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RELATIVELY COARSE SEQUENTIAL CONVERGENCE

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Abstract. We generalize the notion of a coarse sequential convergence compatible with an algebraic structure to a coarse one in a given class of convergences. In particular, we investigate coarseness in the class of all compatible convergences (with unique limits) the restriction of which to a given subset is fixed. We characterize such convergences and study relative coarseness in connection with extensions and completions of groups and rings. E.g., we show that: (i) each relatively coarse dense group precompletion of the group of rational numbers (equipped with the usual metric convergence) is complete; (ii) there are exactly $\exp\exp\omega$ such completions; (iii) the real line is the only one of them the convergence of which is Fréchet. Analogous results hold for the relatively coarse dense field precompletions of the subfield of all complex numbers both coordinates of which are rational numbers.

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This is a continuation of the series of our papers on coarse convergence FRIČ and ZANOLIN [1985], [1990], [19∞], DIKRANJAN, FRIČ and ZANOLIN [1987], SIMON and ZANOLIN [1987], FRIČ [1988], where the background information and the undefined notions can be found. Coarse vector spaces were investigated in JAKUBÍK [1956].

As a rule, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the real numbers, rational numbers, integers and natural numbers (positive integers), respectively, MON denotes the strictly monotone maps of $\mathbb{N}$ into $\mathbb{N}$, if $S = \langle S(n)\rangle \in X^{\mathbb{N}}$ is a sequence of points of X and $s \in$MON, then $S \circ s = \langle S(s(n))\rangle$ denotes the corresponding subsequence of S and if X is equipped with an algebraic structure, then operations in $X^{\mathbb{N}}$ are defined pointwise.

By a sequential convergence, or simply convergence, we understand an FSH-convergence (usual axioms on subsequences, constants and uniqueness of limits), U stands for the Urysohn axiom and L for the compatibility of the convergence and the algebraic operations, e.g., if X is a ring and $\mathbb{L} \subset X^{\mathbb{N}} \times X$ is a compatible FLSH-convergence, then $(S,x),(T,y) \in \mathbb{L}$ implies $(S-T,x-y),(ST,xy) \in \mathbb{L}$. Throughout the paper we consider only commutative groups and rings and a ring need not have a unit.

Let $\mathbb{L} \subset X^{\mathbb{N}} \times X$ be a convergence on X and let A be a subset of X. Then $\mathbb{L} \restriction A = \{(S,x) \in \mathbb{L};\ S \in A^{\mathbb{N}}, x \in A\}$, the more exact notation should be $\mathbb{L} \restriction (A^{\mathbb{N}} \times A)$. Indeed, when studying relative coarseness it is useful to define $\mathbb{L} \restriction (A^{\mathbb{N}} \times B) = \{(S,x) \in \mathbb{L};\ S \in A^{\mathbb{N}}, x \in B\}$, $A, B \subset X$.

Definition 0.1. Let $(X', \mathbb{L}')$ be an FLSH-group (-ring, -field, -vector space), let X be a subgroup (subring, subfield, vector subspace) of X' and let $\mathbb{L} = \mathbb{L}' \restriction X$. Then $(X', \mathbb{L}')$ is said to be an *extension* of $(X, \mathbb{L})$. If X is topologically dense (iterated closure) in $(X', \mathbb{L}')$ and each $\mathbb{L}$-Cauchy sequence $\mathbb{L}'$-converges, then $(X', \mathbb{L}')$ is said to be a *precompletion* of $(X, \mathbb{L})$; if X is dense (first closure) in $(X', \mathbb{L}')$, then we speak of a *dense precompletion.* If a (dense) precompletion is complete, then we speak of a (dense) *completion*, if $\mathbb{L}'$ is $X-$coarse (see Definiton 1.1), then $(X', \mathbb{L}')$ is said to be a *coarse extension, (dense) precompletion* or *completion*, respectively.

Remark 0.2. Let $(X, \mathbb{L})$ be a FLUSH-group and let $(X', \mathbb{L}')$ be its Novák (i.e. categorical) FLUSH-group completion (cf. NOVÁK [1972], FRIČ and KOUTNÍK [1989]). Observe that if $(X'', \mathbb{L}'')$ is a dense precompletion of $(X, \mathbb{L})$, then there exists a continuous isomorphism of $(X', \mathbb{L}')$ onto $(X'', \mathbb{L}'')$ leaving X pointwise fixed. Hence we can identify X' and X'' and consider $\mathbb{L}''$ to be a convergence on X' coarser than $\mathbb{L}'$. Analogous convention will be used for rings.

Remark 0.3. In FRIČ and ZANOLIN [1992] and in other papers on completions of groups and rings the $\mathcal{L}$-notation instead of the FLS-notation is used, e.g. $\mathcal{L}_0^*$-group is a FLUSH-group.

1. RELATIVELY COARSE CONVERGENCE

Definition 1.1. Let X be a group (ring, field, vector space) and let Λ be a class of FLSH-group (-ring, -field, -vecotr space) convergences on X. We say that $\mathbb{L} \in \Lambda$ is *coarse in* Λ if no element in Λ is strictly coarser than $\mathbb{L}$. Let Y be a subgroup (subring, subfield, vector subspace) of X and let $\mathbb{L}$ be an FLSH-group (-ring, -field, -vector space) convergence on X. If Λ is the class of all FLSH-group (-ring, -field, -vector space) convergences $\mathbb{L}'$ on X such that $\mathbb{L}' \restriction Y = \mathbb{L} \restriction Y$ and $\mathbb{L}$ is coarse in Λ, then $\mathbb{L}$ is said to be $\mathbb{L} \restriction Y$*-coarse*, or simply Y-coarse.

Example 1.2. (i) If X is a group and Y is the trivial subgroup of X, then Y-coarseness becomes the usual coarseness (cf. FRIČ-ZANOLIN [1990]).

(ii) Consider the real line (with the usual metric convergence). Since no unbounded sequence of real numbers can converge in any ring FLSH-convergence coarser than the metric one (cf. FRIČ [1989]) and every bounded sequence of real numbers already contains a subsequence converging in the real line, the metric convergence is a $\mathbb{Q}$-coarse FLUSH-ring convergence on R. Later we shall prove (see Corollary 2.1.6) that it is a $\mathbb{Q}$-coarse group convergence on R. This is not quite obvious since the metric convergence on R fails to be a coarse group FLSH-convergence (cf. FRIČ [1989])

Theorem 1.3. *Let $(X, \mathbb{L})$ be an FLSH-group (-ring, -field, -vector space) and let Y be a subgroup (subring, subfield, vector subspace) of X. Then there is a Y-coarse convergence $\mathbb{L}_Y$ on X such that $\mathbb{L} \subset \mathbb{L}_Y$.*

Proof. Let Λ be the set of all FLSH-group (-ring, -field, -vector space) convergences $\mathbb{L}'$ on X such that $\mathbb{L} \restriction Y = \mathbb{L}' \restriction Y$ and $\mathbb{L} \subset \mathbb{L}'$. Let $\{\mathbb{L}_a ; a \in A\}$ be a chain in Λ (partially ordered by inclusion). Since $\bigcup_{a \in A} \mathbb{L}_a$ is clearly an element of Λ, the existence of a Y-coarse convergence $\mathbb{L}_Y$ on X such that $\mathbb{L} \subset \mathbb{L}_Y$ follows from the Kuratowski-Zorn lemma. □

The next lemma explicitly describes some trivial properties of Cauchy sequences with respect to an extension. The proof is omitted.

Lemma 1.4. (i) *Let $(X', \mathbb{L}')$ be an extension of $(X, \mathbb{L})$. If $S \in X^{\mathbb{N}}$ and $(S, x) \in \mathbb{L}'$, then S is an $\mathbb{L}$-Cauchy sequence.*

(ii) *Let $(X', \mathbb{L}')$ be a precompletion of (X, L) and let $\mathbb{L}'_X$ be an X-coarse convergence on X' such that $\mathbb{L}' \subset \mathbb{L}'_X$. Then $\mathbb{L}'_X \restriction (X^{\mathbb{N}} \times X') = \mathbb{L}' \restriction (X^{\mathbb{N}} \times X')$.*

2. GROUPS

2.1. Characterization. Let $(X, \mathbb{L})$ be an FLSH-group. A sequence $S \in X^{\mathbb{N}}$ is called $\mathbb{L}$-free at $x \in X$ if $(S, x) \notin \mathbb{L}$ and there is an FLSH-group convergence $\mathbb{L}'$ on X such that $\mathbb{L} \subset \mathbb{L}'$ and $(S, x) \in \mathbb{L}'$ (cf. FRIČ [1990]). Clearly, S is $\mathbb{L}$-free at x iff $S - \langle x \rangle$ is $\mathbb{L}$-free at 0. Thus $\mathbb{L}$ is coarse iff no $S \in X^{\mathbb{N}}$ is $\mathbb{L}$-free at 0. This is exactly iff for each $S \in X^{\mathbb{N}}$, $(S, 0) \notin \mathbb{L}$, the smallest subgroup of $X^{\mathbb{N}}$ closed with respect to subsequences and containing $\mathbb{L}^{\leftarrow}(0)$ and S (zero sequences of the generated FLS-group) contains a constant nonzero sequence. This leads to the known Coarseness Criterion for FLSH-groups. Since the Urysohn modification $\mathbb{L}^*$ of $\mathbb{L}$ is an FLHS-group convergence coarser than $\mathbb{L}$, we have $\mathbb{L} = \mathbb{L}^*$ whenever $\mathbb{L}$ is coarse (cf. FRIČ and ZANOLIN [1985]). Let Y be a subgroup of X. Consider $(X, \mathbb{L})$ as an extension of

$(Y, \mathbb{L} \restriction Y)$. An analogous reasoning leads us to the following Extension Coarseness Criterion. We leave out the obvious proof.

Theorem 2.1.1. *Let $(X, \mathbb{L})$ be an* FLSH-*group and let Y be a subgroup of X. Then $\mathbb{L}$ is Y-coarse iff*

(CE) *For each $S \in X^{\mathbb{N}}$ any of the following three conditions holds:*

(CE1) $(S, 0) \in \mathbb{L}$;

(CE2) $\langle p \rangle = \sum_{i=1}^{m} \langle z_i \rangle S \circ s_i + T$, *where* $p \in X$, $p \neq 0$, $m \in \mathbb{N}$, $z_i \in \mathbb{Z} \setminus \{0\}$, $s_i \in MON$, $i = 1, \dots, m$, $T \in X^{\mathbb{N}}$, $(T, 0) \in \mathbb{L}$;

(CE3) $U = \sum_{i=1}^{m} \langle z_i \rangle S \circ s_i + T$, *where* $U \in Y^{\mathbb{N}}$, $(U, 0) \notin \mathbb{L}$, $m \in \mathbb{N}$, $z_i \in \mathbb{Z} \setminus \{0\}, s_i \in MON$, $i = 1, \dots, m$, $T \in X^{\mathbb{N}}$, $(T, 0) \in \mathbb{L}$.

Remark 2.1.2. If in Theorem 2.1.1 $\mathbb{L} = \mathbb{L}^*$, then (CE1) can be replaced by a weaker condition

(CE1U) $(S \circ s, 0) \in \mathbb{L}$ for some $s \in$ MON.

Corollary 2.1.3. *Let $(X, \mathbb{L})$ be a* FLUSH-*group and let Y be a dense subgroup of $(X, \mathbb{L})$. Then $\mathbb{L}$ is coarse iff*

(CDEU) *Each $S \in X^{\mathbb{N}}$ satisfies either* (CE1U) *or* (CE3).

P r o o f. Since Y is dense in $(X, \mathbb{L})$, for $p \in X \setminus \{0\}$ there exists $U \in Y^{\mathbb{N}}$ such that $(U, p) \in \mathbb{L}$. Condition (CE2) transforms into (CE3) by putting $\langle p \rangle = U + \langle p \rangle - U$. □

Corollary 2.1.4. *Let $(X, \mathbb{L})$ be a* FLUSH-*group and let Y be a dense subgroup of $(X, \mathbb{L})$. Assume that for each $S \in X^{\mathbb{N}}$ either $(S \circ s, 0) \in \mathbb{L}$ for some $s \in$* MON, *or there are $t \in$* MON *and $U \in Y^{\mathbb{N}}$ such that $(S \circ t - U, 0) \in \mathbb{L}$. Then $\mathbb{L}$ is Y-coarse.*

P r o o f. It follows easily that condition (CDEU) in Corollary 2.1.3 holds true. (Hint: if $(S \circ t, 0) \notin \mathbb{L}$, then $(U, 0) \notin \mathbb{L}$.) □

Corollary 2.1.5. *Let $(X, \mathbb{L})$ be a Fréchet* FLUSH-*group and let Y be a dense subgroup of $(X, \mathbb{L})$. Then $\mathbb{L}$ is Y-coarse.*

P r o o f. We prove that the assumptions of Corollary 2.1.4 are satisfied. Let $S \in X^{\mathbb{N}}$. If $(S \circ s, 0) \in \mathbb{L}$ for some $s \in$ MON, then the assertion is trivial. Let $(S \circ s, 0) \notin \mathbb{L}$ for all $s \in$ MON. Since Y is dense, for each $k \in \mathbb{N}$, there exists $U_k \in Y^{\mathbb{N}}$ such that $(U_k, S(k)) \in \mathbb{L}$, and hence $(U_k - S(k), 0) \in \mathbb{L}$. Since $(X, \mathbb{L})$ is Fréchet, there are $t \in$ MON and $f \in \mathbb{N}^{\mathbb{N}}$ such that $(\langle U_{t(n)}(f(t(n))) - S(t(n)) \rangle, 0) \in \mathbb{L}$. Now it suffices to put $\langle U(n) \rangle = \langle U_{t(n)}(f(t(n))) \rangle$. □

Corollary 2.1.6. *The usual metric convergence on R is $\mathbb{Q}$-coarse.*

Remark 2.1.7. Observe that the usual metric convergence on R fails to be coarse (in the class of all FLSH-group convergences). Indeed the sequence $\langle 2^n \rangle$ does not satisfy the Coarseness Criterion for FLSH-groups.

2.2. Double embedding. In DIKRANJAN, FRIČ and ZANOLIN [1987] a necessary and sufficient condition has been given for a dense subgroup of a coarse FLUSH-group to be coarse (the so-called Density Criterion). In this section we prove an analogous statement for relative coarseness.

Theorem 2.2.1. *Let $(X, \mathbb{L})$ be an* FLSH-*group, let $(X', \mathbb{L}')$ be its extension and let Y be a subgroup of X. Let the $\mathbb{L}'$-closure of Y be a subset of X and let X be dense in $(X', \mathbb{L}')$. Let $\mathbb{L}$ be Y-coarse. Then each nontrivial subgroup of X' intersects X nontrivially.*[1]

Proof. Let $x \in X' \setminus X$. Then there exists $S \in X^{\mathbb{N}}$ such that $(S, x) \in \mathbb{L}'$. Thus $(S, 0) \notin \mathbb{L}$. Hence S satisfies either (CE2) or (CE3). Passing to $\mathbb{L}'$-limits in the two conditions, we get $x \sum_{i=1}^{m} z_i \in X \setminus \{0\}$, i.e. the subgroup of X' generated by x intersects X nontrivially. This completes the proof. □

Theorem 2.2.2. *Let $(X, \mathbb{L})$ be a coarse dense* FLUSH-*group precompletion of its subgroup $(Y, \mathbb{L} \restriction Y)$. If X is divisible, then $(X, \mathbb{L})$ is complete.*

Proof. Let $(X', \mathbb{L}')$ be the Novák FLUSH-group completion of $(X, \mathbb{L})$. Since the assumptions of Theorem 2.2.1 are satisfied, each nontrivial subgroup of X' intersects X nontrivially. Now we proceed as in the proof of Proposition 3.1 in DIKRANJAN, FRIČ and ZANOLIN [1987]. Suppose that, on the contrary $x \in X' \setminus X$. The subgroup of X' generated by x intersects X nontrivially, hence there exists $k \in \mathbb{N}$ such that $kx \in X \setminus \{0\}$. Without loss of generality, we may assume that k is prime. (Indeed, if $k = p_1 p_2 \ldots p_n$ with each p_i prime, $i = 1, \ldots, n$, take the least $m \leqslant n$ such that $p_1 p_2 \ldots p_m x \in X$. Then $x_1 = p_1 p_2 \ldots p_{m-1} x \notin X$ and $p_m x_1 \in X$ (for $m = 1$ set $p_o = 1$), so we can take x_1 instead of x.) Since X is divisible, we have $kX = X$. Consequently, for some $y \in X$ we have $ky = kx$. Now, consider the cyclic subgroup X_0 of X' generated by $x - y$. Since $k(x - y) = 0$ and k is prime, X_0 is simple. From $X_0 \cap X \neq \{0\}$ we get $X_0 \subset X$. Thus $x - y \in X$ and hence also $x \in X$, a contradiction. Thus $X' = X$ and $(X, \mathbb{L})$ is complete. □

[1] In the terminology used in DIKRANJAN, FRIČ and ZANOLIN [1987], X is algebraically essential in X'.

Theorem 2.2.3. *Let $(X, \mathbb{L})$ be an FLSH-group, let $(X', \mathbb{L}')$ be its extension and let Y be a subgroup of X. Let the $\mathbb{L}'$-closure of Y be a subset of X and let X be dense in $(X', \mathbb{L}')$. Let $\mathbb{L}'$ be Y-coarse. Let each nontrivial subgroup of X' intersect X nontrivially. Then $\mathbb{L}$ is Y-coarse.*

Proof. According to Theorem 2.1.1, it suffices to prove that (CE) holds true. Let $S \in X^{\mathbb{N}}$. Let $(S, 0) \notin \mathbb{L}$. Then $(S, 0) \notin \mathbb{L}'$. Since $\mathbb{L}'$ is Y-coarse, there are $m \in \mathbb{N}$, $z_i \in \mathbb{Z} \setminus \{0\}$, $s_i \in \mathrm{MON}$, $i = 1, \dots, m$, and $T \in (X')^{\mathbb{N}}$, $(T, 0) \in \mathbb{L}'$, such that $\sum_{i=1}^{m} \langle z_i \rangle S \circ s_i + T$ is equal either to $\langle p \rangle, p \in X' \setminus \{0\}$, or to $U \in Y^{\mathbb{N}}$, $(U, 0) \notin \mathbb{L}$. In the first case, there exists $z \in \mathbb{Z} \setminus \{0\}$ such that $zp \in X \setminus \{0\}$ and hence $\langle zp \rangle = \sum_{i=1}^{m} \langle zz_i \rangle S \circ s_i + \langle z \rangle T$, where $\langle z \rangle T \in X^{\mathbb{N}}$ and hence $(\langle z \rangle T, 0) \in \mathbb{L}$. In the second case, $U = \sum_{i=1}^{m} \langle z_i \rangle S \circ s_i + T$, where $T \in X^{\mathbb{N}}$ and hence $(T, 0) \in \mathbb{L}$. Thus (CE) is satisfied and the proof is complete. □

Combining Theorem 2.2.1 and Theorem 2.2.3 we get the following Relative Coarseness Density Criterion.

Corollary 2.2.4. *Let $(X, \mathbb{L})$ be an FLSH-group, let $(X', \mathbb{L}')$ be its extension and let Y be a subgroup of X. Let the $\mathbb{L}'$-closure of Y be a subset of X and let X be dense in $(X', \mathbb{L}')$. Let $\mathbb{L}'$ be Y-coarse. Then $\mathbb{L}$ is Y-coarse iff each nontrivial subgroup of X' intersects X nontrivially.*

2.3. Nonstrict completions of $\mathbb{Q}$. Throughout this Section let $\mathbb{M}$ be the usual metric convergence on R and let $\mathbb{L} = \mathbb{M} \restriction \mathbb{Q}$. Let $(R, \mathbb{L}_1^*)$ be the Novák FLUSH-group completion of $(\mathbb{Q}, R)$. In FRIČ and ZANOLIN [1992] it was shown that $(\mathbb{Q}, \mathbb{L})$ has exactly $\exp\exp\omega$ nonhomeomorphic strict FLUSH-group completions $(R, \mathbb{L}_A^*)$, where stricness means that $\mathbb{L}_1^* \subset \mathbb{L}_A^* \subset \mathbb{M}$. In this section we show that $(\mathbb{Q}, \mathbb{L})$ has also $\exp\exp\omega$ nonhomeomorphic FLUSH-group completions of the form $(R, \mathbb{L}'_f)$, where f is a mapping of $\exp\omega$ into $\{0, 1\}$, such that $\mathbb{L}'_f \setminus \mathbb{M} \neq \emptyset$, i.e. $(R, \mathbb{L}'_f)$ fails to be strict, and $\mathbb{L}'_f$ is coarse. Since (up to a homeomorphism pointwise fixed on $\mathbb{Q}$) each dense coarse completion of $(\mathbb{Q}, \mathbb{L})$ is of the form $(R, \mathbb{L}')$ with $\mathbb{L}' \subset R^{\mathbb{N}} \times R$, $(\mathbb{Q}, \mathbb{L})$ has at most $\exp\exp\omega$ nonhomeomorphic coarse dense completions. We also show that, (up to a homeomorphism pointwise fixed on $\mathbb{Q}$) $(R, \mathbb{M})$ is the only coarse dense FLUSH-group completion of $(\mathbb{Q}, \mathbb{L})$ the convergence of which is Fréchet.

Consider R as a vector space over the scalar field $\mathbb{Q}$. Let $\{1\} \cup B$ be a Hamel basis such that $2 < b < 3$ for all $b \in B$ and let $\{S_\alpha ; \alpha \in \exp\omega\}$ be a partition of B into disjoint infinite countable subsets S_α of B. Consider each S_α as a one-to-one sequence. Let f be a mapping of $\exp\omega$ into $\{0, 1\}$.

Lemma 2.3.1. *There is an* FLSH-*group convergence* $\mathbb{L}_f$ *on* R *such that* $\mathbb{L}_1^* \subset \mathbb{L}_f$, $(S_\alpha, f(\alpha)) \in \mathbb{L}_f$ *for all* $\alpha \in \exp\omega$, *and* $\mathbb{L}_f \restriction (\mathbb{Q}^{\mathbb{N}} \times R) = \mathbb{M} \restriction (\mathbb{Q}^{\mathbb{N}} \times R)$.

Proof. Denote $\mathbb{L}_f$ the smallest of all FLS-group convergences on R containing $\mathbb{L}_1^*$ and $(S_\alpha, f(\alpha))$ for all $\alpha \in \exp\omega$ (cf. Theorem 3.3 in FRIČ and ZANOLIN [1986]). Recall that $\mathbb{L}_f^{\leftarrow}(0)$ is the smallest subgroup of $R^{\mathbb{N}}$ closed with respect to subsequences and containing zero sequences of $\mathbb{L}_1^*$ and $(S_\alpha - \langle f(\alpha)\rangle)$ for all $\alpha \in \exp\omega$. More explicitly, a sequence $S \in R^{\mathbb{N}}$ $\mathbb{L}_f$-converges to 0 iff it can be written in the following form

(1) $\langle r - r_n\rangle + \sum_{i=1}^{m(1)} \langle z(1,i)\rangle(S_{\alpha(1)} \circ s(1,i) - \langle f(\alpha(1))\rangle) + \ldots + \sum_{i=1}^{m(1)} \langle z(k,i)\rangle(S_{\alpha(k)} \circ s(k,i) - \langle f(\alpha(k))\rangle$, where $\langle r_n\rangle \in \mathbb{Q}^{\mathbb{N}}$ $\mathbb{M}$-converges to $r, k \in \mathbb{N}$, $m(j) \in \omega$, $z(j,i) \in Z \setminus \{0\}, s(j,i) \in \mathrm{MON}$, $\alpha(j) \in \exp\omega$, and $i = 1,\ldots,m(j)$, $j = 1,\ldots,k$ (as a rule, if $m(j) = 0$, then the sum $\sum_{i=1}^{m(j)} (\ldots)$ represents a zero sequence).

We shall show that $\mathbb{L}_f$ has the desired properties.

1. $\mathbb{L}_f$ has unique limits. This is equivalent to the fact that, besides $\langle 0\rangle$, no constant sequence in R is of the form (1). Contrariwise, suppose that for some $p \in R, p \neq 0, \langle p\rangle$ can be represented in the form (1). Passing to subsequences, if necessary, we can assume that for each $j = 1,\ldots,k$, the sets $\{(S_{\alpha(j)} \circ s(j,i))(n);\ n \in \mathbb{N}\}, i = 1,\ldots,m(j)$ are pairwise disjoint. Since all $(S_{\alpha(j)} \circ s(j,i))(n)$ are now different elements of B and since $\{1\} \cup \{S_\alpha(n) - f(\alpha);\ n \in \mathbb{N}, \alpha \exp\omega\}$ is a Hamel basis of R over $\mathbb{Q}$ as well, this would lead to a contradiction.

2. $\mathbb{L}_f \restriction (\mathbb{Q}^{\mathbb{N}} \times R) = \mathbb{M} \restriction (\mathbb{Q}^{\mathbb{N}} \times R)$. It is easy to see that no $\mathbb{M}$-unbounded sequence in $\mathbb{Q}$ is of the form (1). Further, each $\mathbb{M}$-bounded sequence in $\mathbb{Q}$ contains a subsequence $\mathbb{L}_1^*$-converging and hence also $\mathbb{L}_f$-converging in R. The assertion follows from the uniqueness of $\mathbb{L}_1$-limits. This completes the proof. □

Theorem 2.3.2. *There are* $\exp\exp\omega$ *nonhomeomorphic coarse* FLUSH-*group dense completions of* $(\mathbb{Q}, \mathbb{L})$.

Proof. In view of the introductory remarks of this section, it suffices to prove that for each mapping f of $\exp\omega$ into $\{0,1\}$ the corresponding convergence $\mathbb{L}_f$ can be erlarged to a $\mathbb{Q}$-coarse complete FLUSH-group convergence on R. Let $\mathbb{L}'_f$ be a $\mathbb{Q}$-coarse enlargement of $\mathbb{L}_f$ (cf. Theorem 1.3). Clearly, $\mathbb{L}'_f$ satisfies the Urysohn axiom. Since R is divisible, it follows from Theorem 2.2.2 that $(R, \mathbb{L}'_f)$ is a complete FLUSH-group. According to Lemma 1.4, $(R, \mathbb{L}'_f)$ is a completion of $(\mathbb{Q}, \mathbb{L})$. Finally, it is easy to see that if f and g are different mappings of $\exp\omega$ into $\{0,1\}$, then the corresponding completions $(R, \mathbb{L}'_f)$ and $(R, \mathbb{L}'_g)$ are nonhomeomorphic. This completes the proof. □

Consider any coarse dense completion of $(\mathbb{Q}, \mathbb{L})$, $\mathbb{L} = \mathbb{M} \upharpoonright \mathbb{Q}$. As already printed out, we can assume that the completion is of the form $(R, \mathbb{L}')$, where $\mathbb{L}'$ is $\mathbb{Q}$-coarse.

Lemma 2.3.3. *Let $(R, \mathbb{L}')$ be a Fréchet space. Then $\mathbb{L}' \subset \mathbb{M}$.*

Proof. Let $(S, x) \in \mathbb{L}'$. Then for each $k \in \mathbb{N}$ there exists $S_k \in \mathbb{Q}^{\mathbb{N}}$ such that $(S_k, S(k)) \in \mathbb{M} \cap \mathbb{L}'$. Since $(R, \mathbb{L}')$ is Fréchet, there exists a diagonal subsequence $S_f = \langle S_n(f(n))\rangle$, $f \in \mathbb{N}^{\mathbb{N}}$, a subsequence $S_f \circ s$, $s \in \mathrm{MON}$, of which $\mathbb{L}'$-converges to x. According to Lemma 1.4, $S_f \circ s$ is an $\mathbb{L}$-Cauchy and hence some of its subsequences $S_f \circ s \circ t$ $\mathbb{M}$-converges to x. From this it easily follows that $(S, x) \in \mathbb{M}$. □

Corollary 2.3.4. *$(R, \mathbb{M})$ is (up to a homeomorphism pointwise fixed on $\mathbb{Q}$) the only coarse dense* FLUSH-*group completion of* $(\mathbb{Q}, \mathbb{M} \upharpoonright \mathbb{Q})$ *the convergence of which is Fréchet.*

3. Rings

3.1. Characterization. The necessary and sufficient condition (CR) for an FLSH-ring convergence to be coarse (in the class of all compatible FLSH-ring convergences) is given in Theorem 3.1 in Frič and Zanolin [1990]. The next theorem provides a characterization of a relatively coarse convergence. We omit its straightforward proof.

Theorem 3.1.1. *Let $(X, \mathbb{L})$ be an* FLSH-*ring and let Y be a subring of X. Then $\mathbb{L}$ is Y-coarse iff*

(RCR) *For each $S \in X^{\mathbb{N}}$ any of the following three conditions holds:*

(RCR1) $(S, 0) \in \mathbb{L}$;

(RCR2) $\langle p\rangle = \sum_{i=1}^{m} T(i,1)\dots T(i,k(i))$, *where $p \in X$, $p \neq 0$, $m, k(i) \in \mathbb{N}$, and for each $i = 1, \dots, m$ and each $j = 1, \dots, k(i)$ we have either $T(i,j) = \pm S \circ s$ or $T(i,j) = \langle a\rangle S \circ s$, $s \in \mathrm{MON}$, $a \in X$, or $(T(i,j), 0) \in \mathbb{L}$;*

(RCR3) $U = \sum_{i=1}^{m} T(i,1)\dots T(i,k(i))$, *where $U \in Y^{\mathbb{N}}$, $(U, 0) \notin \mathbb{L}$ and $\sum_{i=1}^{m} (T(i,1)\dots T(i,k(i))$ is as in* (RCR2).

Corollary 3.1.2. *Let $(X, \mathbb{L})$ be an* FLSH-*ring and let Y be a dense subring of $(X, \mathbb{L})$. Then $\mathbb{L}$ is Y-coarse iff*

(RDCR) *Each $S \in X^{\mathbb{N}}$ satisfies either* (RCR1) *or* (RCR3).

P r o o f. Follows easily from Theorem 3.1.1. □

Corollary 3.1.3. *Let $(X, \mathbb{L})$ be a* FLUSH-*ring and let Y be a dense subring of $(X, \mathbb{L})$. Assume that for each $S \in X^{\mathbb{N}}$ either $(S \circ s, 0) \in \mathbb{L}$ for some $s \in$* MON, *or there are $t \in$* MON *and $U \in Y^{\mathbb{N}}$ such that $(S \circ t - U, 0) \in \mathbb{L}$. Then $\mathbb{L}$ is Y-coarse.*

P r o o f. Since each FLUSH-ring convergence is a FLUSH-group convergence, the assertion follows easily from Corollary 2.1.4. □

Corollary 3.1.4. *Let $(X, \mathbb{L})$ be a Fréchet* FLUSH-*ring and let Y be a dense subring of $(X, \mathbb{L})$. Then $\mathbb{L}$ is Y-coarse.*

P r o o f. The assertion follows easily from Corollary 2.1.5. □

Remark 3.1.5. Consider the FLUSH-field $(R, \mathbb{M})$ and its dense subfield $\mathbb{Q}$. Let $S \in R^{\mathbb{N}}$ be an unbounded sequence. Then there exists $s \in$ MON such that $(S^{-1} \circ s, 0) \in \mathbb{M}$. Thus S cannot converge in any FLSH-ring convergence on R coarser than $\mathbb{M}$. Thus $\mathbb{M}$ is not only $\mathbb{Q}$-coarse (follows from Corollary 3.1.3) but also coarse. It might be interesting to characterize such situations in which Y-coarse implies coarse.

3.2. Dense embedding. A method of enlarging an FLSH-ring convergence is described in FRIČ [1990]. Essentially, if $(X', \mathbb{L}')$ is an extension of $(X, \mathbb{L})$ and I is a subring of X' such that $X \cap I = \{0\}$ and I behaves like an ideal with respect to the multiplication by elements of X, then all sequences $S \in X^{\mathbb{N}}$ such that $(S, x) \in \mathbb{L}'$ with $x \in I$ are simultaneously $\mathbb{L}$-free at 0, i.e. there is an FLSH-ring convergence $\mathbb{L}_I$ on X such that $\mathbb{L} \subset \mathbb{L}_I$ and $(S, 0) \in \mathbb{L}_I$ whenever $S \in X^{\mathbb{N}}, x \in I$ and $(S, x) \in \mathbb{L}'$. The construction is very much related to all theorems in this section (variations on "Coarseness Density Criterion" for rings). As pointed out in FRIČ and ZANOLIN[19∞b], FLSH-rings do not satisfy the assumption (C) in BANASCHEWSKI [1974] and hence any Density Criterion for FLSH-rings can be proved only under some additional assumptions. Further, the notion of essentiality for FLSH-rings (if any) should be associated with X-ideals rather than with closed ideals as in the case of topological rings (cf. the paper by B. BANASCHEWSKI cited above).

Lemma 3.2.1. *Let $(X', \mathbb{L}')$ be an* FLSH-*ring and let X be its dense subring. Let $\mathbb{L} = \mathbb{L}' \restriction X$ be coarse. Then for each $x \in X' \setminus X$ there are $a_0 \in X$, $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in X \cup \mathbb{Z}$ such that $a_0 \neq 0, a_n \neq 0$ and $a_n x^n + \ldots + a_1 x + a_0 = 0$.*

P r o o f. Let $x \in X' \setminus X$. Then there exists $S \in X^{\mathbb{N}}$ such that $(S, x) \in \mathbb{L}$. Since $\mathbb{L}$ is coarse, S satisfies (CR2). Passing to $\mathbb{L}'$-limits we get the desired equation. □

Definition 3.2.2. Let $(X, \mathbb{L})$ be an FLSH-ring and let $(X', \mathbb{L}')$ be its extension. A subring I of X' such that $xI \subset I$ for all $x \in X$ is said to be *an X-ideal of X'*.

Remark 3.2.3. Since we do not assume the existence of a unit element, the X-ideal generated by $x \in X'$ will be formally written as $\{a_n x^n + \ldots + a_1 x;\ n \in \mathbb{N}, a_i \in X \cup \mathbb{Z}, i = 1, \ldots, n\}$.

Theorem 3.2.4. *Let $(X', \mathbb{L}')$ be an FLSH-ring and let X be its dense subring. Let $\mathbb{L} = \mathbb{L}' \upharpoonright X$ be coarse. Then each nontrivial X-ideal of X' intersects X nontrivially.*

P r o o f. The assertion follows directly from Lemma 3.2.1. □

Corollary 3.2.5. *Let $(X', \mathbb{L}')$ be an FLSH-ring and let X be its dense subring. Let $\mathbb{L} = \mathbb{L}' \upharpoonright X$ be coarse. Let X and X' be fields. Then X' is an algebraic extension of X.*

Theorem 3.2.6. *Let $(X', \mathbb{L}')$ be an FLSH-ring and let X be its dense subring. Let $\mathbb{L}'$ be sequentially compact and let each nontrivial X-ideal of X' intersect X nontrivially. Then $\mathbb{L} = \mathbb{L}' \upharpoonright X$ is coarse.*

P r o o f. According to Theorem 3 in FRIČ and ZANOLIN [1990], it suffices to prove that each sequence in X satisfies either condition (CR1) or condition (CR2). Let $S \in X$. Then either $(S, 0) \in \mathbb{L}' \upharpoonright X = \mathbb{L}$, or there are $s \in$ MON and $q \in X'$, $q \neq 0$, such that $(S \circ s, q) \in \mathbb{L}'$. In the first case S satisfies (CR1). In the later case either $q \in X$ and then put $p = q$, or $q \in X' \setminus X$. But then there is $p \in X, p \neq 0$, such that $p = a_n q^n + \ldots + a_1 q$, where $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in X \cup \mathbb{Z}$. Hence the sequence $\langle a_n \rangle (S \circ s)^n + \ldots + \langle a_1 \rangle S \circ s$ $\mathbb{L}$-converges to $p \neq 0$. Thus S satisfies (CR2). Hence $\mathbb{L}$ is coarse. □

Remark 3.2.7. It is not sufficient to assume that each closed X-ideal of X' intersects X nontrivially. Indeed, the torus T equipped with the usual metric convergense is sequentially compact, the rational torus $T_{\mathbb{Q}}$ fails to be coarse (this follows easily from the fact that the metric convergence on $\mathbb{Q}$ fails to be coarse in the class of all FLSH-ring convergences, cf. FRIČ [1989] and each nontrivial $T_{\mathbb{Q}}$-ideal of T is dense in T.

Answering a question by J. Novák, CONTESSA and ZANOLIN in [1982] constructed a FLUSH-ring having no completion. The point is that a product of a zero sequence and a Cauchy sequence need not be a zero sequence; the condition is denoted by (C_r) in FRIČ and KOUTNÍK [1989]. In some cases it is necessary not only to guarantee condition (C_r) but also that a product of a sequence and a zero sequence is a zero sequence; such sequences are called bounded in FRIČ and ZANOLIN [1990].

Definition 3.2.8. Let $(X, \mathbb{L})$ be an FLSH-ring. If $(ST, 0) \in \mathbb{L}$ whenever $S \in X^{\mathbb{N}}$ and $(T, 0) \in \mathbb{L}$, then $\mathbb{L}$ is said to be *bounded.*

Theorem 3.2.9. *Let $(X', \mathbb{L}')$ be an* FLSH-*ring and let X be its dense subring. Let $\mathbb{L}'$ be bounded and coarse and let each nontrivial X-ideal of X' intersect X nontrivially. Then $\mathbb{L} = \mathbb{L}' \upharpoonright X$ is coarse in the class Λ of all* FLSH-*ring convergences on X satisfying (C_r).*

P r o o f. Contrariwise, suppose that there are $S \in X^{\mathbb{N}}$ and $\mathbb{L}_S \in \Lambda$ such that $\mathbb{L} \subset \mathbb{L}_S$ and $(S, 0) \in \mathbb{L}_S \setminus \mathbb{L}$. Since $(X', \mathbb{L}')$ is coarse, there exists $q \in X', q \neq 0$, such that $\langle q \rangle$ is of the form

(∗) $\sum\limits_{i=1}^{m} T(i,1) \dots T(i,k(i))$, where $m, k(i) \in \mathbb{N}$, and $T(i,j)$ either $\mathbb{L}'$-converges to 0 or $T(i,j)$ is of the form $\pm S \circ s$, $\langle x \rangle S \circ s$, $x \in X'$, $s \in \mathrm{MON}$, $i = 1, \dots, m$, $j = 1, \dots, k(i)$.

Since the X-ideal generated by q intersects X nontrivially, there are $p \in X, p \neq 0, n \in \mathbb{N}$ and $a_1, \dots, a_n \in X \cup \mathbb{Z}$ such that $p = a_n q^n + \dots + a_1 q$. Hence $\langle p \rangle$ is of the form (∗), too. Now, $\mathbb{L}'$ is bounded and hence $T(i,1) \dots T(i,k(i))$ $\mathbb{L}'$-converges to 0 whenever some $T(i,j)$ does. Further, since X is dense in $(X', \mathbb{L}')$ for each x in $T(i,j) = \langle x \rangle S \circ s$ there exists an $\mathbb{L}$-Cauchy sequence $\mathbb{L}'$-converging to x. Consequently, some finite sum of finite products of subsequences of S and possibly of $\mathbb{L}$-Cauchy sequences $\mathbb{L}'$-converges, hence also $\mathbb{L}$-converges to $p \in X, p \neq 0$. This is a contradiction with the assumption that $\mathbb{L}_S \in \Lambda, \mathbb{L} \subset \mathbb{L}_S$ and $(S, 0) \in \mathbb{L}_S$. This completes the proof. □

Remark 3.2.10. Observe that Theorem 3.2.9 yields a generalization of Proposition 1.4 in DIKRANJAN, FRIČ and ZANOLIN [1987]. Indeed, to each FLSH-group we associate its zero ring and the FLSH-group convergence remains compatible with the zero ring structure and it is bounded. Further, it is coarse iff it is coarse as a ring and the X-ideal generated by $x \in X'$ becomes the periodic group generated by x. Thus the Coarseness Density Criterion for abelian groups is a special case of Theorem 3.2.9.

Corollary 3.2.11. *Let $(X', \mathbb{L}')$ be an* FLSH-*ring and let X be its dense subring. Let $\mathbb{L}'$ be sequentially precompact, coarse and satisfy* (C_r) *and let each nontrivial X-ideal of X' intersect X nontrivially. Then $\mathbb{L} = \mathbb{L}' \upharpoonright X$ is coarse in the class Λ of all* FLSH-*ring convergences on X satisfying* (C_r).

P r o o f. Observe that $\mathbb{L}'$ and hence also $\mathbb{L}$ satisfy the Urysohn axiom. We shall prove that $\mathbb{L}'$ is bounded. The assertion will then follow from Theorem 3.2.9. So, let S be a sequence in X' and let $(T, 0) \in \mathbb{L}'$. Since $\mathbb{L}'$ is sequentially precompact, for each $s \in$ MON there exists $t \in$ MON such that $S \circ s \circ t$ is $\mathbb{L}'$-Cauchy. But $\mathbb{L}'$ satisfies (C_r) and hence $((ST) \circ s \circ t, 0) \in \mathbb{L}'$. Since $\mathbb{L}'$ satisfies the Urysohn axiom, we have $(ST, 0) \in \mathbb{L}'$. This completes the proof. □

Remark 3.2.12. As shown (Under CH) in SIMON and ZANOLIN [1987], an abelian coarse FLUSH-group need not be sequentially precompact. Since each zero ring of an FLSH-group is a bounded FLSH-ring, (under CH) a bounded coarse FLUSH-ring need not be sequentially precompact.

3.3. Nonstrict completions of C_{Q}. Denote C the complex numbers and C_{Q} the subfield of C consisting of all complex numbers $x + iy = [x, y]$ such that both x and y belong to $\mathbb{Q}$. Throughout this Section let $\mathbb{M}$ denote the usual metric convergence on C and let $\mathbb{L} = \mathbb{M} \restriction C_{\mathrm{Q}}$. We show that $(C_{\mathrm{Q}}, \mathbb{L})$ has exactly $\exp\exp\omega$ nonhomeomorphic C_{Q}-coarse dense FLSH-ring (-field) completions and it can be easily verified (cf. Lemma 2.3.3) that, $(C, \mathbb{M})$ is the only one of them the convergence of which is Fréchet.

Relevant information about the categorical FLUSH-ring completions can be found in FRIČ and KOUTNÍK [1989].

Let $(C, \mathbb{L}_1^*)$ be categorial FLUSH-ring completion of $(C_{\mathrm{Q}}, \mathbb{L})$. Let B be a transcendence basis of C over C_{Q} (i.e. a maximal set of algebraically independent complex numbers over C_{Q}) such that $2 < |b| < 3$ for all $b \in B$ and let $\{S_\alpha;\ \alpha \in \exp\omega\}$ be a partition of B into disjoint infinite countable subsets S_α of B. Consider each S_α as a one-to-one sequence. Let f be a mapping of $\exp\omega$ into $\{0, 1\}$.

Lemma 3.3.1. *There is an* FLSH-*ring convergence* $\mathbb{L}_f$ *on* C *such that* $\mathbb{L}_1^* \subset \mathbb{L}_f$, $(S_\alpha, f(\alpha)) \in \mathbb{L}_f$ *for all* $\alpha \in \exp\omega$ *and* $\mathbb{L}_f \restriction (C_{\mathrm{Q}}^{\mathbb{N}} \times C) = \mathbb{M} \restriction (C_{\mathrm{Q}}^{\mathbb{N}} \times C)$.

P r o o f. The assertion can be proved analogously as Lemma 2.3.1. We give only hints. First, enlarge $\mathbb{L}_1^*$ to the smallest FLS-ring convergence $\mathbb{L}_f$ on C such that $\mathbb{L}_1^* \subset \mathbb{L}_f$ and $(S_\alpha, f(\alpha)) \in \mathbb{L}_f$ for each $x \in \exp\omega$. Using the fact that elements of B are algebraically independent, it can be proved that $\mathbb{L}_f$ has unique limits. Observe that no $\mathbb{M}$-unbounded sequence in C_{Q} $\mathbb{L}_f$-converges and each bounded sequence in C_{Q} contains a subsequence $\mathbb{L}_1^*$-converging and hence also $\mathbb{L}_f$-converging in C. Since $\mathbb{L}_f$ has unique limits, we have $\mathbb{L}_f \restriction (C_{\mathrm{Q}}^{\mathbb{N}} \times C) = \mathbb{M} \restriction (C_{\mathrm{Q}}^{\mathbb{N}} \times C)$. □

Theorem 3.3.2. *There are exactly* $\exp\exp\omega$ *nonhomeomorphic coarse* FLUSH-*field dense completions of* $(C_{\mathrm{Q}}, \mathbb{L})$.

Proof. By Lemma 3.3.2, for each mapping f of $\exp\omega$ into $\{0,1\}$, $(C,\mathbb{L}_f)$ is a FLSH-ring dense precompletion of $(C_{\mathrm{Q}},\mathbb{L})$. Let $\mathbb{L}'_f$ be a C_{Q}-coarse FLSH-ring convergence on C coarser than $\mathbb{L}_f$. We claim that each $(C,\mathbb{L}'_f)$ is a coarse dense FLUSH-field completion of $(C_{\mathrm{Q}},\mathbb{L})$. Indeed, it is dense, C_{Q}-coarse and since $\mathbb{L} = \mathbb{L}^*$, we have also $(\mathbb{L}'_f)^* = \mathbb{L}'_f$. Since no $\mathbb{M}$-unbounded sequence $S \in C_{\mathrm{Q}}^{\mathbb{N}}$ can $\mathbb{L}'_f$-converge and each $\mathbb{M}$-bounded sequence $S \in C_{\mathrm{Q}}^{\mathbb{N}}$ contains an $\mathbb{M}$-convergent subsequence, $\mathbb{L}'_f$ is a FLUSH-ring convergence coarse in the class of all FLUSH-ring convergences on C. It follows from Theorem 1.2.1 and Corollary 1.2.7 in FRIČ [1992] that $(C,\mathbb{L}'_f)$ is complete and $\mathbb{L}'_f$ is a FLUSH-field convergence. It is easy to see that if f and g are mapping of $\exp\omega$ into $\{0,1\}$ and $f \neq g$, then $(C,\mathbb{L}'_f)$ and $(C,\mathbb{L}'_g)$ are nonhomeomorphic. Since $(C_{\mathrm{Q}},\mathbb{L})$ cannot have more than $\exp\exp\omega$ nonhomeomorphic dense precompletions, the proof is complete. $\square$

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