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Czechoslovak Mathematical Journal, Vol. 48 (1998), No. 3, 465–472

Persistent URL: http://dml.cz/dmlcz/127433

Czechoslovak Mathematical Journal, 48 (123) (1998), 465–472

EXTENSION OF VECTOR MEASURES

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(Received November 5, 1995)

1. Introduction

A fundamental problem in measure theory is that of finding conditions under which a countably additive vector measure μ on a ring R can be extended to a countably additive measure on a wider class of sets containing R.

The first result states that every closed vector measure μ on a ring R with values in a complete locally convex space X has a unique extension on the algebra $\mathcal{F}$ of locally measurable sets which contains the ring R.

The second result states that a locally bounded vector measure μ on a ring R with values in a weakly complete locally convex space has a (unique) extension on the δ-ring $\mathcal{F}(R)$ generated by R.

Definitions and notation. In all what follows S denotes a nonempty set, R a ring of subsets of S, X a locally convex Hausdorff space, its topology τ being given by a family Γ of seminorms on X in the sense that the family $\{x\colon\ q(x) < \varepsilon\}$, for every $\varepsilon > 0$ and every $q \in \Gamma$, is a sub-base of neighborhoods of zero in Γ. The family of all continuous seminorms can be taken for Γ. Let $\mathcal{L}$ be a class of subsets of S, μ a map from $\mathcal{L}$ into X. We then define: μ is s-bounded, if and only if for every sequence $\{A_n\}$ of mutually disjoint sets from $\mathcal{L}$, we have $\lim\limits_{n\to\infty} \mu(A_n) = 0$.

An X-valued map μ on R is called finitely additive if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever A, B are disjoint sets in R. The map μ is called σ-additive (or countably additive if $\mu(\bigcup\limits_{n=1}^{\infty} A_n) = \sum\limits_{n=1}^{\infty} \mu(A_n)$, whenever $A_1, A_2, \ldots$ are mutually disjoint sets from R such that $\bigcup\limits_{n=1}^{\infty} A_n \in R$.

Let μ be an X-valued finitely additive set function on R and let q be a seminorm defined on X, then the q-variation $q(\mu)$ is defined by

$$q(\mu)(A) = \sup\left\{q\left(\sum_{j=1}^{n} a_j\mu(A_j)\right)\right\}, \quad A \in R$$

where the supremum is taken over all disjoint sets $A_1, \ldots, A_n$ from R such that $A = A_1 \cup \ldots \cup A_n$ and all scalars $a_1, \ldots, a_n$ with $|a_j| \leqslant 1$ for every $j = 1, 2, \ldots, n$.

A locally convex space X is said to be (sequentially) complete if every (ordinary Cauchy sequence) generalized Cauchy sequence is convergent.

Let R be a ring of subsets of a set S. We define an order $A_1 \leqslant A_2$ iff $A_1 \subset A_2$, $A_1, A_2 \in R$. Then R is a directed set with the order "$\leqslant$". A set function $\mu\colon\ R \to X$, where X is a complete locally convex Hausdorff space, is called closed if the image set $\{\mu(A)\colon\ A \in R\}$ of the directed set R converges in X.

The main result of this section is Theorem 1.2 which is a generalization of ([12], Theorem 1).

Lemma 1.1. *Let $\mu\colon\ R \to X$ be a vector measure. Then the following are equivalent:*

(i) *μ is closed;*

(ii) *for every neighborhood U of zero in X there exists $A_0 \in R$ such that, for every $A \in R$ with $A \subset S - A_0$, we have $\mu(A) \in U$.*

P r o o f. ii) $\Rightarrow$ i). $\{\mu(A)\colon\ A \in R\}$ is a Cauchy net in X ([10], Proposition 2).

i) $\Rightarrow$ ii). Let V be an absolutely convex neighborhood of zero in X such that $V + V \subset U$. We set $x_1 = \lim\{\mu(A)\colon\ A \in R\}$. Then x_1 belongs to X. There exists $A_0 \in R$ such that $\mu(A) - x_1 \in V$ for every $A \in R$ with $A_0 \subseteq A$. For every set $A \in R$ with $A \subset S - A_0$ we have $A_0 \subset A \cup A_0$ and therefore $\mu(A \cup A_0) - x_1 \in V$. Then from the relation $\mu(A \cup A_0) - x_1 = \mu(A) + \mu(A_0) - x_1 \in V$ we have

$$\begin{aligned}\mu(A) &= \mu(A \cup A_0) - \mu(A_0) = \mu(A \cup A_0) - x_1 + x_1 - \mu(A_0)\\ &= (\mu(A \cup A_0) - x_1) - (\mu(A_0) - x_1) \in V - V = V + V \subset U.\end{aligned}$$

We put $\mathcal{F} = \{A \subset S$ such that for every set $E \in R$ we have $E \cap A \in R\}$ the locally measurable sets. Then $\mathcal{F}$ is an algebra containing R. If $S \in R$ then we have $\mathcal{F} = R$. $\square$

Theorem 1.2. *Let R be a ring of subsets of S with $S \notin R$, $\mathcal{F}$, the locally measurable sets, X a complete locally convex space and $\mu\colon\ R \to X$ a countably additive set function. If μ is closed, then μ can be extended to a countably additive set function $\widehat{\mu}\colon\ \mathcal{F} \to X$.*

P r o o f. If $A \in \mathcal{F}$ then clearly the set $\{\mu(E \cap A)\colon\ E \in R\}$ is a Cauchy net in X. We define $\widehat{\mu}(A) = \lim\{\mu(E \cap A)\colon\ E \in R\}$. Then $\widehat{\mu}(A) \in X$ and $\widehat{\mu}$ is a finite additive set function. Let $\{A_n\}$ be a sequence from $\mathcal{F}$ with $A_n \cap A_m = \emptyset$ for every $n \neq m$ such that $A = \bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{F}$. Then for every neighborhood U of zero in X there exists a set $E \in R$ such that $\widehat{\mu}(A) - \mu(A \cap E) \in U$ and $\mu(B) \in U$ for every set $B \in R$ with $B \subset S - E$. Since $A \cap E = \bigcup\limits_{n=1}^{\infty} A_n \cap E \in R$ we have $\mu(A \cap E) = \sum\limits_{n=1}^{\infty} \mu(A_n \cap E)$. Then there exists a positive integer n_0 such that

$$\mu(A \cap E) - \sum_{k=1}^{n_0} \mu(A_k \cap E) \in U.$$

For each positive integer k such that $1 \leqslant k \leqslant n_0$ there exists a set $E_k \in R$ with $E \subset E_k$ and $\widehat{\mu}(A_k) - \mu(A_k \cap E_k) \in \frac{1}{n_0} U$. Further,

$$\begin{aligned} \sum_{k=1}^{n_0} (\mu(A_k \cap E_k) - \mu(A_k \cap E)) &= \sum_{k=1}^{n_0} \mu(A_k \cap (E_k - E)) \\ &= \mu\Big(\bigcup_{k=1}^{n_0} A_k \cap (E_k - E)\Big) \end{aligned}$$

and

$$\bigcup_{k=1}^{n_0} A_k \cap (E_k - E) \subset S - E$$

and therefore we have

$$\sum_{k=1}^{n_0} \mu(A_k \cap E_k) - \mu(A_k \cap E) \in U.$$

Then we have

$$\begin{aligned} \widehat{\mu}(A) - \sum_{k=1}^{n_0} \widehat{\mu}(A_k) = (\widehat{\mu}(A) - \mu(A \cap E)) &+ (\mu(A \cap E) - \sum_{k=1}^{n_0} \mu(A_k \cap E) \\ &+ \Big(\sum_{k=1}^{n_0} \mu(A_k \cap E) - \mu(A_k \cap E_k)\Big) \\ &+ \sum_{k=1}^{n_0} \big(\mu(A_k \cap E_k) - \widehat{\mu}(A_k)\big) \in U + U + U + U = 4U. \end{aligned}$$

$\square$

2.

Let Q be a field of subsets of a set S and $\sigma(Q)$ the σ-field generated by Q. If X is a sequentially complete locally convex Hausdorff space and $\mu\colon\ Q \to X$ is a vector measure and $q \in \Gamma$, then for every set $A \in S$ we put

$$\mu_q(A) = \sup\{q(\mu(B))\colon\ B \subseteq A,\ B \in Q\}.$$

Clearly μ_q is monotone, subadditive and $0 \leqslant \mu_q(A) \leqslant +\infty$ for every $A \in Q$.

In this section the main results are Theorems 2.3 and 2.4 which give conditions under which a vector measure μ from a ring R into a sequentially (weakly sequentially) complete locally convex space X can be extended to the δ-ring $\mathcal{F}(R)$.

Proposition 2.1. *Let X be a sequentially complete locally convex Hausdorff space and let $\mu\colon\ Q \to X$ be a countably additive vector measure. The following statements are equivalent:*

(i) *μ has a (necessarily unique) countably additive extension $\widehat{\mu}\colon\ \sigma(Q) \to X$,*

(ii) *μ is s-bounded,*

(iii) *$q(\mu)$ is s-bounded for every $q \in \Gamma$,*

(iv) *μ_q is s-bounded for every $q \in \Gamma$,*

(v) *for every sequence (E_n) of mutually disjoint sets on Q, the series $\sum\limits_{n=1}^{\infty} \mu(E_n)$ converges unconditionally,*

(vi) *for every $p \in \Gamma$ there exists a measure $\lambda_p\colon\ Q \to [0, +\infty)$ such that*

$$\lim_{\lambda_p(A)\to 0} q(\mu(A)) = 0, \quad A \in Q.$$

P r o o f. (iii) $\Leftrightarrow$ (ii) ([7], Proposition 4.1).

(iii) $\Rightarrow$ (iv). From ([8], Lemma II.2) we have

$$\mu_q(A) \leqslant q(\mu)(A) \leqslant 2\mu_q(A),\ A \in Q.$$

For every disjoint sequence $(A_n)_n$, $A_n \in Q$ we have $q(\mu)(A_n) \to 0$ and therefore by (1), $\mu_q(A_n) \to 0$.

(iv) $\Rightarrow$ (iii). It is obvious from (1).

(i) $\Rightarrow$ ii). It is obvious.

ii) $\Rightarrow$ i). Since μ is s-bounded iff $q(\mu)$ is s-bounded, therefore by ([7], Proposition 4.1) for every $q \in \Gamma$ there exists a bounded measure $\lambda_q\colon\ Q \Rightarrow [0, +\infty)$ such that $q(\mu) \ll \lambda_q$. Since $q(\mu(E)) \leqslant q(\mu)(E)$ ([9]) we have that

$$\lim_{\lambda_q(A)\to 0} q(\mu(A)) = 0.$$

By Halmos ([6], §1 Theorem A) λ_q has a unique extension $\widehat{\lambda}_q\colon\ \sigma(Q) \to [0, +\infty)$; we put $d(E_1, E_2) = \widehat{\lambda}_q(E_1 \Delta E_2)$, $E_1, E_2 \in Q$ and consider on $\sigma(Q)$ the uniform structure τ defined by the semi distance d. By Halmos ({19}, theorem D) $Q \subset \sigma(Q)$ is dense in $\sigma(Q)$ for the topology induced by τ.

Since

$$\lim_{\widehat{\lambda}_q(A)\to 0} q(\mu(A)) = \lim_{\lambda_q(A)\to 0} q(\mu(A)) = 0,\ \ A \in Q$$

by ([2], Theorem 7), μ can be extended to a vector measure $\widehat{\mu}\colon\ \sigma(Q) \to X$ such that

$$\lim_{\widehat{\lambda}_q(A)\to 0} q(\mu(A)) = 0,\ \ A \in \sigma(Q).$$

The uniqueness of $\widehat{\mu}$ is immediate by Dinculeanu ([1], Proposition 6).

(v) $\Rightarrow$ (ii). It is obvious.

(ii) $\Rightarrow$ (v). By (ii) $\Rightarrow$ i)) there exists a unique extension $\widehat{\mu}\colon\ \sigma(Q) \to X$. For every disjoint sequence (E_n), $E_n \in Q$ we have $\widehat{\mu}(\bigcup\limits_{n=1}^{\infty} E_n) = \sum\limits_{n=1}^{\infty} \mu(E_n)$ and so $\sum\limits_{n=1}^{\infty} \mu(E_n)$ converges unconditionally.

(i) $\Leftrightarrow$ (vi). ([2], Corollary 1). $\square$

Let R be a ring of subsets of S. Then there exists the smallest δ-ring $\mathcal{F}(R)$ containing R ([1], §1 Proposition 6).

Proposition 2.2. *Let X be a sequentially complete locally convex space and $\mu\colon\ R \to X$ a vector measure. The following statements are equivalent:*

(i) *μ has a (unique) extension $\widehat{\mu}\colon\ \mathcal{F}(R) \to X$.*

(ii) *For every $q \in \Gamma$, $E \in R$, there exists a measure $\lambda_{q,E}\colon\ R \to [0, +\infty)$ such that*

$$\lim_{\lambda_{q,E}(A)\to 0} q(\mu(A)) = 0, \quad A \subset E$$

([2], Theorem 2, Corollary 2).

(iii) *For every set $E \in R$ and every neighborhood $U \in \mathcal{U}$ there exists a positive integer k such that, for every finite sequence (A_i), $1 \leqslant i \leqslant k$ of mutually disjoint sets of R with $\bigcup\limits_{i=1}^{k} A_i \subset E$ there exists a positive integer i_0 $(1 \leqslant i_0 \leqslant k)$ such that $\mu(A_{i_0}) \in U$* ([13], Theorem 1).

(iv) *For every set $E \in R$ and every sequence (E_n) of mutually disjoint sets of R with $E_n \subset E$ $(n = 1, 2, \ldots)$ we have*

$$\lim_{n} \mu(E_n) = 0.$$

P r o o f. We can prove it in the same way as [11]. □

G.G. Gould has proved in [5]that a necessary and sufficient condition for a bounded vector measure μ, taking values in a normed space X, to have a Lebesgue extension is given by the following property:

Property A. If $\{x_n\}$ is a sequence in X whose norms have a positive lower bound, then for an arbitrary positive k there exists a finite subsequence $\{x_{n_k}\}$ such that $\|\sum\limits_k x_{n_k}\| > k$.

It has been proved that all weakly complete spaces, the Hilbert spaces, and the spaces ℓ^p, $1 \leqslant p < +\infty$, satisfy Property A. We make use of Property A in Theorem 2.3.

Theorem 2.3. *Let $\mu\colon R \to X$ be a countably additive vector measure in a sequentially complete locally convex Hausdorff space X with the property*

(A)*: If $\{x_n\}$ is a sequence in X such that there exists a neighborhood U of zero in X with $x_n \notin U$ for every $n \in \mathbb{N}$, then there exists a neighborhood V of zero in X such that for every positive λ there exists a finite subsequence $\{x_{n_k}\}$ with*

$$\sum_k x_{n_k} \notin \lambda V.$$

Then the following statements are equivalent:

(i) *$\mu\colon R \to X$ has a countably additive extension $\widehat{\mu}\colon \mathcal{F}(R) \to X$;*

(ii) *μ is locally bounded over R, that is, for every $q \in \Gamma$ and every $E \in R$, $\mu_q(E) < +\infty$.*

P r o o f. i) $\Rightarrow$ ii). $x^*\widehat{\mu}$ is a scalar measure on the δ-ring $\mathcal{F}(R)$ for every $x^* \in X^*$

By Dinculeanu ([1], §3 Proposition 14) we have $x^*\widehat{\mu}_q(E) = \sup\{|x^*\widehat{\mu}(A)|\colon A \subset E,\ A \in \mathcal{F}(R)\} < +\infty$ for every set $E \in \mathcal{F}(R)$, and by Mackey's theorem $\widehat{\mu}_q(E) < +\infty$. Therefore $\mu_q(E) \leqslant \widehat{\mu}_q(E) < +\infty$ for every $q \in \Gamma$, $E \in R$.

ii) $\Rightarrow$ i). We shall show that (ii) implies (iii) of Proposition 2.2.

If this is false, then there exists a set $E \in R$, a neighborhood $U \in \mathcal{U}$ and a sequence (E_n) of mutually disjoint sets of R with $E_n \subset E$, $n = 1, 2, \ldots$, such that $\mu(E_n) \notin U$ for all n. By property (A), there exists a neighborhood V of zero in X such that for every positive λ there exists a finite subsequence $\mu(E_{k_n})$ with

$$\sum_k \mu(E_{k_n}) \notin \lambda V.$$

Therefore we have a contradiction. □

Theorem 2.4. *Let X be a weakly complete locally convex Hausdorff space and $\mu\colon R \to X$ a vector measure. The following statements are equivalent:*

(i) *$\mu\colon R \to X$ has a countably additive extension $\widehat{\mu}\colon \mathcal{F}(R) \to X$;*

(ii) *μ is locally bounded over R.*

Proof. i) $\Rightarrow$ ii). It is the same as the proof of Theorem 2.3.

ii) $\Rightarrow$ i). We shall show that ii) implies the statement (iii) of Proposition 2.2. If this is false, then there exist a set $E \in R$, a neighborhood $U \in \mathcal{U}$ and a sequence (E_n) of mutually disjoint sets of R with $E_n \subset E$, $n = 1, 2, \ldots$, such that $\mu(E_n) \notin U$ for all $n \in \mathbb{N}$.

By [5], since X is weakly complete it has the property (A) and $X \not\supset c_0$. Indeed, if we suppose that there exists a subspace Y of X which is topologically isomorphic to c_0, then Y is a complete subset of the Hausdorff space X.

Y is closed and, since it is convex, it is weakly closed. Thus Y is weakly sequentially complete, which is impossible since c_0 is not weakly sequentially complete. By property (A) there exists a neighborhood V of zero in X such that for every positive λ there exists a finite subsequence $\{\mu(E_{k_n})\}$ with

$$\sum_k \mu(E_{k_n}) \notin \lambda V,$$

which contradicts the statement (ii). □

Corollary 2.5. *If X is a Frechet space and $q(\mu)(E) < +\infty$ for every $q \in \Gamma$ and $E \in \mathbb{R}$, then μ has a countably additive extension $\widehat{\mu}\colon \mathcal{F}(R) \to X$ ([3]).*

Proof. Since X is Frechet, it is weakly complete. From the inequalities

$$\mu_q(A) \leqslant q(\mu)(A) \leqslant 2\mu_q(A), \quad A \in R$$

we have $\mu_q(A) < +\infty$, $A \in R$. The proof is obvious by Theorem 2.4. □

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