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ON OSCILLATION AND ASYMPTOTIC PROPERTY OF A CLASS OF THIRD ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper, oscillation and asymptotic behaviour of solutions of

$$y''' + a(t)y'' + b(t)y' + c(t)y = 0$$

have been studied under suitable assumptions on the coefficient functions $a, b, c \in C([\sigma, \infty), \mathbb{R})$, $\sigma \in \mathbb{R}$, such that $a(t) \geqslant 0$, $b(t) \leqslant 0$ and $c(t) < 0$.

1. In this paper we study the oscillatory and asymptotic behaviour of solutions of

$$y''' + a(t)y'' + b(t)y' + c(t)y = 0, \tag{1.1}$$

where $a \in C^2\big([\sigma, \infty), \mathbb{R}\big)$, $b \in C^1\big([\sigma, \infty), \mathbb{R}\big)$, $c \in C\big([\sigma, \infty), \mathbb{R}\big)$ and $\sigma \in \mathbb{R}$ is such that $a(t) \geqslant 0$, $b(t) \leqslant 0$ and $c(t) < 0$. Eq. (1.1) may be written as

$$\big(r(t)y''\big)' + q(t)y' + p(t)y = 0, \tag{1.2}$$

where $r(t) = \exp\big(\int_\sigma^t a(s)\,\mathrm{d}s\big)$, $q(t) = b(t)r(t)$ and $p(t) = c(t)r(t)$. When $a(t)$, $b(t)$, $c(t)$ are constants, then Eq. (1.1) takes the form

$$y''' + ay'' + by' + cy = 0, \tag{1.3}$$

where $a \geqslant 0$, $b \leqslant 0$ and $c < 0$.

The motivation for the present work has come from certain observations of oscillatory and asymptotic behaviour of solutions of (1.3). The characteristic equation of (1.3) is

$$m^3 + am^2 + bm + c = 0. \tag{1.4}$$

The transformation $n = m + \frac{a}{3}$ transforms (1.4) to

$$n^3 + 3Hn + G = 0, \tag{1.5}$$

where $H = \frac{1}{3}\big(b - \frac{a^2}{3}\big)$ and $G = c - \frac{ab}{3} + \frac{2a^3}{27}$. We may notice that $H \leqslant 0$ and $-G - 2(-H)^{3/2} > 0$ if and only if

$$-\frac{2a^3}{27} + \frac{ab}{3} - c - \frac{2}{3\sqrt{3}}\Big(\frac{a^2}{3} - b\Big)^{3/2} > 0. \tag{1.6}$$

Since $G^2 + 4H^3 = \big(-G - 2(-H)^{3/2}\big)\big(-G + 2(-H)^{3/2}\big)$, the inequality implies that $G < 0$ and hence $G^2 + 4H^3 > 0$. Thus (1.5) has two imaginary roots and a real root. Consequently, (1.4) has two imaginary roots, say, $\big(\alpha - \frac{a}{3}\big) + \mathrm{i}\beta$ and $\big(\alpha - \frac{a}{3}\big) - \mathrm{i}\beta$ and a real root $\gamma - \frac{a}{3}$, where $\alpha + \mathrm{i}\beta$, $\alpha - \mathrm{i}\beta$ and γ are the roots of (1.5). Since $c < 0$, we have $\gamma - \frac{a}{3} > 0$. Thus (1.3) admits oscillatory solutions. On the other hand, if (1.3) admits an oscillatory solution, then (1.4) has two imaginary roots and a real root. This real root is positive because $c < 0$. Thus (1.5) has two imaginary roots and a positive root. Consequently, $G^2 + 4H^3 > 0$ and $G < 0$. This in turn implies that (1.6) holds. Hence (1.3) admits an oscillatory solution if and only if (1.6) holds. Further, if (1.6) holds, then a basis of the solution space of (1.3) is

$$\Big\{\mathrm{e}^{(\alpha - \frac{a}{3})t}\cos\beta t, \mathrm{e}^{(\alpha - \frac{a}{3})t}\sin\beta t, \mathrm{e}^{(\gamma - \frac{a}{3})t}\Big\}. \tag{1.7}$$

If $y(t) = \lambda_1 \mathrm{e}^{(\alpha - \frac{a}{3})t}\cos\beta t + \lambda_2 \mathrm{e}^{(\alpha - \frac{a}{3})t}\sin\beta t + \lambda_3 \mathrm{e}^{(\gamma - \frac{a}{3})t}$, where λ_1, λ_2, λ_3 are reals such that $\lambda_3 \neq 0$, then $y(t)$ is nonoscillatory because $\gamma - \frac{a}{3} > 0$, $(\alpha - \frac{a}{3}) + \mathrm{i}\beta + (\alpha - \frac{a}{3}) - \mathrm{i}\beta + \gamma - \frac{a}{3} = -a$ implies that $\alpha < 0$ and we may write

$$y(t) = \mathrm{e}^{(\gamma - \frac{a}{3})t}\big[(\lambda_1 \cos\beta t + \lambda_2 \sin\beta t)\mathrm{e}^{(\alpha - \gamma)t} + \lambda_3\big].$$

Hence, if (1.6) holds, then the oscillatory solutions of (1.3) form a two-dimensional subspace of the solution space of (1.3). Further, these oscillatory solutions of (1.3) tend to zero as $t \to \infty$ because $(\alpha - \frac{a}{3}) < 0$. Since $\mathrm{e}^{(\alpha - \frac{a}{3})t}\cos\beta t$ and $\mathrm{e}^{(\alpha - \frac{a}{3})t}\sin\beta t$ are solutions of

$$z'' - 2(\alpha - \frac{a}{3})z' + \Big(\big(\alpha - \frac{a}{3}\big)^2 + \beta^2\Big)z = 0,$$

it follows from Sturm's separation theorem that the zeros of any two linearly independent oscillatory solutions of (1.3) separate on $[\sigma, \infty]$. Moreover, if (1.3) admits an oscillatory solution and if $y(t)$ is a nonoscillatory solution of (1.3), then we may write

$$y(t) = \mathrm{e}^{(\alpha - \frac{a}{3})t}(\mu_1 \cos\beta t + \mu_2 \sin\beta t) + \mu_3 \mathrm{e}^{(\gamma - \frac{a}{3})t},$$

where μ_1, μ_2, μ_3 are reals such that $\mu_3 \neq 0$. As

$$\bigl|y(t)\bigr| \geqslant |\mu_3| \mathrm{e}^{(\gamma - \frac{a}{3})t} - \bigl(|\mu_1| + |\mu_2|\bigr) \mathrm{e}^{(\alpha - \frac{a}{3})t},$$

then $\bigl|y(t)\bigr| \to \infty$ as $t \to \infty$. Conversely, if every nonoscillatory solution of (1.3) tends to $\pm\infty$ as $t \to \infty$, then (1.3) admits an oscillatory solution. Indeed, if all solutions of (1.3) are non-oscillatory, then all roots of (1.4) are positive. Hence the sum of the product of these roots taken two at a time is positive. But Eq. (1.4) implies that this sum $= b \leqslant 0$, a contradiction.

The above observations concerning the behaviour of solutions of Eq. (1.3) may be put in the form of a proposition.

Proposition. *Eq.* (1.3) *admits an oscillatory solution if and only if* (1.6) *holds. If* (1.6) *holds, then oscillatory solutions of* (1.3) *form a two-dimensional subspace of the solution space of* (1.3), *the zeros of any two linearly independent oscillatory solutions of* (1.3) *separate each other on* $[\sigma, \infty)$ *and these oscillatory solutions tend to zero as* $t \to \infty$. *Eq.* (1.3) *admits an oscillatory solution if and only if all nonoscillatory solutions of* (1.3) *tend to* $\pm\infty$ *as* $t \to \infty$. *Further,* (1.3) *admits a positive solution which tends to* ∞ *as* $t \to \infty$ *and whose successive derivatives are positive and tend to* ∞ *as* $t \to \infty$.

The object of this paper is to generalize, as far as possible, the above proposition to Eq. (1.1). In [1], Ahmad and Lazer considered a similar problem for (1.1) with $a(t) \leqslant 0$, $b(t) \leqslant 0$, $c(t) \leqslant 0$. The open question stated by them was answered by Parhi and Das [8] following the techniques used in [5].

We may recall that a function $y \in C\bigl([\sigma, \infty), \mathbb{R}\bigr)$ is said to be oscillatory if it has arbitrarily large zeros in $[\sigma, \infty)$; otherwise, it is said to be nonoscillatory. Eq. (1.1) is said to be oscillatory if it has an oscillatory solution, and it is said to be nonoscillatory if all its solutions are nonoscillatory.

Following Hanan [4], Eq. (1.1) is said to be of Class I or C_I if any solution $y(t)$ of the equation with $y(t_0) = y'(t_0) = 0$, $y''(t_0) > 0$, $t_0 > \sigma$, satisfies $y(t) > 0$ for $\sigma \leqslant t < t_0$. It is said to be of Class II of C_{II} if any solution $y(t)$ of the equation with $y(t_0) = y'(t_0) = 0$, $y''(t_0) > 0$, $t_0 \geqslant \sigma$, satisfies $y(t) > 0$ for $t > t_0$.

The transformation $y = z \exp\bigl(-\frac{1}{3}\int a(t)\,\mathrm{d}t\bigr)$ transforms Eq. (1.1) to the equation

$$z''' + 2A(t)z' + \bigl(A'(t) + B(t)\bigr)z = 0, \tag{1.8}$$

where $A(t) = \frac{1}{2}\bigl(b(t) - a'(t) - \frac{1}{3}a^2(t)\bigr)$ and

$$B(t) = \frac{2}{27}a^3(t) + \frac{1}{6}a''(t) + \frac{1}{3}a(t)a'(t) - \frac{1}{3}a(t)b(t) - \frac{1}{2}b'(t) + c(t).$$

2. In this section we obtain sufficient conditions for oscillation of Eq. (1.1). The adjoint of (1.1) is given by

$$z''' - a(t)z'' + \big(b(t) - 2a'(t)\big)z' - \big(c(t) - b'(t) + a''(t)\big)z = 0. \tag{2.1}$$

The following theorem due to Parhi and Das [7] is needed in the sequel.

Theorem 2.1. *If* $a(t) \leqslant 0$, $b(t) \leqslant 0$, $c(t) > 0$, $b(t) - a'(t) \leqslant 0$ *and*

$$\int_\sigma^\infty \left[\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\Big(\frac{a^2(t)}{3} - b(t) + a'(t)\Big)^{3/2}\right] \mathrm{d}t = \infty,$$

then Eq. (1.1) *is oscillatory.*

Lemma 2.2. *If* $a(t)b(t) + b'(t) - c(t) \leqslant 0$, *then Eq.* (1.1) *is of Class I.*

Proof. Let $y(t)$ be a solution of (1.1) with $y(t_0) = y'(t_0) = 0$ and $y''(t_0) > 0$, where $t_0 > \sigma$. From the continuity of $y''(t)$ it follows that there exists a δ, $0 < \delta < t_0 - \sigma$, such that $y''(t) > 0$ for $t \in [t_0 - \delta, t_0]$. We claim that $y''(t) > 0$ for $t \in [\sigma, t_0]$. If not, there exists a $t_1 \in [\sigma, t_0 - \delta]$ such that $y''(t_1) = 0$ and $y''(t) > 0$ for $t \in (t_1, t_0]$. Thus $y'(t) < 0$ and $y(t) > 0$ for $t \in (t_1, t_0)$. Integrating (1.2) from t_1 to t_0, we obtain

$$0 < r(t_0)y''(t_0) = q(t_1)y(t_1) + \int_{t_1}^{t_0} \big(q'(t) - p(t)\big)y(t)\,\mathrm{d}t < 0,$$

since $q'(t) - p(t) \leqslant 0$, a contradiction. Hence our claim holds. Consequently, $y(t) > 0$ and $y'(t) < 0$ for $t \in [\sigma, t_0)$.

The proof of the lemma is complete. □

Lemma 2.3. *If* $y(t)$ *is a solution of* (1.1) *with* $y(t_0) \geqslant 0$, $y'(t_0) \geqslant 0$ *and* $y''(t_0) > 0$ *for some* $t_0 \geqslant \sigma$, *then* $y(t) > 0$, $y'(t) > 0$ *and* $y''(t) > 0$ *for* $t > t_0$. *Similarly, if* $y(t_0) \leqslant 0$, $y'(t_0) \leqslant 0$ *and* $y''(t_0) < 0$ *for* $t_0 \geqslant \sigma$, *then* $y(t) < 0$, $y'(t) < 0$ *and* $y''(t) < 0$ *for* $t > t_0$.

Proof. Let $y(t)$ be a solution of (1.1) with $y(t_0) \geqslant 0$, $y'(t_0) \geqslant 0$ and $y''(t_0) > 0$ for $t_0 \geqslant \sigma$. So there exists a $\delta > 0$ such that $y''(t) > 0$ for $t \in [t_0, t_0 + \delta)$. If there is a $t_1 \geqslant t_0 + \delta$ such that $y''(t_1) = 0$ and $y''(t) > 0$ for $t_0 \leqslant t < t_1$, then $y'(t) > 0$ and $y(t) > 0$ for $t_0 < t \leqslant t_1$. Multiplying Eq. (1.2) by $y'(t)$ and integrating the resulting identity from t_0 to t_1, we obtain

$$0 < \int_{t_0}^{t_1} r(t)\big(y''(t)\big)^2\,\mathrm{d}t = \int_{t_0}^{t_1} q(t)\big(y'(t)\big)^2\,\mathrm{d}t + \int_{t_0}^{t_1} p(t)y(t)y'(t)\,\mathrm{d}t < 0,$$

a contradiction. Hence $y''(t) > 0$ for $t \geqslant t_0$. Then $y(t) > 0$ and $y'(t) > 0$ for $t > t_0$. The other assertion follows similarly.

Hence the lemma is proved. □

Corollary 2.4. *Eq.* (1.1) *is of Class II.*

Theorem 2.5. *Eq.* (1.1) *admits a positive increasing solution which tends to* ∞ *as* $t \to \infty$. *Further, if* $\int_\sigma^\infty \frac{\mathrm{d}t}{r(t)} = \infty$, *then the derivative of the solution tends to* ∞ *as* $t \to \infty$.

Proof. If $y(t)$ is a solution of (1.1) with $y(t_0) \geqslant 0$, $y'(t_0) \geqslant 0$ and $y''(t_0) > 0$, then Lemma 2.3 implies that $\lim\limits_{t\to\infty} y(t) = \infty$. Since $y(t) > 0$ and $y'(t) > 0$ for $t > t_0$, then $r(t)y''(t)$ is increasing in $[t_0, \infty)$. Thus $\int_\sigma^\infty \frac{\mathrm{d}t}{r(t)} = \infty$ implies that $y'(t) \to \infty$ as $t \to \infty$. This completes the proof of the theorem. □

Theorem 2.6. *If* $b(t) - a'(t) \leqslant 0$, $b(t) - 2a'(t) \leqslant 0$, $c(t) - b'(t) + a''(t) < 0$ *and*

$$\begin{aligned}\int_\sigma^\infty \Big[-\frac{2a^3(t)}{27} + \frac{a(t)b(t)}{3} - c(t) + b'(t) - a''(t) \\ - \frac{2a(t)a'(t)}{3} - \frac{2}{3\sqrt{3}}\Big(\frac{a^2(t)}{3} - b(t) + a'(t)\Big)^{3/2}\Big]\,\mathrm{d}t = \infty,\end{aligned} \tag{2.2}$$

then (1.1) *is oscillatory.*

Proof. It follows from Theorem 2.1 that Eq. (2.1) is oscillatory. Since (1.1) is of C_{II} and its adjoint (2.1) is oscillatory, then (1.1) is oscillatory (see theorem 4.7, Hanan [4]).

The proof of the theorem is complete. □

Remark. We may note that (2.2) reduces to (1.6) if $a(t)$, $b(t)$, $c(t)$ are constants.

A theorem similar to Theorem 2.6 is given in [3] (see Theorem 2.14, p. 39). However, these theorems are not comparable. From the above Remark it is clear that Theorem 2.6 is a generalization of the first part of Proposition in Section 1.1. This cannot be claimed as concerns Theorem 2.14 in [3].

3. This section deals with oscillatory and asymptotic behaviour of solutions of Eq. (1.1).

Lemma 3.1. *If* $y(t)$ *is a nonoscillatory solution of* (1.1), *then there exists a* $t_0 \geqslant \sigma$ *such that either* $y(t)y'(t) < 0$ *or* $y(t)y'(t) > 0$ *for* $t \geqslant t_0$.

Proof. Without any loss of generality we may assume that $y(t) > 0$ for $t \geqslant T \geqslant \sigma$. Let t_1 and $t_2 (T \leqslant t_1 < t_2)$ be two consecutive zeros of $y'(t)$ such that $y'(t) > 0$ for $t \in (t_1, t_2)$. Multiplying (1.1) by $y'(t)$ and integrating the resulting identity from t_1 to t_2, we obtain

$$0 < \int_{t_1}^{t_2} r(t)\big(y''(t)\big)^2\,\mathrm{d}t = \int_{t_1}^{t_2} q(t)\big(y'(t)\big)^2\,\mathrm{d}t + \int_{t_1}^{t_2} p(t)y(t)y'(t)\,\mathrm{d}t < 0,$$

a contradiction. Hence there exists a $t_0 \geqslant T$ such that $y'(t) > 0$ or < 0 for $t \geqslant t_0$. This completes the proof of the lemma. $\square$

Theorem 3.2. *If* (1.1) *has an oscillatory solution, then every nonoscillatory solution* $y(t)$ *of* (1.1) *satisfies the following conditions:*

$$y(t)y'(t) \neq 0,\ \operatorname{sgn} y(t) = \operatorname{sgn} y'(t),\ t \geqslant t_0 \geqslant \sigma$$

and $\lim_{t\to\infty} |y(t)| = \infty$. *If, in addition,* $\int_\sigma^\infty p(t)\,\mathrm{d}t = -\infty$, *then* $y(t)y'(t)y''(t) \neq 0$ *and* $\operatorname{sgn} y(t) = \operatorname{sgn} y'(t) = \operatorname{sgn} y''(t)$, $t \geqslant T_0 \geqslant t_0$.

Proof. Let $y(t) > 0$ for $t \geqslant T \geqslant \sigma$. Let $z(t)$ be an oscillatory solution of (1.1). We claim that $W(t) = y(t)z'(t) - y'(t)z(t)$ must vanish for some value of $t \in [T, \infty)$. If not, then $W(t) \neq 0$ for $t \in [T, \infty)$. Setting $u(t) = z(t)/y(t)$, we obtain $u'(t) = W(t)/y^2(t) \neq 0$ for $t \geqslant T$. If t_1 and $t_2 (T \leqslant t_1 \leqslant t_2)$ are consecutive zeros of $z(t)$, then $u(t_1) = 0$, $u(t_2) = 0$ and $u(t) \neq 0$ for $t \in (t_1, t_2)$. This is impossible since $u'(t) \neq 0$ for $t \geqslant T$. Thus our claim holds. Let $W(a) = 0$ for some $a \in [T, \infty)$. It is possible to obtain c_1 and c_2, not both zero, such that

$$\begin{aligned} c_1 y(a) + c_2 z(a) &= 0, \\ c_1 y'(a) + c_2 z'(a) &= 0 \end{aligned}$$

and

$$c_1 y''(a) + c_2 z''(a) \neq 0,$$

because $y(t)$ and $z(t)$ are linearly independent on $[T, \infty)$. Without any loss of generality we may assume that $c_1 y''(a) + c_2 z''(a) > 0$. Setting $v(t) = c_1 y(t) + c_2 z(t)$, we notice that $v(t)$ is a solution of (1.1) with $v(a) = 0$, $v'(a) = 0$ and $v''(a) > 0$. Proceeding as in Lemma 2.3, one may obtain $v(t) \to \infty$ as $t \to \infty$.

From Lemma 3.1, it follows that $y'(t) > 0$ or < 0 for $t \geqslant t_0 \geqslant T$. If $y'(t) < 0$ for $t \geqslant t_0$, then $\lim_{t\to\infty} y(t) = \lambda$ exists, where $0 \leqslant \lambda < \infty$. Clearly, $c_2 = 0$ implies that $\lim_{t\to\infty} v(t) = c_1\lambda < \infty$, a contradiction. Thus $c_2 \neq 0$. Further, since $\lim_{t\to\infty} c_2 z(t) = \lim_{t\to\infty} v(t) - c_1 \lim_{t\to\infty} y(t) = \infty$, then $\lim_{t\to\infty} z(t) = \infty$ or $-\infty$ provided $c_2 > 0$ or < 0, respectively. In either case we obtain a contradiction since $z(t)$ is oscillatory. Hence $y'(t) > 0$ for $t \geqslant t_0$. Clearly, $c_1 \neq 0$ because $c_1 = 0$ implies that $c_2 \neq 0$ and $v(t) = c_2 z(t)$ is oscillatory, a contradiction. If $c_2 = 0$, then $\lim_{t\to\infty} c_1 y(t) = \lim_{t\to\infty} v(t) = \infty$. As $c_1 < 0$ implies that $y(t) < 0$ for large t, then $c_1 > 0$ and hence $\lim_{t\to\infty} y(t) = \infty$. Suppose that $c_2 \neq 0$. If $\lim_{t\to\infty} y(t)$ exists and its value is finite, then $\lim z(t) = \pm\infty$, contradicting the oscillatory nature of $z(t)$. Thus $\lim_{t\to\infty} y(t) = \infty$.

Suppose that $\int_\sigma^\infty p(t)\,\mathrm{d}t = -\infty$. Since $y(t) > 0$ and $y'(t) > 0$ for $t \geqslant t_0$, then $r(t)y''(t)$ is increasing and hence $y''(t)$ has a constant sign for $t \geqslant T_0 \geqslant t_0$. If $y''(t) < 0$ for $t \geqslant T_0$, then integrating (1.1) from T_0 to t we obtain

$$\begin{aligned} r(t)y''(t) &\geqslant r(T_0)y''(T_0) - \int_{T_0}^t p(s)y(s)\,\mathrm{d}s \\ &\geqslant r(T_0)y''(T_0) - y(T_0)\int_{T_0}^t p(s)\,\mathrm{d}s. \end{aligned}$$

Thus $y''(t) > 0$ for large t, a contradiction. Hence $y''(t) > 0$ for $t \geqslant T_0$ and the proof of the theorem is complete. □

Corollary 3.3. *If* (1.1) *has an oscillatory solution, then every bounded solution of* (1.1) *oscillates.*

Theorem 3.4. *Let $\int_\sigma^\infty p(t)\,\mathrm{d}t = -\infty$. Then Eq.* (1.1) *has an oscillatory solution if and only if every nonoscillatory solution $y(t)$ of* (1.1) *satisfies the conditions*

$$\begin{aligned} y(t)y'(t)y''(t) \neq 0,\ \operatorname{sgn} y(t) = \operatorname{sgn} y'(t) = \operatorname{sgn} y''(t), \\ t \geqslant T_0 \geqslant \sigma \text{ and } \lim_{t\to\infty} \bigl|y(t)\bigr| = \infty. \end{aligned} \tag{3.1}$$

P r o o f. Necessity follows from Theorem 3.2. For sufficiency, we assume that (3.1) holds for every nonoscillatory solution $y(t)$ of (1.1). We shall show that (1.1) admits an oscillatory solution. The proof is similar to that of Theorem 1 in [1], however, it is given here for completeness.

Let z_0, z_1, z_2 be solutions of (1.1) with initial conditions

$$z_k^{(j)}(\sigma) = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$j,k = 0,1,2$. Clearly, z_0, z_1, z_2 are linearly independent. For each positive integer $n > \sigma$ it is possible to determine real numbers a_{0n}, a_{2n}, b_{1n} and b_{2n} such that

$$\begin{aligned} a_{0n}z_0(n) + a_{2n}z_2(n) = 0, \\ b_{1n}z_1(n) + b_{2n}z_2(n) = 0 \end{aligned} \tag{3.2}$$

and $a_{0n}^2 + a_{2n}^2 = 1$, $b_{1n}^2 + b_{2n}^2 = 1$. Define, for each positive integer $n > \sigma$,

$$\begin{aligned} u_n &= a_{0n}z_0 + a_{2n}z_2, \\ v_n &= b_{1n}z_1 + b_{2n}z_2. \end{aligned}$$

Thus u_n and v_n are solutions of (1.1) with $u_n(n) = 0$ and $v_n(n) = 0$ by (3.2). Clearly, there exists a sequence $\langle n_j \rangle$ of positive integers $> \sigma$ such that $a_{0n_j} \to a_0$, $a_{2n_j} \to a_2$, $b_{1n_j} \to b_1$ and $b_{2n_j} \to b_2$ as $n_j \to \infty$ and hence $a_0^2 + a_2^2 = 1$ and $b_1^2 + b_2^2 = 1$. Setting $u = a_0 z_0 + a_2 z_2$ and $v = b_1 z_1 + b_2 z_2$, we notice that u and v are nontrivial solutions of (1.1) and

$$\lim_{n_j \to \infty} u_{n_j}^{(k)} = u^{(k)}, \quad \lim_{n_j \to \infty} v_{n_j}^{(k)} = v^{(k)},$$

$k = 0, 1, 2$, uniformly on any compact subinterval of $[\sigma, \infty)$. We show that both u and v are oscillatory solutions of (1.1). If u is nonoscillatory, then there exists a $T_0 \geqslant \sigma$ such that

$$u(t)u'(t)u''(t) \neq 0, \ \operatorname{sgn} u(t) = \operatorname{sgn} u'(t) = \operatorname{sgn} u''(t), \ t \geqslant T_0$$

and $\lim_{t \to \infty} |u(t)| = \infty$. In particular,

$$u(T_0)u'(T_0)u''(T_0) \neq 0, \ \operatorname{sgn} u(T_0) = \operatorname{sgn} u'(T_0) = \operatorname{sgn} u''(T_0).$$

Hence there exists a positive integer N such that

$$u_{n_j}(T_0)u'_{n_j}(T_0)u''_{n_j}(T_0) \neq 0, \ \operatorname{sgn} u_{n_j}(T_0) = \operatorname{sgn} u'_{n_j}(T_0) = \operatorname{sgn} u''_{n_j}(T_0),$$

for $n_j \geqslant N$. Lemma 2.3 yields that $u_{n_j}(t) \neq 0$ for $n_j \geqslant N$ and $t > T_0$. Thus $u_{n_j}(n_j) \neq 0$ for all $n_j > \max\{N, T_0\}$. This contradicts the fact that $u_n(n) = 0$ for every positive integer $n > \sigma$. Hence $u(t)$ is oscillatory. Similarly, it may be shown that $v(t)$ is oscillatory.

Thus the theorem is proved. □

Remark. The assumption $\int_\sigma^\infty p(t)\,\mathrm{d}t = -\infty$ is not needed in the proof of the sufficiency part of theorem 3.4. Moreover, this condition is satisfied if a, b and c are constants.

Theorem 3.5. *Let $\int_\sigma^\infty p(t)\,\mathrm{d}t = -\infty$. If* (1.1) *admits an oscillatory solution, then there exist two linearly independent oscillatory solutions u and v of* (1.1) *such that any nontrivial linear combination of u and v is also oscillatory and the zeros of u and v separate.*

The proof is similar to that of Theorem 2 due to Ahmad and Lazer [1] and hence is omitted.

Remark. Theorem 3.4 and 3.5 are similar to theorems 6.23 and 6.25, respectively, in [3]. While $\int_\sigma^\infty p(t)\,\mathrm{d}t = -\infty$ is assumed in the former theorems, the disconjugacy of $y'' + a(t)y' + b(t)y = 0$ is assumed in the latter ones. The proof of Theorems 6.23 and 6.25 may be found in Gera [2].

Theorem 3.6. *Suppose that $q'(t)-p(t)\leqslant 0$ but $\not\equiv 0$ in any neighbourhood of infinity. Then Eq.* (1.1) *admits a solution $y(t)$ with the following properties:*

$$y(t)y'(t)y''(t)\neq 0,\ \operatorname{sgn} y(t)=\operatorname{sgn} y''(t)\neq \operatorname{sgn} y'(t),\ t\geqslant\sigma,$$
$$\lim_{t\to\infty} y'(t)=0 \text{ and } \lim_{t\to\infty} y(t)=\lambda,\ -\infty<\lambda<\infty.$$

If, in addition, $\int_\sigma^\infty \big(q'(t)-p(t)\big)\,\mathrm{d}t=-\infty$ and $\lim\limits_{t\to\infty} q(t)=k$, $-\infty<k<0$, then $\lim\limits_{t\to\infty} y(t)=0$.

Proof. For every positive integer $n>\sigma$, let $y_n(t)$ be a solution of (1.1) with initial conditions

$$y_n(n)=0,\ y_n'(n)=0,\ y_n''(n)>0.$$

Since $q'(t)-p(t)=\big(a(t)b(t)+b'(t)-c(t)\big)r(t)$, Lemma 2.2 yields that $y_n(t)>0$, $y_n'(t)<0$ and $y_n''(t)>0$ for $t\in[\sigma,n)$. We may write

$$y_n(t)=c_{1n}u_1(t)+c_{2n}u_2(t)+c_{3n}u_3(t),\ t\in[\sigma,n),$$

where $c_{1n}^2+c_{2n}^2+c_{3n}^2=1$ and $\{u_1,u_2,u_3\}$ is a basis of the solution space of (1.1). The sequence $\langle c_{in}\rangle$, $i=1,2,3$, has a convergent subsequence $\langle c_{in_j}\rangle$ such that $c_{in_j}\to c_i$ and $n_j\to\infty$. Hence $c_1^2+c_2^2+c_3^2=1$. Setting $y(t)=c_1u_1(t)+c_2u_2(t)+c_3u_3(t)$, we see that $y(t)$ is a solution of (1.1) and

$$\lim_{n_j\to\infty} y_{n_j}^{(k)}(t)=y^{(k)}(t),$$

$k=0,1,2$, uniformly on every compact subinterval of $[\sigma,\infty)$. Thus $y(t)>0$, $y'(t)<0$ and $y''(t)>0$ for $t\geqslant\sigma$. As $\lim\limits_{t\to\infty} y'(t)=L$, $-\infty<L<0$, implies that $y(t)<0$ for large t, we have $\lim\limits_{t\to\infty} y'(t)=0$. Clearly, $\lim\limits_{t\to\infty} y(t)=\lambda$, $0\leqslant\lambda<\infty$.

If $\lim\limits_{t\to\infty} y(t)=\lambda$, $\lambda>0$, then integrating (1.1) from σ to t and using the additional conditions we get

$$\begin{aligned}0<r(t)y''(t)&=r(\sigma)y''(\sigma)-q(t)y(t)+q(\sigma)y(\sigma)+\int_\sigma^t \big(q'(s)-p(s)\big)y(s)\,\mathrm{d}s\\&\leqslant r(\sigma)y''(\sigma)-q(t)y(t)+q(\sigma)y(\sigma)+y(t)\int_\sigma^t \big(q'(s)-p(s)\big)\,\mathrm{d}s<0\end{aligned}$$

for large t, a contradiction. Hence the theorem is proved. □

Theorem 3.7. *If $q'(t)-p(t)\leqslant 0$ but $\not\equiv 0$ in any neighbourhood of infinity, then* (1.1) *is nonoscillatory.*

P r o o f. If possible, let (1.1) admit an oscillatory solution. It follows from Theorem 3.2 that every nonoscillatory solution $y(t)$ of (1.1) has the property $|y(t)| \to \infty$ as $t \to \infty$. On the other hand, Theorem 3.6 yields that (1.1) has a nonoscillatory solution $u(t)$ such that $\lim\limits_{t\to\infty} u(t) = \lambda$, $-\infty < \lambda < \infty$. This contradiction completes the proof of the theorem. □

Remark. Theorem 3.7 is the same as Theorem 2.1 in [6]. However, our method of proof is quite different.

Theorem 3.8. *Suppose that* $q'(t) - p(t) \geqslant 0$ *and* $\int_\sigma^\infty \big(q'(t) - p(t)\big)\,\mathrm{d}t = \infty$. *Then Eq.* (1.1) *has an oscillatory solution if and only if every nonoscillatory solution* $y(t)$ *of* (1.1) *satisfies the conditions* (3.1).

P r o o f. The sufficiency part is similar to that of Theorem 3.4. For necessity, one may proceed as in Theorem 3.2 to obtain $y(t)y'(t) \neq 0$, $\operatorname{sgn} y(t) = \operatorname{sgn} y'(t)$ for $t \geqslant t_0 \geqslant \sigma$ and $\lim\limits_{t\to\infty} |y(t)| = \infty$. In order to be definite about the sign of $y''(t)$, we may assume that $y(t) > 0$ for $t \geqslant t_0$. Hence $y'(t) > 0$ for $t \geqslant t_0$. Since $r(t)y''(t)$ is increasing, we have $y''(t) > 0$ or < 0 for $t \geqslant T_0 \geqslant t_0$. If $y''(t) < 0$ for $t \geqslant T_0$, then integration of (1.2) from T_0 to t yields

$$\begin{aligned} r(t)y''(t) &\geqslant r(T_0)y''(T_0) + q(T_0)y(T_0) + \int_{T_0}^t \big(q'(s) - p(s)\big)y(s)\,\mathrm{d}s \\ &\geqslant r(T_0)y''(T_0) + q(T_0)y(T_0) + y(T_0)\int_{T_0}^t \big(q'(s) - p(s)\big)\,\mathrm{d}s. \end{aligned}$$

Hence $y''(t) > 0$ for large t. This contradiction completes the proof of the theorem. □

Example. Consider

$$y''' + \frac{1}{t^2}y'' - \Big(1 + \frac{2}{t^3} - \frac{1}{3t^4}\Big)y' - \Big(\mathrm{e}^t + \frac{2}{3\sqrt{3}}\Big)y = 0.$$

Clearly, the conditions of Theorems 2.6, 3.4 and 3.8 are satisfied. Hence the given equation admits an oscillatory solution and all nonoscillatory solutions of the equation tend to ∞ as $t \to \infty$.

Theorem 3.9. *Suppose that* $q'(t) - p(t) \leqslant 0$, $\int_\sigma^\infty \big(q'(t) - p(t)\big)\,\mathrm{d}t = -\infty$ *and* $\lim\limits_{t\to\infty} q(t) = k$, $-\infty < k < 0$. *Then every solution* $y(t)$ *of* (1.1) *satisfies either*

$$y(t)y'(t)y''(t) \neq 0,\ \operatorname{sgn} y(t) = \operatorname{sgn} y'(t) \neq \operatorname{sgn} y''(t),\ t \geqslant t_0 \geqslant \sigma$$

or

$$y(t)y'(t)y''(t) \neq 0,\ \operatorname{sgn} y(t) = \operatorname{sgn} y''(t) \neq \operatorname{sgn} y'(t),\ t \geqslant t_0 \geqslant \sigma,$$
$$\lim_{t\to\infty} y(t) = 0 \textit{ and } \lim_{t\to\infty} y'(t) = 0.$$

P r o o f. Let $y(t)$ be any solution of (1.1). From Theorem 3.7, it follows that Eq. (1.1) is nonoscillatory and hence $y(t)$ is nonoscillatory. We may assume, without any loss of generality, that $y(t) > 0$ for $t \geqslant T \geqslant \sigma$. Lemma 3.1 yields that $y'(t) > 0$ or < 0 for $t \geqslant T_0 \geqslant T$. If $y'(t) > 0$ for $t \geqslant T_0$, then $r(t)y''(t)$ is increasing and hence $y''(t) > 0$ or < 0 for large t. As $y''(t) > 0$ for large t yields, due to Theorem 3.4, that Eq. (1.1) has an oscillatory solution, a contradiction, we conclude that $y''(t) < 0$ for $t \geqslant t_0 \geqslant T_0$. Thus we have $\operatorname{sgn} y(t) = \operatorname{sgn} y'(t) \neq \operatorname{sgn} y''(t)$ for $t \geqslant t_0$.

Next suppose that $y'(t) < 0$ for $t \geqslant T_0$. If possible, let $y''(t)$ be oscillatory with a sequence of zeros $\langle t_n \rangle$ such that $T_0 < t_1 < t_2 < \ldots$ and $t_n \to \infty$ as $n \to \infty$. Clearly, $\lim\limits_{t\to\infty} y(t) = \alpha$ exists. If $\alpha = 0$, then integrating (1.1) form t_1 to t_n we obtain

$$\begin{aligned} 0 &= r(t_n)y''(t_n) - r(t_1)y''(t_1) + q(t_n)y(t_n) - q(t_1)y(t_1) + \int_{t_1}^{t_n} \big(p(t) - q'(t)\big)y(t)\,\mathrm{d}t \\ &= q(t_n)y(t_n) - q(t_1)y(t_1) + \int_{t_1}^{t_n} \big(p(t) - q'(t)\big)y(t)\,\mathrm{d}t. \end{aligned}$$

If the zeros of y'' and q coincide, then we get a contradiction $0 > 0$ form the above identity. Otherwise, taking limit in

$$0 > q(t_n)y(t_n) - q(t_1)y(t_1)$$

as $n \to \infty$, we obtain $0 \geqslant -q(t_1)y(t_1) > 0$, a contradiction. If $\alpha > 0$, then taking limit as $n \to \infty$ in

$$0 \geqslant q(t_n)y(t_n) - q(t_1)y(t_1) + y(t_n)\int_{t_1}^{t_n} \big(p(t) - q'(t)\big)\,\mathrm{d}t$$

we get a contradiction again. Thus $y''(t) > 0$ or < 0 for large t. As $y''(t) < 0$ for large t implies that $y(t) < 0$ for large t, we have $y''(t) > 0$ for $t \geqslant t_0 \geqslant T_0$. If $\lim\limits_{t\to\infty} y'(t) = \lambda$, $-\infty < \lambda < 0$, then $y(t) < 0$ for large t. Thus $\lambda = 0$. Let $\alpha > 0$. Integrating (1.1) from t_0 to t we obtain

$$\begin{aligned} 0 < r(t)y''(t) &= r(t_0)y''(t_0) - q(t)y(t) + \int_{t_0}^{t} \big(q'(s) - p(s)\big)y(s)\,\mathrm{d}s \\ &\leqslant r(t_0)y''(t_0) - q(t)y(t) + y(t)\int_{t_0}^{t} \big(q'(s) - p(s)\big)\,\mathrm{d}s \end{aligned}$$

and hence $y''(t) < 0$ for large t, a contradiction. Thus $y(t)y'(t)y''(t) \neq 0$, $\operatorname{sgn} y(t) = \operatorname{sgn} y''(t) \neq \operatorname{sgn} y'(t)$, $t \geqslant t_0$, $\lim\limits_{t\to\infty} y(t) = 0$ and $\lim\limits_{t\to\infty} y'(t) = 0$.

The proof of the theorem is complete. □

Remark. The following assertions are yet to be established:

(A) If (1.1) admits an oscillatory solution, then (2.2) holds.

(B) If (1.1) admits an oscillatory solution, then all oscillatory solutions of (1.1) tend to zero as $t \to \infty$. Corollary 3.3 provides an indication in this direction.

(C) If (1.1) has an oscillatory solution, then all oscillatory solutions of (1.1) form a two-dimensional subspace of the solution space of (1.1).

In the conclusion we prove (C) with the assumption of (B).

Theorem 3.10. *Suppose that the existence of an oscillatory solution of* (1.1) *implies that all oscillatory solutions of* (1.1) *tend to zero as* $t \to \infty$. *If* $\int_\sigma^\infty p(t)\,\mathrm{d}t = -\infty$ *and* (1.1) *admits as oscillatory solution, then all oscillatory solutions of* (1.1) *form a two-dimensional subspace of the solution space of* (1.1).

Proof. It follows from Theorem 3.5 that (1.1) admits two linearly independent oscillatory solutions u and v whose linear combination is an oscillatory solution of (1.1). Let $y(t)$ be any oscillatory solution of (1.1). Theorem 2.5 yields that (1.1) admits a positive solution $y_0(t)$ such that $y_0(t) \to \infty$ as $t \to \infty$. Clearly, $\{u, v, y_0\}$ is a basis of the solution space of (1.1). If possible, let $y(t) = c_1u(t) + c_2v(t) + c_3y_0(t)$, where c_1, c_2, c_3 are reals such that $c_3 \neq 0$. Thus $y(t) \to \infty$ or $-\infty$ as $t \to \infty$ provided $c_3 > 0$ or < 0, respectively. In either case we get a contradiction because $y(t)$ is oscillatory. Thus $y(t)$ can be expressed as linear combination of u and v and hence the theorem is proved. □

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