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A NEW PROOF OF A CHARACTERIZATION OF THE SET OF ALL GEODESICS IN A CONNECTED GRAPH

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In [2], the present author gave a characterization of the set of all geodesics (or shortest paths) in a connected graph G. More precisely, he gave a necessary and sufficient condition for a set of paths in G to be the set of all geodesics in G. The proof of necessity is easy and was omitted in [2]. But the proof of sufficiency given there was rather long.

This characterization was partially modified in [3]; the proof given there was also long (in fact, the characterization was derived form a more general theorem proved there). In the present paper we present its new and shorter proof. The proof utilizes a new lemma, which yields a deeper insight into the idea of the characterization.

Let G be a (finite undirected) connected graph (without loops and multiple edges), and let V, E and D denote its vertex set, its edge set and its diameter, respectively. If u, $v \in V$, then $d(u,v)$ denotes the distance between u and v in G. (The letters $g, h, \ldots, n$ will be used to denote integers).

We denote by Σ_N the set of all sequences

$$u_0, \ldots, u_g \tag{0}$$

where $u_0, \ldots, u_g \in V$ and $g \geqslant 0$. Similarly as in [2] and [3], instead of (0) we will write $u_0 \ldots u_g$. Let $\alpha = v_0 \ldots v_h$, where $v_0, \ldots, v_h \in V$ and $h \geqslant 0$. We write $A\alpha = v_0$, $Z\alpha = v_h$, $\|\alpha\| = h$ and

$$\overline{\alpha} = v_h \ldots v_0.$$

Let $\beta = x_0 \ldots x_i$ and $\gamma = y_0 \ldots y_j$, where $x_0, \ldots, x_i$, $y_0, \ldots, y_j \in V$, $i \geqslant 0$ and $j \geqslant 0$; we write

$$\beta\gamma = x_0 \ldots x_i y_0 \ldots y_j.$$

We denote by $*$ the empty sequence in the sense that $*\delta = \delta = \delta*$ for each $\delta \in \Sigma_N$. Put $** = *$ and $\overline{*} = *$. Define $\Sigma = \Sigma_N \cup \{*\}$.

Let $\pi \in \Sigma_N$. We say that π is a path in G if there exist $k \geqslant 0$ and mutually distinct $w_0, \dots, w_k \in V$ such that $\pi = w_0 \dots w_k$ and if $k \geqslant 1$, then

$$\{w_0, w_1\}, \dots, \{w_{k-1}, w_k\} \in E.$$

The set al all paths in G will be denoted by $\mathcal{P}$. If $\mathcal{Q} \subseteq \mathcal{P}$ and $m \geqslant 0$, then we define

$$\mathcal{Q}(m) = \bigl\{\omega \in \mathcal{Q};\ d(A\omega, Z\omega) = m\bigr\}.$$

Obviously, if $\pi \in \mathcal{P}$, then $d(A\pi, Z\pi) \leqslant \|\pi\|$.

Let $\tau \in \Sigma_N$. We say that τ is a *geodesic* (or a shortest path) in G if $\tau \in \mathcal{P}$ and $d(A\tau, Z\tau) = \|\tau\|$.

The following theorem gives a characterization of the set of all geodesics in G.

Theorem ([3]). *Let $\mathcal{R} \subseteq \mathcal{P}$, and let Γ denote the set of all geodesics in G. Then the statements* (1) *and* (2) *are equivalent:*

(1) $\mathcal{R} = \Gamma$.

(2) *$\mathcal{R}$ satisfies the following properties* $\mathbf{A}(\mathcal{R}) - \mathbf{G}(\mathcal{R})$ *(for all u, v, x, $y \in V$ and all φ, $\psi \in \Sigma$):*

$\mathbf{A}(\mathcal{R})$ *if $uv\varphi x \in \mathcal{R}$, then $\{u, x\} \notin E$;*

$\mathbf{B}(\mathcal{R})$ *if $uv\varphi x \in \mathcal{R}$, then $x\overline{\varphi}vu \in \mathcal{R}$;*

$\mathbf{C}(\mathcal{R})$ *if $uv\varphi x \in \mathcal{R}$, then $v\varphi x \in \mathcal{R}$;*

$\mathbf{D}(\mathcal{R})$ *if $uv\varphi x$, $v\psi x \in \mathcal{R}$, then $uv\psi x \in \mathcal{R}$;*

$\mathbf{E}(\mathcal{R})$ *if $uv\varphi x$, $vu\psi y \in \mathcal{R}$ and $\{x, y\} \in E$, then $v\varphi xy \in \mathcal{R}$;*

$\mathbf{F}(\mathcal{R})$ *if $uv\varphi x \in \mathcal{R}$, $\{x, y\} \in E$, $uv\varrho y \notin \mathcal{R}$ for all $\varrho \in \Sigma$ and $u\sigma yx \notin \mathcal{R}$ for all $\sigma \in \Sigma$, then $v\varphi xy \in \mathcal{R}$;*

$\mathbf{G}(\mathcal{R})$ *there exists $\xi \in \mathcal{R}$ such that $A\xi = u$ and $Z\xi = x$.*

We will present a new proof of the theorem. The proof that (1) implies (2) is not complicated and will be omitted here. We only prove that (2) implies (1).

The next lemma yields a deeper insight into the theorem and suggest a new method for proving it.

Lemma. *Let $u_0, u_1, \dots, u_{g+h-1} \in V$, where $\min(g, h) \geqslant 2$. Denote $j = \min(g, h)$ and $u_{g+h} = u_0$, $u_{g+h+1} = u_1$, …, $u_{g+h+j} = u_j$. Moreover, denote*

$$\alpha_i = u_i u_{i+1} \dots u_{i+g} \text{ and } \beta_i = u_{i+g} u_{i+g+1} \dots u_{i+g+h}$$

for each i, $0 \leqslant i \leqslant j$.

Let $\mathcal{Q}$, $\mathcal{T} \subseteq \mathcal{P}$. Assume that $\mathcal{Q}$ satisfies $\mathbf{B}(\mathcal{Q}) - \mathbf{F}(\mathcal{Q})$ *and $\mathcal{T}$ satisfies* $\mathbf{B}(\mathcal{T}) - \mathbf{E}(\mathcal{T})$. *Next, assume that the following conditions* I–IV *hold for all i, $0 \leqslant i \leqslant j$, and all φ, $\psi \in \Sigma$:*

I *if $\alpha_i \in \mathcal{Q}$ and $\beta_i \notin \mathcal{T}$, then $\alpha_i \in \mathcal{T}$;*

II *if $u_i u_{i+1} \varphi u_{i+g+1} \in \mathcal{Q}$, $\beta_i \in \mathcal{T}$ and $\alpha_{i+1} \notin \mathcal{Q}$, then $u_i u_{i+1} \varphi u_{i+g+1} \in \mathcal{T}$;*

III *if $u_i u_{i+1} \varphi u_{i+g} \in \mathcal{T}$ and $u_i u_{i+1} \psi u_{i+g} \in \mathcal{Q}$, then $u_{i+1} \varphi u_{i+g} \in \mathcal{Q}$;*

IV *if $u_{i+g} u_{i+g+1} \varphi u_{i+g+h} \in \mathcal{T}$ and $u_{i+g} u_{i+g+1} \psi u_{i+g+h} \in \mathcal{Q}$, then $u_{i+g+1} \varphi u_{i+g+h} \in \mathcal{Q}$.*

Finally, assume that $\alpha_0 \in \mathcal{Q}$ and $\beta_0 \in \mathcal{T}$. Then $\beta_0 \in \mathcal{Q}$.

P r o o f. Suppose, to the contrary, that $\beta_0 \notin \mathcal{Q}$. Then $\beta_0 \in \mathcal{T} - \mathcal{Q}$. First, we will show that

$$\text{either } \alpha_j \notin \mathcal{Q} \text{ or } \beta_j \notin \mathcal{T}. \tag{3}$$

If $g = h$, then (3) immediately follows from the fact that $\beta_0 \notin \mathcal{Q}$. Next, let $g > h$. Then $j = h$. Suppose that $\alpha_j \in \mathcal{Q}$. Applying $\mathbf{B}(\mathcal{Q})$ to α_0 and $\mathbf{C}(\mathcal{Q})$ to α_j, we get

$$u_g u_{g-1} \ldots u_0,\ u_{g-1} u_g \ldots u_{g+h} \in \mathcal{Q}$$

Recall that $u_{g+h} = u_0$. By $\mathbf{D}(\mathcal{Q})$,

$$u_g u_{g-1} u_g \ldots u_{g+h} \in \mathcal{Q}.$$

Thus $\mathcal{Q} - \mathcal{P} \notin \emptyset$, a contradiction. We get $\alpha_j \notin \mathcal{Q}$ and therefore (3) holds. Finally, let $h > g$. Then $j = g$. In a similar way, we get $\beta_j \notin \mathcal{T}$. Thus (3) holds again.

Recall that $\alpha_0 \in \mathcal{Q}$ and $\beta_0 \in \mathcal{T} - \mathcal{Q}$. Combining this fact with (3), we see that there exists k, $0 \leqslant k < j$ such that

$$\alpha_k \in \mathcal{Q} \text{ and } \beta_k \in \mathcal{T} - \mathcal{Q} \tag{4}$$

and

$$\alpha_{k+1} \notin \mathcal{Q} \text{ or } \beta_{k+1} \notin \mathcal{T} - \mathcal{Q}. \tag{5}$$

Denote $\alpha = \alpha_k$, $\alpha' = \alpha_{k+1}$, $\beta = \beta_k$, $\beta' = \beta_{k+1}$. Clearly, α, $\beta \in \mathcal{P}$. We distinguish two cases:

1. Let $\alpha' \in \mathcal{Q}$. If $\alpha' \notin \mathcal{T}$, then I implies that $\beta' \in \mathcal{T}$. If $\alpha' \in \mathcal{T}$, then combining $\mathbf{B}(\mathcal{T})$ and $\mathbf{E}(\mathcal{T})$, we get $\beta' \in \mathcal{T}$ again. By (5), $\beta' \notin \mathcal{T} - \mathcal{Q}$. Hence $\beta' \in \mathcal{Q}$. By (4), $\alpha \in \mathcal{Q}$. Combining $\mathbf{B}(\mathcal{Q})$ with $\mathbf{E}(\mathcal{Q})$, we get $\beta \in \mathcal{Q}$, which contradicts (4).

2. Let $\alpha' \notin \mathcal{Q}$. Denote $u = u_k$, $v = u_{k+1}$, $x = u_{k+g}$ and $y = u_{k+g+1}$. Clearly, there exist φ, $\tau \in \Sigma$ such that $\alpha = uv\varphi x$ and $\beta = xy\tau u$. Hence $\alpha' = v\varphi xy$. We distinguish two subcases:

2.1. Let $u\sigma yx \notin \mathcal{Q}$ for all $\sigma \in \Sigma$. By virtue of $\mathbf{F}(\mathcal{Q})$, there exists $\varrho \in \Sigma$ such that $uv\varrho y \in \mathcal{Q}$. Since $\beta \in \mathcal{T}$ and $\alpha' \notin \mathcal{Q}$, II implies that $uv\varrho y \in \mathcal{T}$. By $\mathbf{B}(\mathcal{T})$, $y\overline{\varrho}vu \in \mathcal{T}$. Recall that $xy\tau u \in \mathcal{T}$. By $\mathbf{D}(\mathcal{T})$, $xy\overline{\varrho}vu \in \mathcal{T}$ and by $\mathbf{B}(\mathcal{T})$, $uv\varrho yx \in \mathcal{T}$. Since $uv\varphi x \in \mathcal{Q}$, III implies that $v\varrho yx \in \mathcal{Q}$. By $\mathbf{D}(\mathcal{Q})$, $uv\varrho yx \in \mathcal{Q}$ and by $\mathbf{B}(\mathcal{Q})$, $xy\overline{\varrho}vu \in \mathcal{Q}$. Since $xy\tau u \in \mathcal{T}$, IV implies that $y\tau u \in \mathcal{Q}$. By $\mathbf{D}(\mathcal{Q})$, $\beta = xy\tau u \in \mathcal{Q}$, which contradicts (4).

2.2. Let there exist $\sigma \in \Sigma$ such that $u\sigma yx \in \mathcal{Q}$. By $\mathbf{B}(\mathcal{Q})$, $xy\overline{\sigma}u \in \mathcal{Q}$. Since $\beta \in \mathcal{T}$, IV implies that $y\tau u \in \mathcal{Q}$. By $\mathbf{D}(\mathcal{Q})$, $\beta \in \mathcal{Q}$, which contradicts (4) again.

Thus $\beta_0 \in \mathcal{Q}$, which completes the proof of the lemma. $\square$

Proof of the theorem. We will only prove that (2) implies (1). Now, let (2) hold. We will prove that $\Gamma(n) \subseteq \mathcal{R}(n)$ and $\mathcal{R}(n) \subseteq \Gamma(n)$ for every $n \geqslant 0$. We proceed by induction on n. The fact that $\Gamma(0) = \mathcal{R}(0)$ follows from $\mathbf{G}(\mathcal{R})$. The fact that $\Gamma(1) = \mathcal{R}(1)$ follows from $\mathbf{G}(\mathcal{R})$ and $\mathbf{A}(\mathcal{R})$. Let $n \geqslant 2$. Assume that

$$\Gamma(m) = \mathcal{R}(m) \text{ for each } m, 0 \leqslant m < n. \tag{6}$$

The case when $D < n$ is trivial. Suppose that $D \geqslant n$.

Consider an arbitrary $\omega \in \Gamma(n)$. Put $\mathcal{Q} = \mathcal{R}$, $\mathcal{T} = \Gamma$ and $h = n$. Obviously, $\mathbf{G}(\mathcal{Q})$ holds. There exist $u_0, \ldots, u_{g+h-1} \in V$, where $g \geqslant h$, such that

$$u_0 u_1 \ldots u_g \in \mathcal{Q} \text{ and } \omega = u_g u_{g+1} \ldots u_{g+h}, \text{ where } u_{g+h} = u_0. \tag{7}$$

By virtue of (6), I–IV hold. According to the lemma, $\omega \in \mathcal{R}$. We have proved that $\Gamma(n) \subseteq \mathcal{R}(n)$.

Consider an arbitrary $\omega \in \mathcal{R}(n)$. Put $\mathcal{Q} = \Gamma$, $\mathcal{T} = \mathcal{R}$ and $g = n$. There exist $u_0, \ldots, u_{g+h-1} \in V$, where $h \geqslant g$, such that (7) holds. Combining (6) with the fact that $\Gamma(n) \subseteq \mathcal{R}(n)$, we see that I–IV hold. According to the lemma, $\omega \in \Gamma$. We have proved that $\mathcal{R}(n) \subseteq \Gamma(n)$, which completes the proof of the theorem. $\square$

Remark 1. The fact that V is finite was not utilized in any point of our proof.

Remark 2. A different way of characterizing the set of all geodesics in a connected graph can be found in [4].

Remark 3. Some types of graphs can be characterized by counting geodesics. For this topis, see [1].

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