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SINGULAR DIRICHLET BOUNDARY VALUE PROBLEMS II:  
RESONANCE CASE

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*Abstract.* Existence results are established for the resonant problem  $y'' + \lambda_m a y = f(t, y)$  a.e. on  $[0, 1]$  with  $y$  satisfying Dirichlet boundary conditions. The problem is singular since  $f$  is a Carathéodory function,  $a \in L^1_{\text{loc}}(0, 1)$  with  $a > 0$  a.e. on  $[0, 1]$  and  $\int_0^1 x(1-x)a(x) dx < \infty$ .

## 1. INTRODUCTION

This paper presents existence results for the Dirichlet resonant second order problem

$$(1.1) \quad \begin{cases} y'' + \lambda_m a(t) y = f(t, y) & \text{a.e. on } [0, 1], \\ y(0) = y(1) = 0 \end{cases}$$

where  $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function,  $a \in L^1_{\text{loc}}(0, 1)$  with  $a > 0$  a.e. on  $[0, 1]$  and  $\int_0^1 x(1-x)a(x) dx < \infty$ .

**Remark.**  $\lambda_m$  (which is the  $(m+1)^{\text{st}}$  eigenvalue of an appropriate problem) will be described later in the introduction.

Equations of the type (1.1), with  $a \in L^1[0, 1]$ , have been studied extensively in the literature [2–3, 5–8, 10–11]. However very little attention has been given to the case when  $a \notin L^1[0, 1]$  (see [3, 5, 11] for results concerning upper and lower solutions). We remark here that the eigenvalue problem (which is singular) has been studied [1, 4, 9]. In this paper we use a well known technique, initiated by Mawhin and Ward [8] in the early 1980's and extended by Iannacci and Nkashama [6] in the late 1980's, to establish some new existence results for (1.1). The results here rely on a new

existence principle established by the author in [12]. For convenience we now recall the results in [12] which will be used in this paper.

Our first result is an existence principle for

$$(1.2) \quad \begin{cases} y'' + \mu a(t)y = f(t, y) & \text{a.e. on } [0, 1], \\ y(0) = y(1) = 0, \quad \mu \text{ a constant} \end{cases}$$

which was established using fixed point methods. First recall  $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function if

- (i)  $t \mapsto f(t, y)$  is measurable for all  $y \in \mathbb{R}$ ,
- (ii)  $y \mapsto f(t, y)$  is continuous for a.e.  $t \in [0, 1]$ .

**Theorem 1.1.** *Let  $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function with*

$$(1.3) \quad \text{for any } r > 0 \text{ there exists } h_r \in L^1_{\text{loc}}(0, 1) \text{ with } |f(t, u)| \leq h_r(t) \text{ for almost all } t \in [0, 1] \text{ and all } |u| \leq r; \text{ also } \int_0^1 x(1-x)h_r(x) dx < \infty \text{ with } \lim_{t \rightarrow 0^+} t^2(1-t)h_r(t) = 0 \text{ if } \int_0^1 (1-x)h_r(x) dx = \infty \text{ and } \lim_{t \rightarrow 1^-} t(1-t)^2h_r(t) = 0 \text{ if } \int_0^1 xh_r(x) dx = \infty$$

satisfied. Also assume

$$(1.4) \quad a \in L^1_{\text{loc}}(0, 1) \text{ with } a > 0 \text{ a.e. on } (0, 1) \text{ and } \int_0^1 x(1-x)a(x) dx < \infty; \text{ also } \lim_{t \rightarrow 0^+} t^2(1-t)a(t) = 0 \text{ if } \int_0^1 (1-x)a(x) dx = \infty \text{ and } \lim_{t \rightarrow 1^-} t(1-t)^2a(t) = 0 \text{ if } \int_0^1 xa(x) dx = \infty$$

holds and suppose  $\mu$  is such that

$$(1.5) \quad \begin{cases} y'' + \mu a(t)y = 0 & \text{a.e. on } [0, 1], \\ y(0) = y(1) = 0 \end{cases}$$

has only the trivial solution. In addition assume there is a constant  $M_0$ , independent of  $\lambda$ , with  $|y|_0 = \sup_{[0,1]} |y(t)| \neq M_0$  for any solution  $y$  (here  $y \in AC[0, 1] \cap C^1(0, 1)$  with  $y' \in AC_{\text{loc}}(0, 1)$ ) to

$$(1.6)_\lambda \quad \begin{cases} y'' + \mu a(t)y = \lambda f(t, y) & \text{a.e. on } [0, 1], \\ y(0) = y(1) = 0 \end{cases}$$

for each  $\lambda \in (0, 1)$ . Then (1.2) has at least one solution  $u \in AC[0, 1] \cap C^1(0, 1)$  with  $u' \in AC_{\text{loc}}(0, 1)$ .

**Remark.** In [12] we showed that if  $h_r \in L^1[0, 1]$  in (1.3) then in fact the solution  $u$  described above has the additional properties  $au^2 \in L^1[0, 1]$ ,  $u' \in L^2[0, 1]$  and  $\lim_{t \rightarrow 0^+} u(t)u'(t) = \lim_{t \rightarrow 1^-} uu' = 0$ .

We now establish a more general result. Suppose the conditions of theorem 1.1 are satisfied and in addition assume

$$(1.7) \quad \text{there exist } p, q, \text{ with } 0 \leq p, q < \frac{1}{2}, \text{ and } x^p(1-x)^q h_r \in L^1[0, 1]; \text{ here } h_r \text{ is as described in (1.3)}$$

and

$$(1.8) \quad x^{1-p}(1-x)^{1-q} a \in L^1[0, 1]$$

hold. Then the solution  $u$  described in theorem 1.1 satisfies  $au^2 \in L^1[0, 1]$ ,  $u' \in L^2[0, 1]$  and  $\lim_{t \rightarrow 0^+} u(t)u'(t) = \lim_{t \rightarrow 1^-} u(t)u'(t) = 0$ .

Recall [12],

$$u(t) = c_0 w_2(t)(1-t) \int_0^t s w_1(s) f(s, u(s)) ds + c_0 w_1(t) t \int_t^1 (1-s) w_2(s) f(s, u(s)) ds$$

( $c_0, w_1$  and  $w_2$  are as described in theorem 2.1 in [12]). If we show

$$\int_0^1 a(t)(1-t)^2 \left( \int_0^t s h_r(s) ds \right)^2 dt < \infty$$

and

$$\int_0^1 a(t) t^2 \left( \int_t^1 (1-s) h_r(s) ds \right)^2 dt < \infty$$

(here  $h_r$  is as described in (1.3)) then of course  $au^2 \in L^1[0, 1]$ . Immediately we have

$$\begin{aligned} & \int_0^1 a(t)(1-t)^2 \left( \int_0^t s h_r(s) ds \right)^2 dt \\ & \leq \int_0^1 a(t) t^{2(1-p)} (1-t)^{2(1-q)} \left( \int_0^t s^p (1-s)^q h_r(s) ds \right)^2 dt < \infty \end{aligned}$$

so  $au^2 \in L^1[0, 1]$ .

Also recall for  $t \in (0, 1)$  that

$$u'(t) = c_0 y_2'(t) \int_0^t s w_1(s) f(s, u(s)) ds + c_0 y_1'(t) \int_t^1 (1-s) w_2(s) f(s, u(s)) ds.$$

We show

$$(1.9) \quad \int_0^1 \left( y_2'(t) \int_0^t s h_r(s) ds \right)^2 dt < \infty$$

where

$$(1.10) \quad y_2'(t) = 1 + \mu \int_t^1 (1-x)a(x)w_2(x) dx.$$

Now (1.9) is immediate since

$$\begin{aligned} & \int_0^1 \left( y_2'(t) \int_0^t sh_r(s) ds \right)^2 dt \\ & \leq 2 \int_0^1 \left( \int_0^t sh_r(s) ds \right)^2 dt \\ & \quad + 2\mu^2 |w_2|_0^2 \int_0^1 \left( \int_t^1 (1-x)a(x) dx \int_0^t sh_r(s) ds \right)^2 dt \\ & \leq 2 \int_0^1 \frac{1}{(1-t)^{2q}} \left( \int_0^t s(1-s)^q h_r(s) ds \right)^2 dt \\ & \quad + 2\mu^2 |w_2|_0^2 \int_0^1 \left( \int_t^1 (1-x)^{1-q} x^{1-p} a(x) dx \int_0^t s^p (1-s)^q h_r(s) ds \right)^2 dt. \end{aligned}$$

Similarly

$$\int_0^1 \left( y_1'(t) \int_t^1 (1-s)h_r(s) ds \right)^2 dt < \infty$$

so  $u' \in L^1[0, 1]$ . Finally we show  $\lim_{t \rightarrow 0^+} u(t)u'(t) = 0$ . This is immediate once we show

$$(1.11) \quad \lim_{t \rightarrow 0^+} y_2(t) \int_0^t sh_r(s) ds y_2'(t) \int_0^t sh_r(s) ds = 0,$$

$$(1.12) \quad \lim_{t \rightarrow 0^+} y_1(t) \int_t^1 (1-s)h_r(s) ds y_2'(t) \int_0^t sh_r(s) ds = 0,$$

$$(1.13) \quad \lim_{t \rightarrow 0^+} y_2(t) \int_0^t sh_r(s) ds y_1'(t) \int_t^1 (1-s)h_r(s) ds = 0$$

and

$$(1.14) \quad \lim_{t \rightarrow 0^+} y_1(t) \int_t^1 (1-s)h_r(s) ds y_1'(t) \int_t^1 (1-s)h_r(s) ds = 0.$$

Now (1.11) is true since (see (1.10))

$$\begin{aligned} \left( \int_0^t sh_r(s) ds \right)^2 \int_t^1 (1-x)a(x) dx & \leq \left( \int_0^t s^p h_r(s) ds \right)^2 \int_t^1 x^{2(1-p)} (1-x)a(x) dx \\ & \leq \left( \int_0^t s^p h_r(s) ds \right)^2 \int_t^1 x^{1-p} (1-x)^{1-q} a(x) dx \end{aligned}$$

and (1.7) implies  $\lim_{t \rightarrow 0^+} \int_0^t s^p h_r(s) ds = 0$ .

Also we know (1.4) implies  $\lim_{t \rightarrow 0^+} y_1(t) \int_t^1 (1-x)h_r(x) dx = 0$  and thus (1.12) is immediately true since

$$\begin{aligned} & \int_t^1 (1-x)a(x) dx \int_0^t s h_r(s) ds \\ & \leq \int_t^1 x^{1-p}(1-x)^{1-q}a(x) dx \int_0^t s^p(1-s)^q h_r(s) ds < \infty. \end{aligned}$$

Now (1.13) follows from the fact that

$$\begin{aligned} \int_0^t x h_r(x) ds \int_t^1 (1-s)h_r(s) ds & \leq \int_0^t x^p h_r(x) ds \int_t^1 s^{1-p}(1-s)h_r(s) ds \\ & \leq \int_0^t x^p h_r(x) ds \int_t^1 s^p(1-s)^q h_r(s) ds. \end{aligned}$$

Finally (1.14) follows once we show  $\lim_{t \rightarrow 0^+} y_1(t) \left( \int_t^1 (1-s)h_r(s) ds \right)^2 = 0$ . This is immediate since

$$\begin{aligned} t|w_1(t)| \left( \int_t^1 (1-s)h_r(s) ds \right)^2 & = t^{1-2p}|w_1(t)| \left( \int_t^1 s^p(1-s)h_r(s) ds \right)^2 \\ & \leq t^{1-2p}|w_1(t)| \left( \int_t^1 s^p(1-s)^q h_r(s) ds \right)^2. \end{aligned}$$

Thus  $\lim_{t \rightarrow 0^+} u(t)u'(t) = 0$ . Similarly  $\lim_{t \rightarrow 1^-} u(t)u'(t) = 0$ .

Next we recall some results [12] concerning the eigenvalue problem. In particular consider

$$\begin{aligned} Ly & = \lambda y, \quad \text{a.e. on } [0, 1], \\ y(0) & = y(1) = 0 \end{aligned}$$

where  $Ly = -\frac{1}{a}y''$ . Assume (1.4) holds for the remainder of this section. Now  $L_a^2[0, 1]$  denotes the space of functions  $u$  with  $\int_0^1 a(t)|u(t)|^2 dt < \infty$ ; also for  $u, v \in L_a^2[0, 1]$  define  $\langle u, v \rangle = \int_0^1 a(t)u(t)\overline{v(t)} dt$ . Let

$$\begin{aligned} D(L) & = \left\{ u \in C[0, 1]: u \in AC[0, 1], u' \in AC_{\text{loc}}(0, 1) \text{ with } \frac{1}{a}u'' \in L_a^2[0, 1] \right. \\ & \quad \left. u(0) = u(1) = 0 \text{ and } \lim_{t \rightarrow 0^+} u(t)u'(t) = \lim_{t \rightarrow 1^-} u(t)u'(t) = 0 \right\}. \end{aligned}$$

In [12] we showed using the spectral theorem for compact self adjoint operators that  $L$  has a countably infinite number of real eigenvalues  $\lambda_i$  with corresponding

eigenfunctions  $\varphi_i \in D(L)$ . Also the eigenvalues  $\lambda_i$  are simple and  $\lambda_i > 0$  for all  $i$ , so we may arrange the eigenvalues so that

$$\lambda_0 < \lambda_1 < \lambda_2 < \dots$$

In addition the eigenfunctions  $\varphi_i$  may be chosen so that they form an orthonormal set. Also  $\{\varphi_i\}$  form a basis for  $L_a^2[0, 1]$  so if  $r \in L_a^2[0, 1]$  then  $r$  has a fourier series representation

$$r = \sum_{i=0}^{\infty} \langle r, \varphi_i \rangle \varphi_i$$

and  $r$  satisfies Parseval's equality

$$\int_0^1 a|r|^2 dt = \sum_{i=0}^{\infty} |\langle r, \varphi_i \rangle|^2.$$

Next consider functions  $u \in AC[0, 1]$ ,  $u(0) = u(1) = 0$  with  $u \in L_a^2[0, 1]$  and  $u' \in L^2[0, 1]$ . Then  $u = \sum_{i=0}^{\infty} c_i \varphi_i$  where  $c_i = \langle u, \varphi_i \rangle$ . In [12] we showed

$$u' = \sum_{i=0}^{\infty} c_i \varphi_i'$$

(convergence is understood to be in  $L^2$ ).

## 2. EXISTENCE THEORY

In this section we use theorem 1.1 to establish existence results for

$$(2.1) \quad \begin{cases} y'' + \lambda_m a y + y g(t, y) = h(t, y) + v(t) & \text{a.e. on } [0, 1], \\ y(0) = y(1) = 0 \end{cases}$$

where  $m \in \{0, 1, 2, \dots\}$  and  $\lambda_m$  is as described in section 1.

**Remark.** For the remainder of this paper let  $f(t, y) = h(t, y) - y g(t, y)$ .

**Theorem 2.1.** *Let  $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function with*

$$(2.2) \quad \text{for any } r > 0 \text{ there exists } h_r \in L^1[0, 1] \text{ with } |f(t, u)| \leq h_r(t) \text{ for almost all } t \in [0, 1] \text{ and all } |u| \leq r$$

holding. In addition assume (1.4) holds with

$$(2.3) \quad v \in L^1[0, 1] \text{ and there exists } p, 0 \leq p < \frac{1}{2} \text{ with } x^p(1-x)^p a \in L^1[0, 1]$$

(2.4) there exists  $\tau \in C[0, 1]$  with  $a\tau \in L^1[0, 1]$  and  $0 \leq g(t, u) \leq \tau(t)a(t)$  for a.e.  $t \in [0, 1]$  and  $u \in \mathbb{R}$ ; here  $\tau(t) \leq \lambda_{m+1} - \lambda_m$  for a.e.  $t \in [0, 1]$  with  $\tau(t) < \lambda_{m+1} - \lambda_m$  on a subset of  $[0, 1]$  of positive measure

$$(2.5) \quad |h(t, y)| \leq q_1(t) + q_2(t)|y|^\gamma \text{ for a.e. } t \in [0, 1] \text{ with } 0 \leq \gamma < 1$$

and

$$(2.6) \quad q_i \in L^1[0, 1], \quad i = 1, 2$$

are satisfied.

(i) Suppose there exists a constant  $k > \gamma$  with  $1 > k = \frac{\alpha}{\beta}$ , where  $\beta$  is odd and  $\alpha$  is even, and

$$(2.7) \quad 0 < \int_{I^+} [A\varphi_m(t)]^{k+1} \liminf_{x \rightarrow \infty} (x^{1-k} g(t, x)) \, dt \\ + \int_{I^-} [A\varphi_m(t)]^{k+1} \limsup_{x \rightarrow -\infty} (x^{1-k} g(t, x)) \, dt$$

for all constants  $A \neq 0$ ; here  $I^+ = \{t \in [0, 1]: A\varphi_m(t) > 0\}$  and  $I^- = \{t: A\varphi_m(t) < 0\}$ .

Then (2.1) has at least one solution.

(ii) Suppose  $\gamma = 0$  and

$$(2.8) \quad A \int_0^1 v(t)\varphi_m(t) \, dt < A \int_{I^+} \varphi_m(t) \liminf_{x \rightarrow \infty} (x g(t, x)) \, dt \\ + A \int_{I^+} \varphi_m(t) \liminf_{x \rightarrow \infty} [-h(t, x)] \, dt \\ + A \int_{I^-} \varphi_m(t) \limsup_{x \rightarrow -\infty} (x g(t, x)) \, dt \\ + A \int_{I^-} \varphi_m(t) \limsup_{x \rightarrow -\infty} [-h(t, x)] \, dt$$

for all constants  $A \neq 0$ .

Then (2.1) has at least one solution.



Proof. Choose  $\mu$  such that  $\lambda_m < \mu < \lambda_{m+1}$  and look at the boundary value problem

$$(2.9)_\delta \quad \begin{cases} y'' + \mu a y = \delta[h(t, y) + v(t) - y g(t, y) + (\mu - \lambda_m)a y] & \text{a.e. on } [0, 1], \\ y(0) = y(1) = 0 \end{cases}$$

for  $\delta \in (0, 1)$ . Suppose  $y$  is a solution of  $(2.9)_\delta$ . If we show that there exists a constant  $M_0$ , independent of  $\lambda$ , with  $|y|_0 \leq M_0$  then existence of a solution to (2.1) will be guaranteed from theorem 1.1.

Now  $L_a^2[0, 1] = \Omega \oplus \Omega^\perp$  where  $\Omega = \{\varphi_0, \dots, \varphi_m\}$ . Let

$$w = \sum_{i=m+1}^{\infty} c_i \varphi_i, \quad u = \sum_{i=0}^m c_i \varphi_i, \quad u_0 = \sum_{i=0}^{m-1} c_i \varphi_i, \quad u_1 = c_m \varphi_m \quad \text{and} \quad \tilde{y} = w + u_0$$

where  $c_i = \langle y, \varphi_i \rangle$ .

**Remarks.** (i) Notice from section 1 that  $ay^2 \in L^1[0, 1]$  and  $y' \in L^2[0, 1]$ .

(ii) Note  $\tilde{y} = y - u_1$  and  $y = w + u$ .

Multiply the differential equation in  $(2.9)_\delta$  by  $u - w$  and integrate from 0 to 1 to obtain

$$\begin{aligned} & \int_0^1 ([w']^2 - [\lambda_m a + (1 - \delta)(\mu - \lambda_m)a + \delta g(t, y)]w^2) dt \\ & \quad - \int_0^1 ([u']^2 - [\lambda_m a + (1 - \delta)(\mu - \lambda_m)a + \delta g(t, y)]u^2) dt \\ & = \delta \int_0^1 [h(t, y) + v(t)](u - w) dt. \end{aligned}$$

**Remark.** Now  $w - u = y - 2u$  and we claim  $\lim_{t \rightarrow 0^+} y'(y - 2u) = 0$ . To see this we show  $\lim_{t \rightarrow 0^+} u(t)y'(t) = 0$ . This will be established by showing  $\lim_{t \rightarrow 0^+} \varphi_i(t)y'(t) = 0$  for  $i = 0, 1, \dots, m$ . Fix  $i \in \{0, 1, \dots, m\}$ . Then

$$\varphi_i(t) = \lambda_i (1 - t) \int_0^t s a(s) \varphi_i(s) ds + \lambda_i t \int_t^1 (1 - s) a(s) \varphi_i(s) ds$$

and

$$\begin{aligned} y'(t) &= \delta c_0 y_2'(t) \int_0^t s w_1(s) [f(s, y(s)) + v(s) + (\mu - \lambda_m)a(s)y(s)] ds \\ & \quad + \delta c_0 y_1'(t) \int_t^1 (1 - s) w_2(s) [f(s, y(s)) + v(s) + (\mu - \lambda_m)a(s)y(s)] ds. \end{aligned}$$

Now  $\lim_{t \rightarrow 0^+} \varphi_i(t)y'(t) = 0$  if we show

$$\lim_{t \rightarrow 0^+} \int_0^t xa(x) dx \int_t^1 (1-s)a(s) ds = 0 \quad \text{and} \quad \lim_{t \rightarrow 0^+} t \left( \int_t^1 (1-x)a(x) dx \right)^2 = 0.$$

The first limit is immediate since

$$\begin{aligned} \int_0^t xa(x) dx \int_t^1 (1-s)a(s) ds &\leq \int_0^t x^p a(x) dx \int_t^1 s^{1-p}(1-s)a(s) ds \\ &\leq \int_0^t x^p a(x) dx \int_0^1 s^p(1-s)^p a(s) ds \end{aligned}$$

and noting (2.3) implies  $\lim_{t \rightarrow 0^+} \int_0^t x^p a(x) dx = 0$ ; the second limit follows since

$$t \left( \int_t^1 (1-x)a(x) dx \right)^2 \leq t^{1-2p} \left( \int_0^1 x^p(1-x)a(x) dx \right)^2$$

Thus  $\lim_{t \rightarrow 0^+} y'(y-2u) = 0$ . Similarly  $\lim_{t \rightarrow 1^-} y'(y-2u) = 0$ .

Now use (2.4) to obtain

$$\begin{aligned} (2.10) \quad &\int_0^1 ([w']^2 - [\lambda_m + \tau(t)]aw^2) dt - (\mu - \lambda_m) \int_0^1 aw^2 dt \\ &- \int_0^1 ([u'_0]^2 - \lambda_m au_0^2) dt \leq \int_0^1 |[h(t,y) + v(t)](u-w)| dt. \end{aligned}$$

Let

$$R(\tilde{y}) = \int_0^1 ([w']^2 - [\lambda_m + \tau(t)]aw^2) dt - \int_0^1 ([u'_0]^2 - \lambda_m au_0^2) dt.$$

We claim that there exists  $\varepsilon > 0$  with

$$(2.11) \quad R(\tilde{y}) \geq \varepsilon (\|w\|_a^2 + \|w'\|^2 + \|u_0\|_a^2 + \|u'_0\|^2);$$

here  $\|z\|_a^2 = \int_0^1 az^2 dt$  and  $\|z'\|^2 = \int_0^1 [z']^2 dt$  where  $z = w$  or  $u_0$ . Next notice a standard argument, using (2.4), implies  $R(\tilde{y}) \geq 0$  and if  $R(\tilde{y}) = 0$  then  $\tilde{y} = 0$ .

If (2.11) is not true then there exists a sequence  $\{\tilde{y}_n\} = \{w_n + u_{0,n}\}$  with

$$(2.12) \quad \|w_n\|_a^2 + \|w'_n\|^2 + \|u_{0,n}\|_a^2 + \|u'_{0,n}\|^2 = 1$$

and

$$(2.13) \quad R(\tilde{y}_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now (2.12) implies that there is a subsequence  $S$  of integers with

$$(2.14) \quad w_n \rightarrow w \text{ in } C[0,1] \text{ and } w'_n \rightharpoonup w' \text{ in } L^2[0,1] \text{ as } n \rightarrow \infty \text{ in } S$$

and

$$(2.15) \quad u_{0,n} \rightarrow u_0 \text{ in } C[0,1] \text{ and } u'_{0,n} \rightharpoonup u'_0 \text{ in } L^2[0,1] \text{ as } n \rightarrow \infty \text{ in } S;$$

here  $\rightharpoonup$  denotes weak convergence.

Since weak and strong convergence are the same in finite dimensional spaces we have

$$(2.16) \quad u_{0,n} \rightarrow u_0 \text{ in } W^{1,2}[0,1] \text{ as } n \rightarrow \infty \text{ in } S.$$

Also

$$(2.17) \quad \int_0^1 [w']^2 dt \leq \liminf \int_0^1 [w'_n]^2 dt.$$

Now (2.13), (2.14), (2.15), (2.16), (2.17) (and the Lebesgue dominated convergence theorem with  $ay^2 \in L^1[0,1]$  and  $\tau \in C[0,1]$ ) and the fact that  $\liminf[s_n + t_n] \geq \liminf s_n + \liminf t_n$  for sequences  $s_n$  and  $t_n$ , yields (with  $\tilde{y} = w + u_0$ ),

$$\begin{aligned} R(\tilde{y}) &\leq \liminf \int_0^1 [w'_n]^2 dt + \lim \int_0^1 [-a(t)](\lambda_m + \tau(t))w_n^2 dt \\ &\quad - \lim \int_0^1 [u'_{0,n}]^2 dt + \lim \int_0^1 a(t)(\lambda_m)u_{0,n}^2 dt \leq \liminf R(\tilde{y}_n) \leq 0. \end{aligned}$$

Hence  $\tilde{y} = 0$ . However

$$\begin{aligned} &\|w_n\|_a^2 + \|w'_n\|^2 + \|u_{0,n}\|_a^2 + \|u'_{0,n}\|^2 \\ &= R(\tilde{y}_n) + \int_0^1 [2(u'_{0,n})^2 + a u_{0,n}^2 - \lambda_m a u_{0,n}^2 + a w_n^2 \\ &\quad + (\lambda_m + \tau) a w_n^2] dt \rightarrow 0 \text{ as } n \rightarrow \infty \text{ in } S, \end{aligned}$$

which is impossible. Hence (2.11) holds for some  $\varepsilon > 0$  (note  $\varepsilon$  is independent of  $\mu$ ).

Put (2.11) into (2.10) to obtain

$$\begin{aligned} &\varepsilon (\|w\|_a^2 + \|w'\|^2 + \|u_0\|_a^2 + \|u'_0\|^2) - (\mu - \lambda_m) \int_0^1 a w^2 dt \\ &\leq \int_0^1 |h(t, y) + v(t)| |u - w| dt. \end{aligned}$$

Now choose  $\mu = \lambda_m + \frac{\varepsilon\lambda_{m+1}}{2}$  where  $\varepsilon$  is chosen sufficiently small so that  $\lambda_m + \frac{\varepsilon\lambda_{m+1}}{2} < \lambda_{m+1}$ . Then since  $(\mu - \lambda_m)\|w\|_a^2 \leq \frac{(\mu - \lambda_m)}{\lambda_{m+1}}\|w'\|^2$  we have

$$(2.18) \quad \frac{\varepsilon}{2} (\|w\|_a^2 + \|w'\|^2 + \|u_0\|_a^2 + \|u'_0\|^2) \leq \int_0^1 |h(t, y) + v(t)| |u - w| dt.$$

Notice also for  $t \in [0, 1]$  that

$$|w(t) - u(t)| = \left| \int_0^t [w'(s) - u'(s)] ds \right| \leq \|w' - u'\| \quad \text{and} \quad |y(t)| \leq \|w' + u'\|.$$

Thus

$$(2.19) \quad \begin{aligned} \int_0^1 q_1(t) |w - u| dt &\leq |w - u|_0 \int_0^1 q_1 dt \leq \|w' - u'\| \int_0^1 q_1 dt \\ &= (\|w'\|^2 + \|u'_0\|^2 + \|u'_1\|^2)^{\frac{1}{2}} \int_0^1 q_1 dt \end{aligned}$$

$$(2.20) \quad \int_0^1 |v(t)| |w - u| dt \leq (\|w'\|^2 + \|u'_0\|^2 + \|u'_1\|^2)^{\frac{1}{2}} \int_0^1 |v| dt$$

and

$$(2.21) \quad \begin{aligned} \int_0^1 q_2(t) |w - u| |y|^\gamma dt &\leq |w - u|_0 |y|_0^\gamma \int_0^1 q_2 dt \\ &\leq (\|w'\|^2 + \|u'_0\|^2 + \|u'_1\|^2)^{\frac{\gamma+1}{2}} \int_0^1 q_2 dt \end{aligned}$$

since

$$\|w' - u'\|^2 = \|w'\|^2 + \|u'\|^2 = \|y'\|^2 = \|\tilde{y}'\|^2 + \|u'_1\|^2 = \|w'\|^2 + \|u'_0\|^2 + \|u'_1\|^2.$$

Put (2.19), (2.20) and (2.21) into (2.18) to obtain

$$\begin{aligned} &\frac{\varepsilon}{2} (\|w\|_a^2 + \|w'\|^2 + \|u_0\|_a^2 + \|u'_0\|^2) \\ &\leq (\|w'\|^2 + \|u'_0\|^2 + \|u'_1\|^2)^{\frac{1}{2}} \left( \int_0^1 q_1 dt + \int_0^1 |v| dt \right) \\ &\quad + (\|w'\|^2 + \|u'_0\|^2 + \|u'_1\|^2)^{\frac{\gamma+1}{2}} \int_0^1 q_2 dt. \end{aligned}$$

Now since  $0 \leq \gamma < 1$  there exist constants  $A_1$  and  $A_2$  with

$$(2.22) \quad \|w\|_a^2 + \|u_0\|_a^2 + \|w'\|^2 + \|u'_0\|^2 \leq A_1 + A_2 \|u'_1\|^{\gamma+1}.$$

Consequently

$$(2.23) \quad \|\tilde{y}\|_a^2 + \|\tilde{y}'\|^2 \leq A_1 + A_2 \|u'_1\|^{\gamma+1}.$$

We next claim that (2.23) implies that there is a constant  $M_0 > 0$  with

$$(2.24) \quad \|y\|_a^2 + \|y'\|^2 \leq M_0.$$

Suppose the claim is false. Then there is a sequence  $(\delta_n)$  in  $(0, 1)$  and a sequence  $(y_n)$  with for a.e.  $t \in [0, 1]$ ,

$$(2.25) \quad y_n'' + \lambda_m a y_n + (1 - \delta_n)(\mu - \lambda_m)a y_n + \delta_n y_n g(t, y_n) = \delta_n [h(t, y_n) + v(t)]$$

and

$$(2.26) \quad \|y_n\|_a^2 + \|y_n'\|^2 \rightarrow \infty.$$

From (2.23), with  $y_n = \tilde{y}_n + u_{1,n}$ , we have

$$(2.27) \quad \|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2 \rightarrow \infty.$$

Also (2.23) and (2.27) imply

$$(2.28) \quad \frac{\|\tilde{y}_n\|_a^2 + \|\tilde{y}'_n\|^2}{\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2} \rightarrow 0.$$

Let

$$r_n = \frac{y_n}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{1}{2}}} = \tilde{r}_n + r_{1,n}$$

where

$$\tilde{r}_n = \frac{\tilde{y}_n}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{1}{2}}} \quad \text{and} \quad r_{1,n} = \frac{u_{1,n}}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{1}{2}}}.$$

Now (2.28) implies there is a subsequence  $S^*$  of integers with

$$(2.29) \quad \tilde{r}_n \rightarrow 0 \text{ in } C[0, 1] \text{ and } \tilde{r}'_n \rightarrow 0 \text{ in } L^2[0, 1] \text{ as } n \rightarrow \infty \text{ in } S^*.$$

Now

$$\frac{\|y_n\|_a^2 + \|y_n'\|^2}{\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2} = 1 + \frac{\|\tilde{y}_n\|_a^2 + \|\tilde{y}'_n\|^2}{\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2}$$

together with (2.28) and (2.29) implies that there is a subsequence  $S$  of integers and a constant  $A \neq 0$  with

$$(2.30) \quad r_{1,n} \rightarrow A\varphi_m \text{ in } C[0,1] \text{ as } n \rightarrow \infty \text{ in } S$$

and

$$(2.31) \quad r_n \rightarrow A\varphi_m \text{ in } C[0,1] \text{ as } n \rightarrow \infty \text{ in } S.$$

**Remark.** Note as well that  $r_n \rightarrow A\varphi_m$  in  $W^{1,2}[0,1]$  as  $n \rightarrow \infty$  in  $S$  and  $r_{1,n} \rightarrow A\varphi_m$  in  $W^{1,2}[0,1]$  as  $n \rightarrow \infty$  in  $S$ .

Multiply (2.25) by  $\frac{r_{1,n}}{\delta_n}$  and integrate from 0 to 1 (noting that  $\int_0^1 [r_{1,n}y_n'' + \lambda_m r_{1,n} a y_n] dt = 0$ ), to obtain

$$(2.32) \quad \int_0^1 v r_{1,n} dt = - \int_0^1 r_{1,n} h(t, y_n) dt + \int_0^1 r_{1,n} y_n g(t, y_n) dt \\ + \frac{(1 - \delta_n)(\mu - \lambda_m)}{\delta_n (\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{1}{2}}} \int_0^1 a u_{1,n}^2 dt$$

and so

$$(2.33) \quad \int_0^1 v r_{1,n} dt \geq - \int_0^1 r_{1,n} h(t, y_n) dt + \int_0^1 r_{1,n} y_n g(t, y_n) dt.$$

**Case (i).** Suppose (2.7) holds.

Multiply (2.33) by  $(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{-\frac{k}{2}}$  to obtain

$$\frac{1}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k}{2}}} \int_0^1 v r_{1,n} dt \geq - \int_0^1 \frac{r_{1,n} h(t, y_n)}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k}{2}}} dt \\ + \int_0^1 \frac{r_{1,n} y_n g(t, y_n)}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k}{2}}} dt.$$

Now since (2.30) holds then there exists a constant  $B_0$  with  $|r_{1,n}|_0 \leq B_0$  for all  $n$  so we may apply the Lebesgue dominated convergence theorem to deduce

$$(2.34) \quad \liminf \int_0^1 v r_{1,n} dt = \lim \int_0^1 v r_{1,n} dt = A \int_0^1 v \varphi_m dt.$$

Now (2.27) and (2.34) imply

$$\liminf \int_0^1 \frac{v r_{1,n}}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k}{2}}} dt = 0.$$

and so

$$(2.35) \quad 0 \geq \liminf \int_0^1 \frac{r_{1,n}[-h(t, y_n)]}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k}{2}}} dt + \liminf \int_0^1 \frac{r_{1,n} y_n g(t, y_n)}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k}{2}}} dt.$$

**Remark.** In (2.35) we have  $n \rightarrow \infty$  in  $S$ .

Now

$$(2.36) \quad \liminf \int_0^1 \frac{r_{1,n}[-h(t, y_n)]}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k}{2}}} dt = 0$$

since

$$\begin{aligned} 0 &\leq \int_0^1 \frac{|r_{1,n} h(t, y_n)|}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k}{2}}} dt \\ &\leq \frac{1}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k+1}{2}}} \left( \int_0^1 q_1 |u_{1,n}| dt + \int_0^1 q_2 |u_{1,n}| |y_n|^\gamma dt \right) \end{aligned}$$

together with

$$\int_0^1 q_1 |u_{1,n}| dt \leq |u_{1,n}|_0 \int_0^1 q_1 dt \leq (\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{1}{2}} \int_0^1 q_1 dt$$

and (here we use (2.23)),

$$\begin{aligned} \int_0^1 q_2 |y_n|^\gamma |u_{1,n}| dt &\leq (\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{1}{2}} (\|y_n\|_a^2 + \|y'_n\|^2)^{\frac{\gamma}{2}} \int_0^1 q_2 dt \\ &\leq (\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2 + \|\tilde{y}_n\|_a^2 + \|\tilde{y}'_n\|^2)^{\frac{\gamma+1}{2}} \int_0^1 q_2 dt \\ &\leq (\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2 + A_1 + A_2 \|u'_{1,n}\|^{\gamma+1})^{\frac{\gamma+1}{2}} \int_0^1 q_2 dt \end{aligned}$$

implies that (2.36) is true (note  $k > \gamma$  and  $\gamma < 1$ ). This together with (2.35) yields

$$(2.37) \quad 0 \geq \liminf \int_{I^+} \frac{r_{1,n} y_n g(t, y_n)}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k}{2}}} dt + \liminf \int_{I^-} \frac{r_{1,n} y_n g(t, y_n)}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k}{2}}} dt.$$

**Remark.** Notice since  $\varphi_m$  has at most a finite number of zero's  $[1,4]$  and  $a\tau \in L^1[0, 1]$  then, with  $J = [0, 1]/\{I^+ \cup I^-\}$ , we have

$$\int_J |u_{1,n} y_n| a \tau dt \leq |u_{1,n}|_0 |y_n|_0 \int_J a \tau dt = 0 \quad \text{for each } n \in S.$$

Consequently

$$\int_J \frac{r_{1,n} y_n g(t, y_n)}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k}{2}}} dt = 0 \quad \text{for each } n \in S.$$

We next want to apply Fatou's lemma in (2.37). To justify this we need to show that there exists  $\varrho \in L^1[0, 1]$  with

$$(2.38) \quad \frac{r_{1,n} y_n g(t, y_n)}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k}{2}}} \geq \varrho(t) \quad \text{for a.e. } t \in [0, 1].$$

To see this first notice

$$[\tilde{y}_n(t)]^2 \leq \|\tilde{y}_n\|^2 + \|\tilde{y}'_n\|^2 \leq A_1 + A_2 \|u'_{1,n}\|^{\gamma+1}$$

so there exists  $A_3 > 0$  (since  $k > \gamma$  and (2.27) holds) with

$$(2.39) \quad \frac{[\tilde{y}_n(t)]^2}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k+1}{2}}} \leq A_3.$$

Also for a.e.  $t \in [0, 1]$ ,

$$\begin{aligned} \frac{r_{1,n} y_n g(t, y_n)}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k}{2}}} &= \frac{1}{2} \frac{g(t, y_n)[y_n^2 + u_{1,n}^2 - \tilde{y}_n^2]}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k+1}{2}}} \geq -\frac{1}{2} \frac{g(t, y_n) \tilde{y}_n^2}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k+1}{2}}} \\ &\geq -\frac{1}{2} \frac{a(t) \tau(t) \tilde{y}_n^2}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k+1}{2}}} \geq -\frac{1}{2} A_3 a(t) \tau(t). \end{aligned}$$

Thus (2.38) is true since  $a \tau \in L^1[0, 1]$ . Apply Fatou's lemma to (2.37) to obtain

$$\begin{aligned} 0 &\geq \int_{I^+} \liminf \left( \frac{r_{1,n} y_n g(t, y_n)}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k}{2}}} \right) dt + \int_{I^-} \liminf \left( \frac{r_{1,n} y_n g(t, y_n)}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{k}{2}}} \right) dt \\ &= \int_{I^+} \liminf \left( r_{1,n} \left( \frac{y_n}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{1}{2}}} \right)^k y_n^{1-k} g(t, y_n) \right) dt \\ &\quad + \int_{I^-} \liminf \left( r_{1,n} \left( \frac{-y_n}{(\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{1}{2}}} \right)^k [-[-y_n]^{1-k} g(t, y_n)] \right) dt. \end{aligned}$$

This together with (2.30) and (2.31) implies

$$(2.40) \quad \begin{aligned} 0 &\geq \int_{I^+} [A\varphi_m(t)]^{k+1} \liminf (y_n^{1-k} g(t, y_n)) dt \\ &\quad + \int_{I^-} [A\varphi_m(t)]^{k+1} \limsup (y_n^{1-k} g(t, y_n)) dt. \end{aligned}$$

**Remark.** Note  $(-1)^k = (-1)^{\frac{\alpha}{\beta}} = 1$  and  $-(-1)^{1-k} = 1$ .



Let  $t \in I^+$ . Then (2.31) implies there is an integer  $n_1$  (i.e.  $n_1(t)$ ) with

$$(2.41) \quad y_n(t) \geq \frac{1}{2} A \varphi_m(t) (\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{1}{2}} \quad \text{for } n \geq n_1.$$

Let  $t \in I^-$ . There is an integer  $n_2$  with

$$(2.42) \quad y_n(t) \leq \frac{1}{2} A \varphi_m(t) (\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{1}{2}} \quad \text{for } n \geq n_2.$$

This together with (2.40) implies

$$\begin{aligned} 0 &\geq \int_{I^+} [A\varphi_m(t)]^{k+1} \liminf_{x \rightarrow \infty} (x^{1-k} g(t, x)) \, dt \\ &\quad + \int_{I^-} [A\varphi_m(t)]^{k+1} \limsup_{x \rightarrow -\infty} (x^{1-k} g(t, x)) \, dt, \end{aligned}$$

which contradicts (2.7). Thus (2.24) holds and so  $|y|_0^2 \leq M_0$ . Theorem 1.1 now guarantees that (2.1) has a solution.

**Case (ii).** Suppose (2.8) holds.

Now (2.33) together with (2.34) yields

$$\begin{aligned} A \int_0^1 v \varphi_m \, dt &\geq \liminf \int_{I^+} r_{1,n} y_n g(t, y_n) \, dt + \liminf \int_{I^-} r_{1,n} y_n g(t, y_n) \, dt \\ &\quad + \liminf \int_{I^+} r_{1,n} [-h(t, y_n)] \, dt + \liminf \int_{I^-} r_{1,n} [-h(t, y_n)] \, dt. \end{aligned}$$

As in case (i) there exists  $\varrho \in L^1[0, 1]$  with

$$r_{1,n} y_n g(t, y_n) \geq \varrho(t) \quad \text{for a.e. } t \in [0, 1]$$

and so Fatou's lemma together with (2.41) and (2.42) implies

$$\begin{aligned} A \int_0^1 v \varphi_m \, dt &\geq A \int_{I^+} \varphi_m(t) \liminf_{x \rightarrow \infty} (xg(t, x)) \, dt + A \int_{I^-} \varphi_m(t) \limsup_{x \rightarrow -\infty} (xg(t, x)) \, dt \\ &\quad + \liminf \int_{I^+} r_{1,n} [-h(t, y_n)] \, dt + \liminf \int_{I^-} r_{1,n} [-h(t, y_n)] \, dt. \end{aligned}$$

Also  $|h(t, y)| \leq q_1(t) + q_2(t)$  a.e. so we may apply Fatou's lemma (together with (2.41) and (2.42)) to deduce

$$\begin{aligned} A \int_0^1 v(t)\varphi_m(t) dt &\geq A \int_{I^+} \varphi_m(t) \liminf_{x \rightarrow \infty} (xg(t, x)) dt \\ &\quad + A \int_{I^-} \varphi_m(t) \limsup_{x \rightarrow -\infty} (xg(t, x)) dt \\ &\quad + A \int_{I^+} \varphi_m(t) \liminf_{x \rightarrow \infty} [-h(t, x)] dt \\ &\quad + A \int_{I^-} \varphi_m(t) \limsup_{x \rightarrow -\infty} [-h(t, x)] dt, \end{aligned}$$

which contradicts (2.8). Thus (2.24) holds.  $\square$

**Remark.** One can obtain in addition a result if  $k = \gamma$  in case (i). Of course (2.36) does not necessarily hold in this case so we need to adjust (2.7) using the ideas in case (ii).

We now obtain an extra existence result if  $\int_0^1 v\varphi_m dt = 0$  and  $a \in L^1[0, 1]$ .

**Theorem 2.2.** *Let  $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function and assume (1.4), (2.2), (2.3), (2.4), (2.5) and (2.6) are satisfied. In addition assume*

$$(2.43) \quad \int_0^1 v(t)\varphi_m(t) dt = 0 \quad \text{and} \quad a \in L^1[0, 1],$$

$$(2.44) \quad h(t, u) \leq 0 \quad \text{for a.e. } t \in [0, 1], u > 0$$

and

$$(2.45) \quad h(t, u) \geq 0 \quad \text{for a.e. } t \in [0, 1], u < 0$$

hold. Then (2.1) has a solution.

**Proof.** Suppose  $y$  is a solution to (2.9) $_{\delta}$ . Follow the argument in theorem 2.1 to equation (2.32). In this case (since (2.43) is true) we have

$$\begin{aligned} (2.46) \quad 0 &= - \int_0^1 r_{1,n} h(t, y_n) dt + \int_0^1 r_{1,n} y_n g(t, y_n) dt \\ &\quad + \frac{(1 - \delta_n)(\mu - \lambda_m)}{\delta_n (\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{1}{2}}} \int_0^1 a u_{1,n}^2 dt. \end{aligned}$$

Now [1,4 section 4] implies  $\varphi_m$  has a finite number of zeros so

$$(2.47) \quad 0 > \int_{I^+} (r_{1,n} y_n g(t, y_n) + r_{1,n}[-h(t, y_n)]) dt \\ + \int_{I^-} (r_{1,n} y_n g(t, y_n) + r_{1,n}[-h(t, y_n)]) dt.$$

We now claim that for  $t \in I^+$  there exists an integer  $n_1$  (independent of  $t$ ) with

$$(2.48) \quad r_{1,n}(t) > 0 \quad \text{and} \quad y_n(t) > 0 \quad \text{for} \quad n \geq n_1 \quad \text{and} \quad n \in S.$$

Similarly we claim for  $t \in I^-$  there is a integer  $n_2$  (independent of  $t$ ) with

$$(2.49) \quad r_{1,n}(t) < 0 \quad \text{and} \quad y_n(t) < 0 \quad \text{for} \quad n \geq n_2 \quad \text{and} \quad n \in S.$$

To see the first part of (2.48) let  $r_{1,n} = \beta_{1,n} \varphi_m$ . Fix  $t_1 \in I^+$ . Then (2.30) implies that there is an integer  $m_1$  (i.e  $m_1(t_1)$ ) with

$$r_{1,n}(t_1) \geq \frac{1}{2} A \varphi_m(t_1) (\|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2)^{\frac{1}{2}} \quad \text{for} \quad n \geq m_1.$$

Thus  $\frac{\beta_{1,n}}{A} > 0$  for  $n \geq m_1$  since  $\frac{\beta_{1,n}}{A} A \varphi_m(t_1) = r_{1,n}(t_1) > 0$  for  $n \geq m_1$ . Consequently for any  $s \in I^+$  we have  $r_{1,n}(s) = \frac{\beta_{1,n}}{A} A \varphi_m(s) > 0$  for  $n \geq m_1$ . The proof of the second part of (2.48) is more involved. Notice (2.25), (2.27), (2.31) and the ideas used in (2.19), (2.20) and (2.21) immediately guarantee the existence of  $\omega \in L^1[0, 1]$  ( $\omega$  independent of  $n$ ) with, for  $n$  sufficiently large,

$$(2.50) \quad |r''_n(t)| \leq \omega(t) \quad \text{for a.e.} \quad t \in [0, 1].$$

**Remark.**  $a \in L^1[0, 1]$  is needed to guarantee  $\omega \in L^1[0, 1]$ .

Notice also [10] since  $a \in L^1[0, 1]$  that  $\varphi_m \in C^1[0, 1]$ . Now (2.50) implies (see also (2.31)) that there is a subsequence  $S^*$  of integers with

$$(2.51) \quad r_n \rightarrow A \varphi_m \quad \text{in} \quad C^1[0, 1] \quad \text{as} \quad n \rightarrow \infty \quad \text{in} \quad S^*.$$

Notice  $\varphi'_m(0) \neq 0$  (i.e. if  $\varphi'_m(0) = 0$  then  $\varphi''_m = -\lambda_m a \varphi_m$ ,  $\varphi_m(0) = \varphi'_m(0) = 0$  which implies  $\varphi_m \equiv 0$ , a contradiction). Also we know  $\varphi_m$  has a finite number of zero's in  $(0, 1)$ ; let  $s_i$  denote these zero's and let  $s_0$  be the smallest one. Without loss of generality assume  $A \varphi_m > 0$  on  $(0, s_0)$  with  $\varphi_m(0) = \varphi_m(s_0) = 0$ . Since  $\varphi''_m = -\lambda_m a \varphi_m$  a.e. then  $A \varphi'_m(0) > 0$ . Thus there exists  $t_1 \in (0, s_0)$  with  $A \varphi'_m(t) \geq A \frac{1}{2} \varphi'_m(0)$  for  $t \in [0, t_1]$ . Also (2.51) implies that there is an integer  $k_1$  with  $r'_n(t) -$

$A\varphi'_m(t) > -\frac{1}{4}A\varphi'_m(0)$  for  $n \geq k_1$ . Thus  $r'_n(t) > \frac{1}{4}A\varphi'_m(0)$  for  $n \geq k_1$  and  $t \in [0, t_1]$ . Consequently

$$\begin{aligned} y_n(t) &= \int_0^t y'_n(s) \, ds \\ &= \int_0^t r'_n(s) \, ds \left( \|u_{1,n}\|_a^2 + \|u'_{1,n}\|^2 \right)^{\frac{1}{2}} > 0 \text{ for } n \geq k_1 \text{ and } t \in (0, t_1]. \end{aligned}$$

Similarly there exists  $t_2$  with  $t_1 < t_2 < s_0$  and there exists an integer  $k_2$  with  $y_n(t) > 0$  for  $n \geq k_2$  and  $t \in [t_2, s_0]$  (note in this case for a fixed  $t_3 \in (t_2, s_0)$  then (2.41) implies that  $y_n(t_3) > 0$  for  $n$  sufficiently large; now consider  $t \in [t_2, t_3)$  and  $t \in (t_3, s_0)$ ). Finally since  $r_n \rightarrow A\varphi_m$  in  $C[0, 1]$  as  $n \rightarrow \infty$  in  $S^*$  there exists (since  $\min_{[t_1, t_2]} A\varphi_m(t) > 0$ ) an integer  $k_3$  with  $y_n(t) > 0$  for  $n \geq k_3$  and  $t \in [t_1, t_2]$ . Consequently for  $n \geq \max\{k_1, k_2, k_3\}$  we have  $y_n(t) > 0$  on  $(0, s_0)$ . Since  $\varphi_m$  has only a finite number of zero's  $s_i$  in  $(0, 1)$  then there exists an integer  $n_1$  with  $y_n(t) > 0$  for  $n \geq n_1$  and  $t \in I^+$ , so (2.48) is true. A similar argument shows (2.49) is also true.

Now (2.44), (2.45), (2.48) and (2.49) yield for  $n \in S^*$  and  $n \geq \max\{n_1, n_2\}$ ,

$$\begin{aligned} 0 &\leq \int_{I^+} (r_{1,n} y_n g(t, y_n) + r_{1,n} [-h(t, y_n)]) \, dt \\ &\quad + \int_{I^-} (r_{1,n} y_n g(t, y_n) + r_{1,n} [-h(t, y_n)]) \, dt. \end{aligned}$$

This contradicts (2.47). □

The above results have “dual” versions. We will just give the dual version of theorem 2.1.

**Theorem 2.3.** *Let  $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function and assume (1.4), (2.2), (2.3), (2.5) and (2.6) hold. In addition suppose*

(2.52) *there exists  $\tau \in C[0, 1]$  with  $a\tau \in L^1[0, 1]$  and  $-\tau(t)a(t) \leq g(t, u) \leq 0$  for a.e.  $t \in [0, 1]$  and  $u \in \mathbb{R}$ ; here  $\tau(t) \leq \lambda_m - \lambda_{m-1}$  for a.e.  $t \in [0, 1]$  with  $\tau(t) < \lambda_m - \lambda_{m-1}$  on a subset of  $[0, 1]$  of positive measure*

*is satisfied.*

(i) *Suppose there exists a constant  $k > \gamma$  with  $1 > k = \frac{\alpha}{\beta}$ , where  $\beta$  is odd and  $\alpha$  is even, and*

$$\begin{aligned} (2.53) \quad 0 &> \int_{I^+} [A\varphi_m(t)]^{k+1} \limsup_{x \rightarrow \infty} (x^{1-k} g(t, x)) \, dt \\ &\quad + \int_{I^-} [A\varphi_m(t)]^{k+1} \liminf_{x \rightarrow -\infty} (x^{1-k} g(t, x)) \, dt \end{aligned}$$

for all constants  $A \neq 0$ ; here  $I^+ = \{t \in [0, 1]: A\varphi_m(t) > 0\}$  and  $I^- = \{t: A\varphi_m(t) < 0\}$ .

Then (2.1) has at least one solution.

(ii) Suppose  $\gamma = 0$  and

$$(2.54) \quad \begin{aligned} A \int_0^1 v(t)\varphi_m(t) dt &> A \int_{I^+} \varphi_m(t) \limsup_{x \rightarrow \infty} (xg(t, x)) dt \\ &+ A \int_{I^+} \varphi_m(t) \limsup_{x \rightarrow \infty} [-h(t, x)] dt \\ &+ A \int_{I^-} \varphi_m(t) \liminf_{x \rightarrow -\infty} (xg(t, x)) dt \\ &+ A \int_{I^-} \varphi_m(t) \liminf_{x \rightarrow -\infty} [-h(t, x)] dt \end{aligned}$$

for all constants  $A \neq 0$ .

Then (2.1) has at least one solution.

*Proof.* In this case choose  $\mu$  such that  $\lambda_{m-1} < \mu < \lambda_m$  and suppose  $y$  is a solution to

$$\begin{aligned} y'' + \mu a y &= \delta[h(t, y) + v(t) - y g(t, y) + (\mu - \lambda_m) a y] \quad \text{a.e. on } [0, 1], \\ y(0) &= y(1) = 0 \end{aligned}$$

for  $\delta \in (0, 1)$ . Let

$$w = \sum_{i=m}^{\infty} c_i \varphi_i, \quad u = \sum_{i=0}^{m-1} c_i \varphi_i, \quad w_0 = \sum_{i=m+1}^{\infty} c_i \varphi_i, \quad \text{and} \quad w_1 = c_m \varphi_m$$

in this case. The same type of analysis as that in theorem 2.1 establishes the result.  $\square$

**Example.** Let  $f: [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  and  $v, a: [0, \infty) \rightarrow \mathbb{R}$ . Suppose  $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function with (2.2) holding. In addition assume (1.4) and (2.3) hold and also that  $f$  has the decomposition  $f(t, u) = h(t, u) - y g(t, u)$ . Now suppose (2.4), (2.5) and (2.6) are satisfied and that there exists a constant  $k > \gamma$ , where  $1 > k = \frac{\alpha}{\beta}$  with  $\beta$  odd and  $\alpha$  even, with (2.7) holding. Finally assume

$$(2.55) \quad f(t, 0) + v(t) = 0 \quad \text{and} \quad a(t) \in \mathbb{R} \quad \text{for a.e. } t \geq 1$$

is satisfied. Then

$$(2.56) \quad \begin{aligned} y'' + \lambda_m a y &= f(t, y) + v(t) \quad \text{a.e. on } [0, \infty), \\ y(0) &= y(\infty) = 0 \end{aligned}$$

has a solution in  $C[0, \infty)$ .

To see this notice theorem 2.1 (i) guarantees that

$$\begin{aligned}y'' + \lambda_m a y &= f(t, y) + v(t) \quad \text{a.e. on } [0, 1], \\y(0) &= y(1) = 0\end{aligned}$$

has a solution  $y \in C[0, 1]$  (the other smoothness properties in theorem 2.1 (i) also hold). Let

$$y^* = \begin{cases} y, & 0 \leq t \leq 1, \\ 0, & t \geq 1. \end{cases}$$

Notice  $y^* \in C[0, \infty)$  and  $y^*$  satisfies  $y'' + \lambda_m a y = f(t, y) + v(t)$  a.e. on  $[0, \infty)$ .

**Remark.** We can also obtain analogue results for (2.56) when the conditions of theorem 2.1 (ii) or theorem 2.2 are satisfied.

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