(s)}, u_2^{(s)}, \ldots)$, $v = (y_1, y_2, \ldots)$. Hence, h_p is a $(p+2)$-ary operation on B and, moreover,

$$\begin{aligned} &h_1(u, v) = g_2^m\big(d(u^{(1)}, g_2^m(d(u^{(2)}, v)))\big), \\ &h_{p+1}(u^{(1)}, \ldots, u^{(p+2)}, v) = h_1\big(u^{(1)}, h_p(u^{(2)}, u^{(3)}, \ldots, u^{(p+2)}, v)\big) \end{aligned}$$

(where $g_2^0(x) = x, g_2^m(x) = g_2\big(g_2^{m-1}(x)\big)$), thus all h_p are term operations of $\mathscr{B}$.

For $Q \in \operatorname{Quord}\mathscr{A}$ we put $\varphi(Q) = Q^*$, where $Q^* \subseteq B \times B$ and $\langle u, v\rangle \in Q^*$ iff $\langle x_k, y_k\rangle \in Q$ for $k = 1, 2 \ldots$. It is easy to show that for each $Q \in \operatorname{Quord}\mathscr{A}$, $\varphi(Q)$ is reflexive and transitive. Further, $Q_1 \subseteq Q_2$ iff $\varphi(Q_1) \subseteq \varphi(Q_2)$, thus φ is an injection. Let us prove $\varphi(Q) \in \operatorname{Quord}\mathscr{B}$:

Let $\langle u, v\rangle \in Q^* = \varphi(Q)$. Then $\langle x_k, y_k\rangle \in Q$ for all $k \in \mathbb{N}$ whence $\langle g_1(u), g_1(v)\rangle \in Q^*$ and $\langle g_2(u), g_2(v)\rangle \in Q^*$. Also $\langle u, v\rangle \in Q^*$, $\langle w, t\rangle \in Q^*$ imply

$$\langle d(u, w), d(v, t)\rangle \in Q^*$$

directly by the definition of d. Thus $Q^* \in \operatorname{Quord}\mathscr{B}$.

It remains to show that φ is a surjection of $\operatorname{Quord}\mathscr{A}$ onto $\operatorname{Quord}\mathscr{B}$. Suppose $R \in \operatorname{Quord}\mathscr{B}$. Put $Q = \{\langle x, y\rangle \in A \times A;\ \langle (x, x, x, \ldots), (y, y, y, \ldots)\rangle \in R\}$. Trivially, Q is reflexive and transitive. Suppose $\langle u, v\rangle \in R$ for $u = (x_1, x_2, \ldots)$, $v = (y_1, y_2, \ldots)$. Since R has the substitution property with respect to g_1, g_2, d we obtain $\langle g_1 g_2^{k-1}(u), g_1 g_2^{k-1}(v)\rangle \in R$, i.e. $\langle (x_k, x_k, \ldots), (y_k, y_k, \ldots)\rangle \in R$. Hence $\langle x_k, y_k\rangle \in Q$ for all $k \in \mathbb{N}$. Conversely, let $u = (x_1, x_2, \ldots)$, $v = (y_1, y_2, \ldots)$ and $\langle x_k, y_k\rangle \in Q$ for all $k \in \mathbb{N}$. Let $p \in \mathbb{N}$ be such a number that for all $i > p$ both sequences u, v are constant, and for $k = 1, 2, \ldots, p$ we put

$$x^{(k)} = (x_k, x_k, x_k, \ldots), \qquad y^{(k)} = (y_k, y_k, y_k, \ldots).$$

Then $\langle x^{(k)}, y^{(k)}\rangle \in R$ for $k = 1, \ldots, p$ and

$$u = h_p(x^{(1)}, \ldots, x^{(p)}), \qquad v = h_p(y^{(1)}, \ldots, y^{(p)}),$$

whence $\langle u, v\rangle \in R$. Thus $\varphi(Q) = R$. It remains to prove the substitution property of Q. Suppose $\langle x_1, y_1\rangle \in Q, \ldots, \langle x_n, y_n\rangle \in Q$ and for an arbitrary $a \in A$ put

$$u = (x_1, x_2, \ldots, x_n, a, a, \ldots), \qquad v = (y_1, y_2, \ldots, y_n, a, a, \ldots).$$

Then $u, v \in B$ and $\langle u, v\rangle \in R$. Hence $\langle d(u, u), d(v, v)\rangle \in R$, which implies

$$\langle d(u, u)_k, d(v, v)_k\rangle \in Q \quad \text{for all} \quad k \in \mathbb{N}.$$

This gives $\langle f_i(x_1, \ldots, x_n), f_i(y_1, \ldots, y_n)\rangle \in Q$ by the definition of d. Thus $Q \in \operatorname{Quord}\mathscr{A}$. $\square$

An element $\alpha \in A$ is called a "zero of $\mathscr{A} = (A, F)$" if for each n-ary $f \in F$ and each $i \in \{1, \ldots, n\}$ and all $a_1, \ldots, a_n \in A$ such that $a_i = \alpha$ we have

$$f(a_1, \ldots, a_{i-1}, \alpha, a_{i+1}, \ldots, a_n) = \alpha.$$

Theorem 5. *For every countable (finite) algebra $\mathscr{A}$ with zero there exists a countable (finite) algebra $\mathscr{B}$ with only two unary operations such that* $\operatorname{Quord}\mathscr{A} \cong \operatorname{Quord}\mathscr{B}$.

P r o o f. Suppose that $\mathscr{A}$ is countable. Similarly as in the proof of Theorem 2, we can consider (without loss of generality) that all operations of $\mathscr{A}$ are unary. For each quadruple $\gamma = \langle a, b, c, d\rangle \in A^4$, the function $f\colon A \to A$ is called γ-compatible if $f(a) = c$ and $f(b) = d$. A quadruple $\gamma = \langle a, b, c, d\rangle \in A^4$ is called accessible if there exists a term function $f(x)$ of $\mathscr{A}$ such that $f(a) = c$ and $f(b) = d$. Let $\Gamma(\mathscr{A})$ be the set of all accessible $\gamma \in A^4$ and let Ψ be a function which maps every $\gamma \in \Gamma(\mathscr{A})$ onto some γ-compatible term of $\mathscr{A}$. Then, of course, $\operatorname{Quord}\mathscr{A} \cong \operatorname{Quord}\mathscr{B}$ for $\mathscr{B} = (A, \Psi(\Gamma(\mathscr{A})))$. Hence $\mathscr{A}$ can be considered to be of countable signature, i.e. $\mathscr{A} = (A; f_1(x), f_2(x), \ldots)$. Suppose now that for each $n \in \mathbb{N}$ we have $f_n(\alpha) = \alpha$, where α is a zero of $\mathscr{A}$. Take the class $\{A_1, A_2, \ldots\}$ of sets A_i with $|A| = |A_i|$, $A_0 = A$ and $A_i \cap A_j = \{\alpha\}$ for each $i, j \in \mathbb{N}$, $i \neq j$. Put $B = \bigcup\limits_{i=0}^{\infty} A_i$. Let h be a mapping of B into itself such that $h(\alpha) = \alpha$ and h maps A_i bijectively on A_{i+1}. Let k be a mapping of B into itself such that $k(h(b)) = b$ for each $b \in B$ and $k(a) = a$ for $a \in A_0$. Introduce $g\colon B \to B$ as follows:

$$(*)\qquad g(b) = \begin{cases} \alpha & \text{if } b \in A_0,\\ h(a) & \text{if } b = h(a) \text{ for some } a \in A_0,\\ f_{i-1}(a) & \text{if } b = h^i(a) \text{ for some } a \in A_0 \text{ and some } i > 1. \end{cases}$$

Consider an algebra $\mathscr{B} = (B; \{h, k, g\})$. Evidently α is the (unique) zero of $\mathscr{B}$. For every subset $E \subseteq B^2$ we consider the following properties of the quadruple $\lambda = \langle \mathscr{A}, \mathscr{B}, E, \alpha\rangle$:

(a) $E \cap A^2 \in \operatorname{Quord}\mathscr{A}$;

(b) for each $n, m \in \mathbb{N}$ we have $\langle h^m(a), h^n(b)\rangle \in E \Rightarrow \langle h^n(a), h^n(b)\rangle \in E$ for every a, b of A;

(c) for each $m, n \in \mathbb{N}$, $m \neq n$ and each $a, b \in A$, $\langle h^m(a), h^n(b)\rangle \in E \Rightarrow \langle a, \alpha\rangle \in E$ and $\langle \alpha, b\rangle \in E$;

(d) for each $a, b \in A$ and each $m, n \in \mathbb{N}$, $\langle a, \alpha\rangle \in E$ and $\langle \alpha, b\rangle \in E \Rightarrow \langle h^m(a), h^n(b)\rangle \in E$;

(e) for each $a, b \in A$ and each $m, n \in \mathbb{N}$, $\langle h^m(a), h^n(b)\rangle \in E \Rightarrow \langle a, b\rangle \in E$.

Of course, for every $E \in \operatorname{Quord}\mathscr{B}$ the quadruple $\langle \mathscr{A}, \mathscr{B}, E, \alpha\rangle$ satisfies (a)–(e). It is routine to verify the converse, i.e. that for $\langle \mathscr{A}, \mathscr{B}, E, \alpha\rangle$ satisfying (a)–(e) we have $E \in \operatorname{Quord}\mathscr{B}$.

For $T \subseteq A^2$ let the symbol $\mathscr{B}(T)$ denote the quasiorder on $\mathscr{B}$ generated by T.

We prove $\mathscr{B}(A^2 \cap E) = E$ for every $E \in \operatorname{Quord} \mathscr{B}$. For this we need only to show $E \subseteq \mathscr{B}(A^2 \cap E)$. Let $\langle h^m(a), h^n(b)\rangle \in E$ for some $a, b \in A$ and $m, n \in \mathbb{N}$. Then $\langle a, b\rangle = \langle h^{n+m}(h^m(a)), k^{n+m}(h^n(b))\rangle \in E \cap A^2$. If $m = n$ then $\langle h^m(a), h^n(b)\rangle \in \mathscr{B}(A^2 \cap E)$. If $m \neq n$ and e.g. $m < n$ then

$$\langle \alpha, b\rangle = \langle k(g(k^{n-1}(h^m(a)))), k(g(k^{n-1}(h^n(b))))\rangle \in A^2 \cap E$$

and

$$\langle a, \alpha\rangle = \langle k(g(k^{m-1}(h^m(a)))), k(g(k^{n-1}(h^n(b))))\rangle \in A^2 \cap E.$$

Hence $\langle h^m(a), \alpha\rangle = \langle h^m(a), h^m(\alpha)\rangle \in \mathscr{B}(A^2 \cap E)$ and $\langle \alpha, h^n(b)\rangle = \langle h^n(\alpha), h^n(b)\rangle \in \mathscr{B}(A^2 \cap E)$. This yields $\langle h^m(a), h^n(b)\rangle \in \mathscr{B}(A^2 \cap E)$.

Analogously we can prove $A^2 \cap \mathscr{B}(Q) = Q$ for every $Q \in \operatorname{Quord} \mathscr{A}$.

The previous equalities imply $\operatorname{Quord} \mathscr{A} \cong \operatorname{Quord} \mathscr{B}$, i.e. for every countable $\mathscr{A}$ there exists $\mathscr{B}$ with only three unary operations such that they have isomorphic lattices of quasiorders.

Moreover, if $\mathscr{A}$ is finite, we can consider only a finite number of unary operations on $\mathscr{A}$.

Hence, we can consider only algebras $\mathscr{A}$ which are countable or finite and whose similarity types are finite. Let $\mathscr{A} = (A; f_1(x), \dots, f_k(x))$ be such an algebra. Let $A_0, A_1, \dots, A_{k+1}$ be a collection of sets with $|A_i| = |A_0|$, $A_0 = A$, $A_i \cap A_j = \{\alpha\}$ for all $i, j \in \{0, \dots, k+1\}$, $i \neq j$. We set $B = A_0 \cup A_1 \cup \ldots \cup A_{k+1}$. Let h be a bijection of B onto itself such that $h(A_i) = A_{i+1}$ for $i = 0, \dots, k$ and $h(A_{k+1}) = A_0$ and h^{k+2} is the identity mapping on B. Further, let $h(\alpha) = \alpha$. The mapping g can be defined by the aabove formula $(*)$. We can easily verify that $\operatorname{Quord} \mathscr{A} \cong \mathit{Quord}\,(B; \{h, g\})$. $\square$

Theorem 6. *For every algebra $\mathscr{A}$ with zero whose lattice* $\operatorname{Quord} \mathscr{A}$ *has only a countable set of compact elements there exists an algebra $\mathscr{B}$ with only two unary operations such that* $\operatorname{Quord} \mathscr{A} \cong \operatorname{Quord} \mathscr{B}$.

P r o o f. Let $\mathscr{A} = (A, G)$ be an algebra with zero such that $\operatorname{Quord} \mathscr{A}$ contains only countable many compact elements. We can construct an algebra $\mathscr{C}' = (C', G')$ where $C' \subseteq A$ and $G' \subseteq G$ and $\operatorname{Quord} \mathscr{A} \cong \operatorname{Quord} \mathscr{C}'$ for countable sets C' and G'. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of finite subsets of A^2 such that $\{\mathscr{A}(X_n);\ n \in \mathbb{N}\}$ is a set of all compact quasiorders of $\mathscr{A}$ and $\mathscr{A}(X_i) \neq \mathscr{A}(X_j)$ for $i \neq j$. Of course, $\mathscr{A}(X_i)$ means a quasiorder generated by the finite set X_i. Let C_0 be a countable set of elements of A which are entries of pairs of elements of $\bigcup\limits_{i \in \mathbb{N}} X_i$ and containing elements a_{mn}, b_{mn} where $\langle a_{mn}, b_{mn}\rangle$ is a fixed pair of $\mathscr{A}(X_m) \setminus \mathscr{A}(X_n)$ provided it is a non–void set. Set $G_0 = \emptyset$. By induction we construct sets $C_n \subseteq A$ and $G_n \subseteq G$ as follows: suppose $\mathscr{C}_n = (C_n, G_n)$ is done and X is an arbitrary finite subset of C_n^2.

Evidently, $\mathscr{C}_n(X) \subseteq C_n^2 \cap \mathscr{A}(X)$. To any $\langle a, b\rangle \in (C_n^2 \cap \mathscr{A}(X)) \setminus \mathscr{C}_n(X)$ we assign a subset $g(a,b) \subseteq A^2$ and a finite collection $G_{a,b}$ of functions of G such that every subset of A^2 containing $g(a,b)$ and $\mathscr{C}_n(X)$ and closed under all functions of $G_{a,b}$ contains also the pair $\langle a, b\rangle$.

Let D_{n+1} be a set which consists of elements of C_n and of all elements contained in all pairs of $g(a,b)$, where X is an arbitrary finite subset of C_n^2 and $\langle a, b\rangle \in (C_n^2 \cap \mathscr{A}(X)) \setminus \mathscr{C}_n(X)$. Now, we take for C_{n+1} the closure of D_{n+1} with respect to all operations of

$$G_{n+1} = G_n \cup \bigcup \{G_{a,b};\ \langle a, b\rangle \in (C_n^2 \cap \mathscr{A}(X)) \setminus \mathscr{C}_n(X) \text{ for a finite } X \subseteq C_n^2\}.$$

Since both C_n and G_n are countable, also C_{n+1} and G_{n+1} have this property. Put

$$C' = \bigcup_{n\in\mathbb{N}} C_n, \quad G' = \bigcup_{n\in\mathbb{N}} G_n \quad \text{and} \quad \mathscr{C} = (C', G').$$

Since C' contains all elements of all pairs of $\bigcup_{n\in\mathbb{N}} X_n$, hence $\mathscr{C}'(X_n)$ are pairwise distinct quasiorders of $\mathscr{C}'$. Let X be a finite subset of $(C')^2$. By our construction of $\mathscr{C}'$, we have $\mathscr{C}'(X) = (C')^2 \cap \mathscr{A}(X)$. Because $\mathscr{A}(X) = \mathscr{A}(X_m)$ for some $m \in \mathbb{N}$, we conclude

$$\mathscr{C}'(X) = (C')^2 \cap \mathscr{A}(X) = (C')^2 \cap \mathscr{A}(X_m) = \mathscr{C}'(X_m).$$

Hence, the quasiorders of the form $\mathscr{C}'(X_m)$, $m \in \mathbb{N}$, are all compact quasiorders of $\mathscr{C}'$. Moreover, $\mathscr{A}(X_n) \subseteq \mathscr{A}(X_m)$ if and only if $\mathscr{C}'(X_n) \subseteq \mathscr{C}'(X_m)$, thus the semilattices of compact quasiorders on $\mathscr{A}$ and on $\mathscr{C}'$ are isomorphic. This yields $\operatorname{Quord}\mathscr{A} \cong \operatorname{Quord}\mathscr{C}'$. By applying Theorem 5 to the algebra $\mathscr{C}'$ we obtain an algebra $\mathscr{B}$ as required. □

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