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ON COVERS IN THE LATTICE OF REPRESENTABLE ℓ-VARIETIES

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In the work [1] the first example of representable ℓ-variety $\mathscr{V}$ without covers in the lattice of representable ℓ-varieties $\mathbb{L}_0$ was discovered. In connection with this result the natural question on the existence of new examples of representable ℓ-varieties with this property arises.

In this paper the existence of at least five representable ℓ-varieties without covers in the lattice of representable ℓ-varieties $\mathbb{L}_0$ is shown (Theorems 1, 2, 4). Some properties of these ℓ-varieties are described (Theorems 3, 5, 6).

1. Preliminaries

In this paper $\mathbb{N}$ denotes the set of natural numbers, $[b,a]=b^{-1}a^{-1}ba$; $|x|=x\vee x^{-1}$. $x\ll y(x,y>e)$ denotes $x^n\leqslant y$ for all $n\in N$. If $|x|^n\geqslant |y|$ and $|x|\leqslant |y|^m$ for some $n,m\in\mathbb{N}$, then the elements x,y are archimedean equivalent and this fact is denoted by $x\sim_a y$.

The ℓ-variety $\mathscr{R}$ defined by the identity

$$(x\wedge y^{-1}x^{-1}y)\vee e=e \tag{1}$$

is called the ℓ-variety of representable ℓ-groups. Any ℓ-variety $\mathscr{X}$ in which the identity (1) is valid is called a representable ℓ-variety. Since each ℓ-group in $\mathscr{R}$ is a subdirect product of totally ordered groups, any ℓ-variety $\mathscr{X}$, $\mathscr{X}\subseteq\mathscr{R}$ is uniquely determined by the totally ordered groups contained in $\mathscr{X}$ (in fact, any subdirectly irreducible ℓ-group of $\mathscr{R}$ is totally ordered). The set $\mathbb{L}_0$ of all representable ℓ-varieties is a complete lattice under naturally defined operations of join and meet [2].

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Let $\mathscr{V}_1, \mathscr{V}_2 \in \mathbb{L}_0$. $\mathscr{V}_1$ is said to cover $\mathscr{V}_2$ in the lattice $\mathbb{L}_0$ if $\mathscr{V}_1 \supseteq \mathscr{V}_2$, $\mathscr{V}_1 \neq \mathscr{V}_2$ and the inclusions $\mathscr{V}_1 \supseteq \mathscr{U} \supseteq \mathscr{V}_2$, where $\mathscr{U} \in \mathbb{L}_0$, imply $\mathscr{V}_1 = \mathscr{U}$ or $\mathscr{V}_2 = \mathscr{U}$.

The basic facts on groups and ℓ-groups can be found in [2, 3] and [4, 5] respectively.

Let A_β be a subgroup of the additive group of reals, let $1 \neq \beta$ be a positive real number such that $a \in A_\beta$ implies βa, $\beta^{-1} a \in A_\beta$. Let B_β be an infinite cyclic subgroup of the multiplicative group of positive reals generated by the number β. Then the set $T_\beta = \{(r, a) \mid r \in B_\beta, a \in A_\beta\}$ with the operation of multiplication defined by the rule

$$(r, a)(r', a') = (rr', ra + a')$$

is a group. The group T_β is a totally ordered group under the lexicographic order: $(r, a) \geqslant 0$ if $r = \beta^p$ and $p > 0$ or $p = 0$ and $a \geqslant 0$.

Lemma 1 [1]. *Let G be a nonabelian totally ordered group with a convex archimedean normal subgroup A such that the quotient group G/A is an infinite cyclic group. Then G is isomorphic to a totally ordered group T_β for some positive real number $\beta \neq 1$ and for some subgroup A_β of the additive group of reals.*

Lemma 2 [1]. *Let $\mathscr{U}_\beta = \mathrm{var}_\ell(T_\beta)$ and $\mathscr{U}_{\beta^m} = \mathrm{var}_\ell(T_{\beta^m})$ for $m \geqslant 2$. Then $\mathscr{U}_{\beta^m} \subseteq \mathscr{U}_\beta$ and $\mathscr{U}_{\beta^m} \neq \mathscr{U}_\beta$.*

Corollary. $T_\beta \in \mathscr{U}_\beta \setminus U_{\beta^m}$.

In the work [6] the automorphism φ of order 2 of the lattice of ℓ-varieties $\mathbb{L}$ is defined. It is also described how to rewrite the basis of identities of any ℓ-variety $\mathscr{X}$ to the basis of identities of the ℓ-variety $\varphi(\mathscr{X})$. More precisely, with any ℓ-group G we associate the ℓ-group G^R which is obtained from G by reversing order, and with any ℓ-variety $\varphi(\mathscr{X})$ we associate the ℓ-variety $\varphi(\mathscr{X}) = \mathscr{X}^R = \{G^R \mid G \in \mathscr{X}\}$.

Proposition 1. $(T_\beta)^R \cong T_{\beta^{-1}}$.

The proof is straightforward. □

2. New examples of ℓ-varieties without covers

In this section new examples of representable ℓ-varieties without covers in the lattice of representable ℓ-varieties $\mathbb{L}_0$ will be constructed.

Let $\mathscr{H}$ be the ℓ-variety defined by the identities

$$(x \wedge y^{-1}x^{-1}y) \vee e = e, \tag{1}$$

$$\begin{aligned} &\big|\big(|[x,y]|^2 \vee (|x| \vee |y|)|[x,y]|(|x| \vee |y|)^{-1}\big)|[x,y]|^{-2}\big| \\ &\wedge \big|\big(|[x,y]|^m \wedge (|x| \vee |y|)|[x,y]|(|x| \vee |y|)^{-1}\big)|[x,y]|^{-m}\big| = e \\ &\qquad (m \in \mathbb{N}; m \geqslant 3). \end{aligned} \tag{2}$$

Lemma 3. *Let β be a positive real number such that $0 < \beta < 1$. Then 1) $T_\beta \notin \mathscr{H}$, 2) $\mathscr{H} \nsupseteq \mathscr{U}_\beta = \operatorname{var}_\ell(T_\beta)$.*

Proof. Let $0 < \beta < 1$. Then there are $t, m \in \mathbb{N}$ such that $2 < \beta^{-t} < m$. We claim that the identities of the ℓ-variety $\mathscr{H}$ are not valid in T_β where $x = (\beta^{-t}, c)$, $y = (\beta^{-t}, 0)$, $c > 0$. Then $|[x,y]| = (1, c(\beta^{-t} - 1)) \neq e$ in view of $\beta^{-t} > 2$. Let $c(\beta^{-t} - 1) = d$. Thus, $|[x,y]|^2 = (1, 2d)$, $|x| \vee |y| = (\beta^t, -c\beta^t) \vee (\beta^t, 0) = (\beta^t, 0)$. Therefore, $(|x| \vee |y|)|[x,y]|(|x| \vee |y|)^{-1} = (\beta^t, 0)(1, d)(\beta^{-t}, 0) = (1, \beta^{-t}d)$. It is clear that $(1, 2d) < (1, \beta^{-t}d) < (1, md)$. Hence, $T_\beta \notin \mathscr{H}$ for any real number β, $0 < \beta < 1$, and $\mathscr{H} \nsupseteq \mathscr{U}_\beta = \operatorname{var}_\ell(T_\beta)$. □

Corollary 1. *Let β be a positive real number such that $0 < \beta < 1$. Then $\mathscr{H} \nsupseteq \mathscr{U}_\beta^m = \operatorname{var}_\ell(T_{\beta^m})$ for any positive integer m.*

The proof is similar to that of Lemma 3. □

Lemma 4. *Let β be a positive real number such that $\beta > 1$. Then $T_\beta \in \mathscr{H}$.*

Proof. Let $x, y \in T_\beta$. Then $x = (\beta^{t_1}, c)$, $y = (\beta^{t_2}, d)$ and $[x,y] = (1, c(\beta^{t_2} - 1) + d(1 - \beta^{t_1}))$. Let $c(\beta^{t_2} - 1) + d(1 - \beta^{t_1}) = a$. Then $|[x,y]| = (1, |a|)$, $|x| \vee |y| = (\beta^t, k)$ where $t > 0$ or $t = 0$, $k \geqslant 0$ and $(|x| \vee |y|)|[x,y]|(|x| \vee |y|)^{-1} = (\beta^t, k)(1, |a|)(\beta^{-t}, -k\beta^{-t}) = (1, |a|\beta^{-t})$.

Therefore,

$$|[x,y]|^2 \vee (|x| \vee |y|)|[x,y]|(|x| \vee |y|)^{-1} = (1, 2|a|) \vee (1, |a|\beta^{-t}) = (1, 2|a|).$$

Since $\beta > 1$, it follows that the identities of the ℓ-variety $\mathscr{H}$ are valid in T_β. □

Theorem 1. *The ℓ-variety $\mathscr{H}$ has no covers in the lattice $\mathbb{L}_0$.*

P r o o f. Assume, on the contrary, that there is an ℓ-variety $\overline{\mathscr{H}} \in \mathbb{L}_0$ which covers $\mathscr{H}$. Since $\overline{\mathscr{H}}$ is a representable ℓ-variety, there is a totally ordered group $G \in \overline{\mathscr{H}} \setminus \mathscr{H}$ such that the identities of the ℓ-variety $\mathscr{H}$ are not valid in it. Therefore, there are x_0, $y_0 \in G$ and a natural number $m, m \geqslant 3$ such that

$$\begin{aligned} (3) \qquad & (|x_0| \vee |y_0|)|[x_0, y_0]|(|x_0| \vee |y_0|)^{-1} > |[x_0, y_0]|^2, \\ & |[x_0, y_0]|^m > (|x_0| \vee |y_0|)|[x_0, y_0]|(|x_0| \vee |y_0|)^{-1}. \end{aligned}$$

This clearly yields $|[x_0, y_0]| \sim_a (|x_0| \vee |y_0|)|[x_0, y_0]|(|x_0| \vee |y_0|)^{-1}$. Thus, the jump $G_\alpha \prec \overline{G}_\alpha$ in the system of convex subgroups of G determined by the element $|[x_0, y_0]|$ is invariant under conjugation by $(|x_0| \vee |y_0|)^{-1}$ and $\overline{G}_\alpha / G_\alpha$ is isomorphic to a subgroup of the additive group of real numbers. By Hölder's Theorem [2, Theorem 3.2.1.] the automorphism of conjugation by $(|x_0| \vee |y_0|)$ is the multiplication by some positive real number β. From (3) we have $\beta < 1$.

Let $G_1 = \mathrm{gp}(\overline{G}_\alpha, (|x_0| \vee |y_0|))$ be the subgroup of G generated by $\overline{G}_\alpha$ and $(|x_0| \vee |y_0|)$. Then $\overline{G}_\alpha \lhd G_1$,

$$G_1/G_\alpha \lhd \overline{G}_\alpha/G_\alpha,$$

where $\overline{G}_\alpha/G_\alpha$ is a normal convex archimedean subgroup. By the Homomorphism Theorem we have

$$G_1/G_\alpha/\overline{G}_\alpha/G_\alpha \cong G_1/\overline{G}_\alpha \cong \overline{(|x_0| \vee |y_0|)},$$

where $\overline{|x_0| \vee |y_0|} = |x_0|\overline{G}_\alpha \vee |y_0|\overline{G}_\alpha$ and $\overline{(|x_0| \vee |y_0|)}$ denotes the infinite cyclic group generated by the element $\overline{|x_0| \vee |y_0|}$. From Lemma 1 it follows that $G_1/G_\alpha \cong T_\beta$ where $0 < \beta < 1$.

Hence, the ℓ-variety $\overline{\mathscr{H}}$ contains the ℓ-variety $\mathscr{U}_\beta = \mathrm{var}_\ell(T_\beta)$ for some positive real number β such that $\beta < 1$.

By Lemma 2 there is an ℓ-variety $\mathscr{U}_{\beta^m}$ such that $\mathscr{U}_\beta \supset \mathscr{U}_{\beta^m}$. By Lemma 3 and Corollary of Lemma 3, $\mathscr{U}_\beta \nsubseteq \mathscr{H}$, $\mathscr{U}_{\beta^m} \nsubseteq \mathscr{H}$. According to Lemma 3 and Corollary of Lemma 2, we have $T_\beta \notin \mathscr{H}$, $\mathscr{U}_{\beta^m}$. Therefore, $\overline{\mathscr{H}} \supseteq \mathscr{U}_\beta \vee \mathscr{H} \supset \mathscr{H}$ and $\overline{\mathscr{H}} \supseteq \mathscr{U}_{\beta^m} \vee \mathscr{H} \supset \mathscr{H}$. Since $\overline{\mathscr{H}}$ covers $\mathscr{H}$, it follows that $\overline{\mathscr{H}} = \mathscr{U}_\beta \vee \mathscr{H} = \mathscr{U}_{\beta^m} \vee \mathscr{H}$. By Proposition 9.1.1 from the book [2] we have $T_\beta \in \mathscr{U}_{\beta^m}$ or $T_\beta \in \mathscr{H}$. These inclusions contradict Lemma 3 and Corollary 1 of Lemma 3. □

M. Anderson, M. Darnel, T. Feil in their work [7] introduced (for some other purposes) the representable ℓ-variety $\mathscr{C}$ which is defined by the following identical inequalities:

$$(4) \qquad ([b, a] \vee e) \wedge b \ll b \vee a^{-1}ba, \quad \text{for all} \quad e \leqslant b \leqslant a.$$

Now we will prove that the ℓ-variety $\mathscr{C}$ has no covers in the lattice of representable ℓ-varieties $\mathbb{L}_0$.

Our proof starts with rewriting the system of identical inequalities (4) defining the ℓ-variety $\mathscr{C}$ in the standard form of identities

$$(5)\qquad \begin{aligned}(([|x|,|x|\vee|y|]\vee e)\wedge|x|)^n\wedge(|x|\vee(|x|\vee|y|)^{-1}|x|(|x|\vee|y|))\\ =(([|x|,|x|\vee|y|]\vee e)\wedge|x|)^n,\ n\in\mathbb{N}.\end{aligned}$$

Lemma 5. *Let β be any positive real number such that $\beta<1$. Then $T_\beta\in\mathscr{C}$.*

P r o o f. Let $y,x\in T_\beta$. Then $|x|\vee|y|=(\beta^{t_1},c)$, $|x|=(\beta^{t_2},d)$.

Case 1. Let $0<t_2\leqslant t_1$. Then

$$\begin{aligned}[|x|,|x|\vee|y|]&=(\beta^{-t_2},-d\beta^{-t_2})(\beta^{-t_1},-c\beta^{-t_1})(\beta^{t_2},d)(\beta^{t_1},c)\\ &=(1,d(\beta^{t_1}-1)+c(1-\beta^{t_2})).\end{aligned}$$

Let $d(\beta^{t_1}-1)+c(1-\beta^{t_2})=\bar c$. Then $[|x|,|x|\vee|y|]=(1,\bar c)$ and

$$\begin{aligned}([|x|,|x|\vee|y|]\vee e)\wedge|x|&=(1,\bar c\vee 0),\\ (|x|\vee|y|)^{-1}|x|(|x|\vee|y|)&=(\beta^{t_2},c(1-\beta^{t_2})+d\beta^{t_1}).\end{aligned}$$

Let $c(1-\beta^{t_2})+d\beta^{t_1}=\bar d$. Then $(|x|\vee|y|)^{-1}|y|(|x|\vee|y|)=(\beta^{t_2},\bar d)$ and

$$|x|\vee(|x|\vee|y|)^{-1}|x|(|x|\vee|y|)=(\beta^{t_2},d\vee\bar d).$$

Thus, $(1,\bar c\vee 0)\ll(\beta^{t_2},d\vee\bar d)$.

Case 2. Let now $0=t_2\leqslant t_1$. Calculations similar to the previous ones prove this case.

Thus, $T_\beta\in\mathscr{C}$ in view of $0<\beta<1$. □

Lemma 6. *Let β be a positive real number such that $\beta>1$. Then $T_\beta\notin\mathscr{C}$ and $\mathscr{C}\not\supseteq\mathscr{U}_\beta=\operatorname{var}_\ell(T_\beta)$.*

P r o o f. Let $\beta>1$. Then there are $t,n\in\mathbb{N}$, such that $2<\beta^t<n$. The direct verification shows that the identities (5) are violated in T_β. In fact, let $|x|=(1,d)$, $|x|\vee|y|=(\beta^t,c)$.

Then:

$$\begin{aligned}
[(1,d),(\beta^t,c)] &= (1,d(\beta^t-1)),\\
(1,d(\beta^t-1)) \vee (1,0) &= (1,d(\beta^t-1)),\\
(1,d(\beta^t-1)) \wedge (1,d) &= (1,d),\\
(\beta^t,c)^{-1}(1,d)(\beta^t,c) &= (1,d\beta^t),\\
(1,d\beta^t) \vee (1,d) &= (1,d\beta^t).
\end{aligned}$$

Since $\beta^t < n$, we have

$$(1,nd) \wedge (1,d\beta^t) = (1,d\beta^t), \qquad (1,nd) \neq (1,d\beta^t).$$

Therefore,

$$\begin{gathered}
(([(1,d),(\beta^t,c)] \vee e) \wedge (1,d))^n \wedge ((1,d) \vee (\beta^t,c)^{-1}(1,d)(\beta^t,c))\\
\neq (([(1,d),(\beta^t,c)] \vee e) \wedge (1,d))^n
\end{gathered}$$

and $T_\beta \notin \mathscr{C}$ for any positive real number such that $\beta > 1$. □

Corollary 1. *For any positive integer $m \geqslant 1$ and positive real number $\beta > 1$ we have $\mathscr{U}_{\beta^m} = \mathrm{var}_\ell(T_{\beta^m}) \nsubseteq \mathscr{C}$.*

The proof follows immediately from Lemma 6. □

Theorem 2. *The ℓ-variety $\mathscr{C}$ has no covers in the lattice of representable ℓ-varieties $\mathbb{L}_0$.*

Proof. Assume, on the contrary, that there is an ℓ-variety $\overline{\mathscr{C}} \in \mathbb{L}_0$ such that $\overline{\mathscr{C}}$ covers $\mathscr{C}$. Since $\overline{\mathscr{C}}$ is a representable ℓ-variety, there is a totally ordered group G such that $G \in \overline{\mathscr{C}} \setminus \mathscr{C}$. Thus, there are $x_0, y_0 \in G$ and a positive integer $n > 1$ such that

$$\begin{gathered}
(([|x_0|,|x_0| \vee |y_0|] \vee e) \wedge |x_0|)^n \wedge (|x_0| \vee (|x_0| \vee |y_0|)^{-1}|x_0|(|x_0| \vee |y_0|))\\
\neq (([|x_0|,|x_0| \vee |y_0|] \vee e) \wedge |x_0|)^n.
\end{gathered} \tag{6}$$

Since

$$([|x_0|,|x_0| \vee |y_0|] \vee e) \wedge |x_0| \leqslant |x_0|, (|x_0| \vee (|x_0| \vee |y_0|)^{-1}|x_0|(|x_0| \vee |y_0|) \geqslant |x_0|,$$

we have

$$([|x_0|,|x_0| \vee |y_0|] \vee e) \wedge |x_0| \ngeqslant |x_0| \vee (|x_0| \vee |y_0|)^{-1}|x_0|(|x_0| \vee |y_0|).$$

From this we deduce that

$$(([|x_0|,|x_0|\vee|y_0|]\vee e)\wedge|x_0|)\sim_a(x_0\vee(|x_0|\vee|y_0|)^{-1}|x_0|(|x_0|\vee|y_0|)).$$

Case 1. $[|x_0|,|x_0|\vee|y_0|]=e$. Since $e^n\leqslant|x_0|$, it follows that the inequality (6) is violated.

Case 2. $[|x_0|,|x_0|\vee|y_0|]<e$. Then the inequality (6) is violated, too.

Case 3. $[|x_0|,|x_0|\vee|y_0|]>e$. Then $(|x_0|\vee|y_0|)^{-1}|x_0|(|x_0|\vee|y_0|)>|x_0|$. If $(|x_0|\vee|y_0|)\sim_a|x_0|$, then $[|x_0|,|x_0|\vee|y_0|]\ll(|x_0|\vee|y_0|)\vee|x_0|\sim_a|x_0|$, and the inequality (6) is violated. This implies that $|x_0|\ll(|x_0|\vee|y_0|)$. If $|x_0|\ll(|x_0|\vee|y_0|)^{-1}|x_0|(|x_0|\vee|y_0|)$, then the inequality (6) is not valid. Hence, $|x_0|\sim_a(|x_0|\vee|y_0|)^{-1}|x_0|(|x_0|\vee|y_0|)$. Consequently, the jump $G_\alpha\prec\overline{G}_\alpha$ in the system of convex subgroups of G defined by the element $|x_0|$ is invariant under conjugation by $(|x_0|\vee|y_0|)$, and $\overline{G}_\alpha/G_\alpha$ is isomorphic to a subgroup of the additive group of real numbers. By Hölder's Theorem [2, Theorem 3.2.1.] the automorphism of conjugation by $(|x_0|\vee|y_0|)$ is the multiplication by some positive real number $\beta>0$. Hence, $|\bar{x}_0|=r$ and $|\bar{x}_0|^{(|x_0|\vee|y_0|)}=\beta r$. If $\beta=1$, then $|x_0|^{(|x_0|\vee|y_0|)}G_\alpha=|x_0|G_\alpha$ and $|x_0|^{-1}|x_0|^{(|x_0|\vee|y_0|)}G_\alpha=G_\alpha$. Then $[|x_0|,|x_0|\vee|y_0|]\ll|x_0|,|x_0|^{(|x_0|\vee|y_0|)}$. Since $([|x_0|,|x_0|\vee|y_0|]\vee e)\wedge|x_0|=[|x_0|,|x_0|\vee|y_0|]\wedge|x_0|=[|x_0|,|x_0|\vee|y_0|]\ll|x_0|\vee(|x_0|\vee|y_0|)^{-1}|x_0|(|x_0|\vee|y_0|)$, the inequality (6) is violated. Thus, $\beta\neq1$.

Now arguments similar to the proof of Theorem 1 show that $G_1/G_\alpha\cong T_\beta$. Since $|x_0|<(|x_0|\vee|y_0|)^{-1}|x_0|(|x_0|\vee|y_0|)$, we can conclude that $\beta>1$. Hence, the ℓ-variety $\overline{\mathscr{C}}$ contains the ℓ-variety $\mathscr{U}_\beta=\mathrm{var}_\ell(T_\beta)$ for some positive real number β such that $\beta>1$.

By Lemma 2 there exists an ℓ-variety $\mathscr{U}_{\beta^m}$ such that $\mathscr{U}_\beta\supset\mathscr{U}_{\beta^m}$. By Lemma 6 and Corollary of Lemma 6 we have $\mathscr{U}_\beta\not\subseteq\mathscr{C}$, $\mathscr{U}_{\beta^m}\not\subseteq\mathscr{C}$. According to Lemma 6 and Corollary of Lemma 2, we have $T_\beta\notin\mathscr{C}$, $\mathscr{U}_{\beta^m}$. Therefore, $\overline{\mathscr{C}}\supseteq\mathscr{U}_\beta\vee\mathscr{C}\supset\mathscr{C}$ and $\overline{\mathscr{C}}\supseteq\mathscr{U}_{\beta^m}\vee\mathscr{C}\supset\mathscr{C}$. Since $\overline{\mathscr{C}}$ covers $\mathscr{C}$, it follows that $\overline{\mathscr{C}}=\mathscr{U}_\beta\vee\mathscr{C}=\mathscr{U}_{\beta^m}\vee\mathscr{C}$. By Proposition 9.1.1 from the book [2] we have $T_\beta\in\mathscr{U}_{\beta^m}$ or $T_\beta\in\mathscr{C}$. These inclusions contradict Lemma 6 and Corollary 1 of Lemma 6. □

Lemmas 3, 6 imply that $\mathscr{H}\neq\mathscr{C}$.

Let $\mathscr{V}$ [1] be the ℓ-variety defined by the following infinite basis of identities:

$$(7)\qquad \begin{aligned}&(x\wedge y^{-1}x^{-1}y)\vee e=e,\\ &|(|[x,y]|^2\vee y^{-1}|[x,y]|y)|[x,y]|^{-2}|\wedge|(|[x,y]|^2\vee x^{-1}|[x,y]|x)|[x,y]|^{-2}|\\ &\wedge|((|x|\vee|y|)^{-1}|[x,y]|(|x|\vee|y|)\wedge|[x,y]|^n)|[x,y]|^{-n}|\\ &\wedge|((|x|\vee|y|)|[x,y]|(|x|\vee|y|)^{-1}\wedge|[x,y]|^m)|[x,y]|^{-m}|=e\\ &\qquad(m,n\in\mathbb{N};n,m\geqslant2).\end{aligned}$$

It is known [1] or [2] (Lemma 12.5.8) that $\mathscr{V}$ has no covers in the lattice $\mathbb{L}_0$ and $T_\beta \notin \mathscr{V}$ for any positive real number β, $\beta \neq 1$. Hence, $\mathscr{V} \neq \mathscr{H}$, $\mathscr{V} \neq \mathscr{C}$.

Let φ be the automorphism of order 2 of the lattice of ℓ-varieties $\mathbb{L}$ which is defined in [6].

Proposition 2. $\varphi(\mathscr{V}) = \mathscr{V}$.

Proof. In [6] the method of rewriting the basis of identities of any ℓ-variety $\mathscr{X}$ to the basis of identities of the ℓ-variety $\varphi(\mathscr{X})$ is described. Now the direct application of this method shows that the bases of the ℓ-varieties $\varphi(\mathscr{V})$ and V are the same. □

Now let us consider the ℓ-varieties $\varphi(\mathscr{C})$ and $\varphi(\mathscr{H})$. Since $\varphi(\mathscr{R}) = \mathscr{R}$, it is clear that these ℓ-varieties have no covers in the lattice of representable ℓ-varieties $\mathbb{L}_0$, and therefore, we have five possible different representable ℓ-varieties without covers in the lattice $\mathbb{L}_0$.

3. Properties of l-varieties $\mathscr{V}$, $\mathscr{C}$, $\mathscr{H}$, $\varphi(\mathscr{C})$, $\varphi(\mathscr{H})$

In this section we will prove that all these ℓ-varieties $\mathscr{V}, \mathscr{C}, \mathscr{H}, \varphi(\mathscr{C}), \varphi(\mathscr{H})$ are distinct and we will also establish some of its properties.

Proposition 3. *Let G_1, G_2 be totally ordered groups from the ℓ-variety $\mathscr{C}(\varphi(\mathscr{C}))$. Then the lexicographic product $G_1 \overleftarrow{\times} G_2$ is contained $\mathscr{C}(\varphi(\mathscr{C}))$.*

Proof. Let $G_1, G_2 \in \mathscr{C}$ and $b, a \in G_1 \overleftarrow{\times} G_2$ be such that $e \leqslant b \leqslant a$. Then $b = (b_1, b_2)$, $a = (a_1, a_2)$ for some $b_1, a_1 \in G_1$ and $b_2, a_2 \in G_2$. Thus, $[b, a] = ([b_1, a_1], [b_2, a_2])$.

We claim that the following inequalities are valid in $G_1 \overleftarrow{\times} G_2$:

$$(8) \qquad (([b_1, a_1], [b_2, a_2]) \vee e) \wedge (b_1, b_2) \ll (b_1, b_2) \vee (a_1^{-1} b_1 a_1, a_2^{-1} b_2 a_2).$$

Let $[b_2, a_2] \neq e$, then the validity of the system of identities (8) on the elements b, a is equivalent to the validity of (6) on the elements $b_2, a_2 \in G_2$. Since $G_2 \in \mathscr{C}$, it follows that the system (6) is true.

Let now $[b_2, a_2] = e$, then $b_2 = a_2^{-1} b_2 a_2$.

The group $G_1 \overleftarrow{\times} G_2$ is a totally ordered group under the lexicographic order. Therefore, if $b_2 > e$ in G_2, then $(b_1, b_2) > (g_1, e)$ in the group $G_1 \overleftarrow{\times} G_2$ for any element $g_1 \in G_1$. Thus

$$(([b_1, a_1], e) \vee e) \wedge (b_1, b_2) = ([b_1, a_1] \vee e, e) \wedge (b_1, b_2) = ([b_1, a_1] \vee e, e).$$

If $b_2 \neq e$, then the system of inequalities (8) has the following form:

$$([b_1, a_1] \vee e, e) \ll (b_1 \vee a_1^{-1} b_1 a_1, b_2). \tag{9}$$

The validity of (9) is evident.

If $b_2 = e$, the verification of (8) is reduced to its verification on the elements $b_1, a_1 \in G_1$. Since $G_1 \in \mathscr{C}$, it follows that the system (8) is true.

Therefore, the elements b, a satisfy the system of identities (5) of the ℓ-variety $\mathscr{C}$, and $G_1 \overleftarrow{\times} G_2 \in \mathscr{C}$.

Now let us assume that $G_1, G_2 \in \varphi(\mathscr{C})$. Then $G_1^R, G_2^R \in \varphi^2(\mathscr{C}) = \mathscr{C}$, and by the previous arguments $G_1^R \overleftarrow{\times} G_2^R \in \mathscr{C}$.

Direct verification shows that $(G_1 \overleftarrow{\times} G_2)^R = G_1^R \overleftarrow{\times} G_2^R$. From the above it follows that $(G_1 \overleftarrow{\times} G_2)^R \in \mathscr{C}$ and $(G_1 \overleftarrow{\times} G_2) \in \varphi(\mathscr{C})$. □

Theorem 3. *The ℓ-variety $\mathscr{V}$ is strictly contained in the ℓ-variety $\mathscr{H}$.*

P r o o f. Since $\mathscr{V}$ is a representable ℓ-variety, it suffices to show that any totally ordered group of the ℓ-variety $\mathscr{V}$ belongs to the ℓ-variety $\mathscr{H}$.

On the contrary, assume that there exists a totally ordered group $G \in \mathscr{V} \setminus \mathscr{H}$ such that the identities of the ℓ-variety $\mathscr{H}$ are not valid in it. Therefore, there are $x_0, y_0 \in G$ and a natural number m such that

$$\begin{aligned} &(|x_0| \vee |y_0|)|[x_0, y_0]|(|x_0| \vee |y_0|)^{-1} > |[x_0, y_0]|^2, \\ &|[x_0, y_0]|^m > (|x_0| \vee |y_0|)|[x_0, y_0]|(|x_0| \vee |y_0|)^{-1}. \end{aligned} \tag{10}$$

Hence, $|[x_0, y_0]| \sim_a (|x_0| \vee |y_0|)|[x_0, y_0]|(|x_0| \vee |y_0|)^{-1}$.

As in the proof of Theorem 1, this yields that $T_\beta \in \mathscr{V}$ for some positive real number $\beta < 1$, which is impossible by Lemma 12.5.7 form the book [2].

Consequently, $\mathscr{V} \subseteq \mathscr{H}$ and by Lemma 4, the ℓ-variety $\mathscr{V}$ is strictly contained in $\mathscr{H}$. □

Theorem 4. *All ℓ-varieties $\mathscr{V}$, $\mathscr{C}$, $\mathscr{H}$, $\varphi(\mathscr{C})$, $\varphi(\mathscr{H})$ are distinct.*

P r o o f. By Lemma 5, $T_\beta \in \mathscr{C}$ for any positive $\beta, \beta < 1$. Then Proposition 1 implies that $(T_\beta)^R \cong T_{\beta^{-1}} \in \varphi(\mathscr{C})$. Similarly, by Lemma 4, $T_\beta \in \mathscr{H}$ for any positive β, $1 < \beta$ and $(T_\beta)^R \cong T_{\beta^{-1}} \in \varphi(\mathscr{H})$. By Lemma 12.5.8 form the book [2] we obtain the inequalities $\mathscr{V} \neq \mathscr{C}, \varphi(\mathscr{C}), \mathscr{H}, \varphi(\mathscr{H})$.

From Lemma 3 it follows that $\mathscr{H} \neq \varphi(\mathscr{H})$ and Lemmas 5 and 6 imply $\mathscr{C} \neq \varphi(\mathscr{C})$. By the same argument $\mathscr{H} \neq \mathscr{C}$ and $\varphi(\mathscr{H}) \neq \varphi(\mathscr{C})$.

So we need only to prove the remaining cases $\varphi(\mathscr{H}) \neq \mathscr{C}$ and $\mathscr{H} \neq \varphi(\mathscr{C})$.

Let $T_3 \overleftarrow{\times} T_3$ be the lexicographic product of two totally ordered groups T_3. By Proposition 3, $T_3 \overleftarrow{\times} T_3 \in \varphi(\mathscr{C})$. Direct verification shows that the identity

$$|(|[x,y]|^2 \vee (|x| \vee |y|)|[x,y]|(|x| \vee |y|)^{-1})|[x,y]|^{-2}| \\ \wedge |(|[x,y]|^5 \wedge (|x| \vee |y|)|[x,y]|(|x| \vee |y|)^{-1})|[x,y]|^{-5}| = e$$

is violated in $T_3 \overleftarrow{\times} T_3$ on $x = ((1,4),(1,0))$, $y = ((\frac{1}{3},4),(3,0))$.

Thus, $T_3 \overleftarrow{\times} T_3 \in \varphi(\mathscr{C}) \setminus \mathscr{H}$ and $\varphi(\mathscr{C}) \neq \mathscr{H}$. Since φ is an automorphism of the lattice of ℓ-varieties $\mathbb{L}$, it follows that $\varphi(\mathscr{H}) \neq \mathscr{C}$. □

It is worth pointing out that the ℓ-variety $\mathscr{V}$ is strictly contained in the ℓ-variety $\mathscr{C}$. This fact is proved in [8].

Theorem 5. $\mathscr{V} = \mathscr{C} \wedge \mathscr{H} = \mathscr{C} \wedge \varphi(\mathscr{C}) = \mathscr{H} \wedge \varphi(\mathscr{H}) = \varphi(\mathscr{C}) \wedge \varphi(\mathscr{H})$.

P r o o f. We first prove that $(\mathscr{C} \wedge \mathscr{H}) \subseteq \mathscr{V}$. Assume, on the contrary, that there is a totally ordered group $G \in (\mathscr{C} \wedge \mathscr{H}) \setminus \mathscr{V}$. Thus, there are $x_0, y_0 \in G$ and natural numbers m, n such that

1) $|[x_0,y_0]|^2 < y_0^{-1}|[x_0,y_0]|y_0$;
2) $|[x_0,y_0]|^2 < x_0^{-1}|[x_0,y_0]|x_0$;
3) $|[x_0,y_0]|^n > (|x_0| \vee |y_0|)^{-1}|[x_0,y_0]|(|x_0| \vee |y_0|)$;
4) $|[x_0,y_0]|^m > (|x_0| \vee |y_0|)|[x_0,y_0]|(|x_0| \vee |y_0|)^{-1}$.

Let $|x_0| < |y_0|$. Then 3) and 4) can be rewritten in the form

3.1) $|y_0|^{-1}|[x_0,y_0]||y_0| < |[x_0,y_0]|^n$,
4.1) $|y_0||[x_0,y_0]||y_0|^{-1} < |[x_0,y_0]|^m$.

Hence,

$$|[x_0,y_0]| < |y_0|^{-1}|[x_0,y_0]|^m|y_0| = (|y_0|^{-1}|[x_0,y_0]||y_0|)^m < |[x_0,y_0]|^{mn}.$$

Therefore, the elements $|[x_0,y_0]|$ and $|y_0|^{-1}|[x_0,y_0]||y_0|$ are archimedean equivalent. Consider the jump $G_\alpha \prec \overline{G}_\alpha$ in the system of convex subgroups of G defined by the element $|[x_0,y_0]|$. As in the proof of Theorem 1, it yields that $T_\beta \in (\mathscr{C} \wedge \mathscr{H})$ for some positive β, $\beta \neq 1$. This fact contradicts Lemmas 3, 6. Thus, $(\mathscr{C} \wedge \mathscr{H}) \subseteq \mathscr{V}$. The converse statement is obvious.

The other equalities are proved similarly. □

Theorem 6. *The ℓ-varieties $\mathscr{V}$, $\mathscr{C}$, $\mathscr{H}$, $\varphi(\mathscr{C})$, $\varphi(\mathscr{H})$ have the following properties: first, they have no independent basis of identities, and second, they contain all representable covers of the abelian ℓ-variety.*

P r o o f. The first property follows from Proposition 12.7.1 [2]. The second follows immediately from the distributivity of the lattice of ℓ-varieties $\mathbb{L}$ and from the non-existence of covers in the lattice of representable ℓ-varieties $\mathbb{L}_0$ of all these ℓ-varieties. $\square$

Remark. Theorem 1 was proved by the first author, Theorems 2, 3 by the second and all other results were obtained in common discussions.

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