Czechoslovak Mathematical Journal

Graham D. Barbour; James G. Raftery

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Czechoslovak Mathematical Journal, Vol. 47 (1997), No. 2, 317–325

Persistent URL: `http://dml.cz/dmlcz/127358`

Czechoslovak Mathematical Journal, 47 (122) 1997, Praha

ON THE DEGREES OF PERMUTABILITY OF SUBREGULAR VARIETIES

Graham D. Barbour and James G. Raftery, Durban

(Received February 10, 1995)

Two distinct Mal'cev conditions for point regular varieties (which we shall call Fichtner's first and second theorems) were given in [Fic68] and [Fic70]. The latter refines the former in that it replaces certain terms of unspecified arity by quaternary terms, and the former is deducible from the latter. Consequently, the style of the latter condition is generally considered preferable. Similar characterizations of congruence regular, weakly regular and (most generally) subregular varieties may be formulated in either of the two styles, e.g., [Csá70], [Wil70], [Hag73], [DMS87], [Dud87]. A subregular variety which realises either style of Mal'cev condition in n equations is congruence n-permutable. In greater generality, [DMS87, Theorem 3.5] appears to offer a converse, viz. that for any n, an n-permutable subregular variety must realise the appropriate variant of Fichtner's second scheme in at most n equations, making the n-line scheme a Mal'cev condition for 'subregular and n-permutable' varieties. For each n, similar Mal'cev conditions for n-permutable varieties with each of the other regularity properties would follow.

We point out here, however, that [DMS87, Theorem 3.5] is false. There is therefore no satisfactorily proved Mal'cev characterization of even 'point regular and n-permutable' varieties (for fixed n) in the literature, other than unwieldy conjunctions of independent Mal'cev conditions for these two properties. The purpose of this note is to provide such a result as a consequence of a more general Mal'cev characterization of 'subregular n-permutable' varieties (Corollary 5). It seems unavoidable that the result be in the style of Fichtner's first theorem, i.e., that it involve terms of unspecified arity, but using n-permutability, we are able to replace the original $(2 + 2m)$-ary Fichtner terms by $(2 + m)$-ary ones. We see no reason to expect an analogue of our result in the style of Fichtner's second theorem to be true, though an example showing this would be hard to construct.

[DMS87, Theorem 3.5] claims that for any integer $n \geqslant 2$, an $(n+1)$-permutable variety K satisfies a quasi-identity $f_1(\vec{x}) \approx g_1(\vec{x}) \,\&\, \ldots \,\&\, f_m(\vec{x}) \approx g_m(\vec{x}) \to r(\vec{x}) \approx s(\vec{x})$ (in variables $\vec{x} = x_1, \ldots, x_p$) if and only if there exist an integer $k \leqslant n-1$, $(p+1)$-ary terms $t_1, \ldots, t_k$ and pairs $\langle u_1, v_1\rangle, \ldots, \langle u_k, v_k\rangle \in \{\langle f_i, g_i\rangle \colon i = 1, \ldots, m\}$ such that K satisfies the identities

$$\begin{gathered} r(\vec{x}) \approx t_1(\vec{x}, u_1(\vec{x})), \\ t_j(\vec{x}, v_j(\vec{x})) \approx t_{j+1}(\vec{x}, u_{j+1}(\vec{x})) \quad (j = 1, \ldots, k-1), \\ t_k(\vec{x}, v_k(\vec{x})) \approx s(\vec{x}). \end{gathered}$$

Over a variety, any such equational scheme certainly entails the quasi-identity, but the converse is false. The congruence permutable variety of Boolean algebras satisfies $x' \wedge y \approx 0 \,\&\, y' \wedge x \approx 0 \to x \approx y$. By the above claim, it should satisfy $x \approx t(x, y, u(x, y))$ and $y \approx t(x, y, v(x, y))$, and therefore also $u(x, y) \approx v(x, y) \to x \approx y$, for some ternary term t, where $\langle u(x, y), v(x, y)\rangle \in \{\langle x' \wedge y, 0\rangle, \langle y' \wedge x, 0\rangle\}$. By symmetry of the variables, Boolean algebras should satisfy $x' \wedge y \approx 0 \to x \approx y$, which they do not.

An $(n+1)$-permutable variety satisfying the aforementioned quasi-identity must also satisfy a scheme of $k+1$ equations of the above form for *some* k, by the arguments given in [DMS87], but the example shows that the minimum number of equations in such a scheme need not be bounded by the variety's degree of permutability. To correct this result, we need the following preliminaries.

Consider a variety K of algebras and an algebra $\mathbf{A} = \langle A; \ldots\rangle \in K$. Let τ be a binary reflexive relation on A, compatible with the fundamental operations of $\mathbf{A}$. We call τ *a tolerance* (resp. a *quasiorder*) on $\mathbf{A}$ if τ is symmetric (resp. transitive). If $a \in A$ then $\{b \in A \colon \langle a, b\rangle \in \tau\}$ is denoted by a/τ. Let $\mathrm{Ref}\mathbf{A}$, $\mathrm{Tol}\mathbf{A}$ and $\mathrm{Con}\mathbf{A}$ be, respectively, the lattice of all reflexive compatible relations on $\mathbf{A}$, the tolerance lattice and the congruence lattice of $\mathbf{A}$. (All of these are ordered by set inclusion and are algebraic.) We use $R^{\mathbf{A}}$, $T^{\mathbf{A}}$ and $\Theta^{\mathbf{A}}$ to denote, respectively, the algebraic closure operators on the power set of A^2 associated with the algebraic closure systems $\mathrm{Ref}\mathbf{A}$, $\mathrm{Tol}\mathbf{A}$ and $\mathrm{Con}\mathbf{A}$. If $X = \{\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\} \subseteq A^2$, we write $R^{\mathbf{A}}(X)$ as $R^{\mathbf{A}}(\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle)$ and $R^{\mathbf{A}}(\langle a, b\rangle)$ as $R^{\mathbf{A}}(a, b)$; similar conventions apply to $T^{\mathbf{A}}$ and $\Theta^{\mathbf{A}}$.

For $\tau, \eta \subseteq A \times A$, we denote the relational product $\tau \circ \eta$ by $\tau\eta$ and we define $\tau^{[0]} = \mathrm{id}_A := \{\langle a, a\rangle \colon a \in A\}$, and $\tau^{[n+1]} = \tau^{[n]}\tau$ ($n \in \omega$). The transitive closure $\bigcup_{n \in \omega} \tau^{[n]}$ of a tolerance τ on $\mathbf{A}$ is just $\Theta^{\mathbf{A}}(\tau)$. The least positive $n \in \omega$, if it exists, such that $\tau^{[n]}$ is a congruence for every $\tau \in \mathrm{Tol}\mathbf{A}$, is called the *tolerance number* of $\mathbf{A}$ and is denoted by $\mathrm{tn}(\mathbf{A})$. We also write $\mathrm{tn}(K) = n$ if n is the least positive integer such that $\mathrm{tn}(\mathbf{B}) \leqslant n$ for all $\mathbf{B} \in K$.

Lemma 1. *Let $a_1, \ldots, a_n, b_1, \ldots, b_n, c, d \in A$.*

(i) [Dud83] $\langle c, d\rangle \in R^{\mathbf{A}}(\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle)$ *if and only if there is an n-ary polynomial G on $\mathbf{A}$ such that $c = G(a_1, \ldots, a_n)$ and $d = G(b_1, \ldots, b_n)$.*

(ii) [Cha81] $\langle c, d\rangle \in T^{\mathbf{A}}(\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle)$ *if and only if there is a $2n$-ary polynomial G on $\mathbf{A}$ such that $c = G(a_1, \ldots, a_n, b_1, \ldots, b_n)$ and $d = G(b_1, \ldots, b_n, a_1, \ldots, a_n)$.*

Theorem 2.

(i) [HM73], [CR83] *A variety K of algebras is congruence $(n+1)$-permutable if and only if $\operatorname{tn}(K) \leqslant n$.*

(ii) [Hag73, Theorem 1, Corollary 4] *If η is a reflexive compatible binary relation on an algebra $\mathbf{A}$ in a congruence $(n+1)$-permutable variety K then $\eta^{[n]}$ is a quasiorder on $\mathbf{A}$ and every quasiorder on an algebra in K is a congruence, hence $\eta^{[n]}$ is $\Theta^{\mathbf{A}}(\eta)$.*

Given a set X of variables, $\mathbf{T} = \mathbf{T}(X) = \langle T(X); \ldots\rangle$ denotes the term algebra and $\mathbf{F} = \mathbf{F}_K(\overline{X}) = \langle F_K(\overline{X}); \ldots\rangle$ the K-free algebra over X. Thus, $\mathbf{F}$ is the factor algebra of $\mathbf{T}$ modulo the congruence that identifies all pairs of terms $t, s \in T(X)$ for which K satisfies $t \approx s$. The image $t^{\mathbf{F}}(\bar{x}_1, \ldots, \bar{x}_p) \in F_K(\overline{X})$ of $t = t(x_1, \ldots, x_p) \in T(X)$ is denoted by $\bar{t}$ (so $t \in \bar{t}$). In particular, $\mathbf{F}_K(\overline{X})$ is freely generated by $\overline{X} := \{\bar{x} \colon x \in X\}$. The following result corrects [DMS87, Theorem 3.5].

Theorem 3. *Let K be a variety and let f_i, g_i, r and s be terms in variables $\vec{x} = x_1, \ldots, x_p$, for $i = 1, \ldots, m$.*

(a) *The following conditions are equivalent:*

(i) *K satisfies the quasi-identity*

$$f_1(\vec{x}) \approx g_1(\vec{x}) \,\&\, \ldots \,\&\, f_m(\vec{x}) \approx g_m(\vec{x}) \to r(\vec{x}) \approx s(\vec{x}).$$

(ii) *For some positive integer k, there exist $(2m+p)$-ary terms $t_1 \ldots, t_k$ such that for $j = 1, \ldots, k-1$, K satisfies the identities*

$$\begin{aligned} r(\vec{x}) &\approx t_1(\vec{x}, f_1(\vec{x}), \ldots, f_m(\vec{x}), g_1(\vec{x}), \ldots, g_m(\vec{x})),\\ t_j(\vec{x}, g_1(\vec{x}), &\ldots, g_m(\vec{x}), f_1(\vec{x}), \ldots, f_m(\vec{x}))\\ &\approx t_{j+1}(\vec{x}, f_1(\vec{x}), \ldots, f_m(\vec{x}), g_1(\vec{x}), \ldots, g_m(\vec{x})),\\ t_k(\vec{x}, g_1(\vec{x}), &\ldots, g_m(\vec{x}), f_1(\vec{x}), \ldots, f_m(\vec{x})) \approx s(\vec{x}). \end{aligned}$$

(b) *Suppose that the equivalent conditions of* (a) *hold and that K is congruence $(n+1)$-permutable. Then we may choose $k \leqslant n$ in* (ii). *More strongly, in this* *case, there exist a positive integer $k \leqslant n$ and $(m+p)$-ary terms $t_1, \ldots, t_k$ such that for $j = 1, \ldots, k-1$, K satisfies the identities*

$$r(\vec{x}) \approx t_1(\vec{x}, f_1(\vec{x}), \ldots, f_m(\vec{x})),$$
$$t_j(\vec{x}, g_1(\vec{x}), \ldots, g_m(\vec{x})) \approx t_{j+1}(\vec{x}, f_1(\vec{x}), \ldots, f_m(\vec{x})),$$
$$t_k(\vec{x}, g_1(\vec{x}), \ldots, g_m(\vec{x})) \approx s(\vec{x}).$$

Proof. (a) (i) $\Rightarrow$ (ii): Let $\mathbf{T} = \mathbf{T}_K(\vec{x}) = \langle T; \ldots \rangle$ and $\mathbf{F} = \mathbf{F}_K(\bar{x}_1, \ldots, \bar{x}_p) = \langle F; \ldots \rangle$. Since K is a variety, (i) clearly implies that $\bigvee_{i=1}^{m} \Theta^{\mathbf{F}}(\bar{f}_i, \bar{g}_i) \supseteq \Theta^{\mathbf{F}}(\bar{r}, \bar{s})$. We therefore have

$$\langle \bar{r}, \bar{s} \rangle \in \Theta^{\mathbf{F}}(\langle \bar{f}_1, \bar{g}_1 \rangle, \ldots, \langle \bar{f}_m, \bar{g}_m \rangle) = \bigcup_{j \in \omega} \tau^{[j]},$$

where $\tau = T^{\mathbf{F}}(\langle \bar{f}_1, \bar{g}_1 \rangle, \ldots, \langle \bar{f}_m, \bar{g}_m \rangle)$. Choose $k \in \omega$ such that $\langle \bar{r}, \bar{s} \rangle \in \tau^{[k]}$. Note that if K is $(n+1)$-permutable then we can choose $k \leqslant n$, since $\operatorname{tn}(K) \leqslant n$, by Theorem 2(i).

There exist $\bar{c}_0, \ldots, \bar{c}_k \in F$ such that $\bar{r} = \bar{c}_0 \tau \bar{c}_1 \tau \ldots \tau \bar{c}_k = \bar{s}$. For $j = 1, \ldots, k$, since $\bar{c}_{j-1} \tau \bar{c}_j$, there exists, by Lemma 1(ii), a $2m$-ary polynomial G_j on $\mathbf{F}$ such that

$$(\dagger) \qquad \bar{c}_{j-1} = G_j(\bar{f}_1, \ldots, \bar{f}_m, \bar{g}_1, \ldots, \bar{g}_m) \text{ and } \bar{c}_j = G_j(\bar{g}_1, \ldots, \bar{g}_m, \bar{f}_1, \ldots, \bar{f}_m).$$

Thus, there exist a $(2m+q)$-ary term s_j and $\bar{a}_{j1}, \ldots, \bar{a}_{jq} \in F$ such that

$$G_j(d_1, \ldots, d_m, e_1, \ldots, e_m) = s_j^{\mathbf{F}}(d_1, \ldots, d_m, e_1, \ldots, e_m, \bar{a}_{j1}, \ldots, \bar{a}_{jq})$$

for all $\vec{d}, \vec{e} \in F^m$. Choosing terms $a_{j1}, \ldots, a_{jq} \in T$ with $a_{jl} \in \bar{a}_{jl}$ for all l, define a $(2m+p)$-ary term t_j by

$$\begin{aligned} &t_j(x_1, \ldots, x_p, z_1, \ldots, z_m, w_1, \ldots, w_m) \\ &\qquad = s_j(z_1, \ldots, z_m, w_1, \ldots, w_m, a_{j1}(x_1, \ldots, x_p), \ldots, a_{jq}(x_1, \ldots, x_p)). \end{aligned}$$

We have

$$\begin{aligned} &t_j^{\mathbf{F}}(\bar{x}_1, \ldots, \bar{x}_p, f_1^{\mathbf{F}}(\bar{x}_1, \ldots, \bar{x}_p), \ldots, f_m^{\mathbf{F}}(\bar{x}_1, \ldots, \bar{x}_p), g_1^{\mathbf{F}}(\bar{x}_1, \ldots, \bar{x}_p) \ldots, g_m^{\mathbf{F}}(\bar{x}_1, \ldots, \bar{x}_p)) \\ &\quad = s_j^{\mathbf{F}}(\bar{f}_1, \ldots, \bar{f}_m, \bar{g}_1, \ldots, \bar{g}_m, \bar{a}_{j1}, \ldots, \bar{a}_{jq}) \\ &\quad = G_j(\bar{f}_1, \ldots, \bar{f}_m, \bar{g}_1, \ldots, \bar{g}_m) \end{aligned}$$

for $j = 1, \ldots, k$, and the same is true if we interchange f_i and g_i for all i. This, together with $(\dagger)$, yields that $\mathbf{F}$ satisfies the identities of (ii). Consequently, all algebras in K satisfy these identities.

(ii) $\Rightarrow$ (i) follows readily from the identities of (ii).

(b) We use the argument of (a) (i) $\Rightarrow$ (ii), replacing τ by $\eta = R^{\mathbf{F}}(\langle \bar{f}_1, \bar{g}_1\rangle, \ldots, \langle \bar{f}_m, \bar{g}_m\rangle)$ and using Lemma 1(i) rather than (ii), to obtain the equations of (b); note that $\eta^{[n]}$ is a congruence, by Theorem 2(ii), so we may choose $k \leqslant n$. Clearly, these equations entail the quasi-identity of (a)(i). □

In fact the minimum length (taken over all K-quasi-identities) of the equational scheme in Theorem 3(b) *characterizes* K's degree of permutability, by the following converse of 3(b). If for every K-quasi-identity as in 3(a)(i), there exist $k \leqslant n$ and $t_1, \ldots, t_k$ such that K satisfies the equations of 3(b), then K is congruence $(n+1)$-permutable. For applying 3(b) to $y \approx x \rightarrow x \approx y$ yields K-identities $x \approx t_1(x, y, y)$, $t_i(x, y, x) \approx t_{i+1}(x, y, y)$ $(i < k)$ and $t_k(x, y, x) \approx y$. The Hagemann-style Mal'cev condition for $(k+1)$-permutability [HM73] is realised by K when we set $q_i(x, y, z) = t_i(x, z, y)$ $(i \leqslant k)$.

We say that an algebra $\mathbf{A}$ is *regular with respect to elements* $a_1, \ldots, a_n \in A$ provided that for any $\theta, \varphi \in \mathrm{Con}\mathbf{A}$, if $a_r/\theta = a_r/\varphi$ for $r = 1, \ldots, n$, then $\theta = \varphi$. If $g_1, \ldots, g_n$ are unary terms, we say that $\mathbf{A}$ is *regular with respect to* $g_1, \ldots, g_n$ if for any $a \in A$, $\mathbf{A}$ is regular with respect to $g_1^{\mathbf{A}}(a), \ldots, g_n^{\mathbf{A}}(a)$. A variety K is called *regular with respect to* $g_1, \ldots, g_n$ if all members of K have this property. An algebra $\mathbf{A}$ (resp. a variety K) is called *congruence regular* if it is regular with respect to the single unary term $g(z) = z$, i.e., any congruence on $\mathbf{A}$ (resp. on any $\mathbf{B} \in K$) is determined by the congruence class of any element of A (resp. B). An algebra $\mathbf{A}$ (resp. a variety K) is called *weakly regular* if it is regular with respect to a finite set of unary terms g_r that are essentially nullary, by which we mean that $\mathbf{A}$ (resp. K) satisfies $g_r(x) \approx g_r(y)$ for each r. In this case, if we define nullary $0_r = g_r(x)$ for each r, we say that $\mathbf{A}$ (resp. K) is *weakly regular with respect to* $0_1, \ldots, 0_n$. Finally, $\mathbf{A}$ (resp. K) is called *point regular with respect to* 0, or just 0-*regular*, if it is weakly regular with respect to a single equationally defined constant term 0.

Corollary 4. *(a) The following conditions on a variety K with unary terms $g_1, \ldots, g_n$ are equivalent:*

(i) *K is regular with respect to $g_1, \ldots, g_n$.*

(ii) *For some positive integer m, there exist ternary terms $p_1, \ldots, p_m$ and a function $r \mapsto i_r$ from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$ such that K satisfies*

$$p_1(x, y, z) \approx g_{i_1}(z) \,\&\, p_2(x, y, z) \approx g_{i_2}(z) \,\&\, \ldots \,\&\, p_m(x, y, z) \approx g_{i_m}(z) \leftrightarrow x \approx y.$$

(iii) *For some positive integers m, k, there exist ternary terms $p_1, \ldots, p_m$, $(m+3)$-ary terms $t_1, \ldots, t_k$ and a function $r \mapsto i_r$ from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$ such that for*

$j = 1, \dots, k-1$ and $r = 1, \dots, m$, K satisfies the identities $p_r(x,x,z) \approx g_{i_r}(z)$ and

$$
\begin{aligned}
& x \approx t_1(x,y,z,g_{i_1}(z),\dots,g_{i_m}(z)), \\
(*_n) \quad & t_j(x,y,z,p_1(x,y,z),\dots,p_m(x,y,z)) \approx t_{j+1}(x,y,z,g_{i_1}(z),\dots,g_{i_m}(z)), \\
& t_k(x,y,z,p_1(x,y,z),\dots,p_m(x,y,z)) \approx y.
\end{aligned}
$$

(b) *If the conditions of* (a) *hold then the smallest positive integer k for which $(*)_n$ holds is the tolerance number* $\mathrm{tn}(K)$ *of K, i.e., it is the smallest k such that K is congruence $(k+1)$-permutable.*

Proof. (a) (i)⇔(ii) is essentially [DMS87, Theorem 2.2] and (iii)⇒(i) is obvious.

(ii)⇒(iii) By [DMS87, Theorem 3.9], any variety satisfying a quasi-identity of the form

$$f_1(x,y,\vec{z}) \approx g_1(x,y,\vec{z}) \,\&\, \dots \,\&\, f_m(x,y,\vec{z}) \approx g_m(x,y,\vec{z}) \to x \approx y,$$

as well as the identities $f_r(x,x,\tilde{z}) \approx h_r(z) \approx g_r(x,x,\tilde{z})$, for suitable terms f_r, g_r and (unary) h_r (where $\tilde{z}$ abbreviates $z, z, \dots, z$) is congruence modular and congruence n-permutable for some integer $n > 1$. Clearly, we have a special case of these conditions here, so the result follows from Theorem 3(b).

(b) It suffices, by Theorem 3, to show that K is congruence $(k+1)$-permutable. Let

$$q_j(x,y,z) = t_j(x,z,z,p_1(y,z,z),\dots,p_m(y,z,z)) \quad (1 \leqslant j \leqslant k)$$

and observe that K satisfies the Hagemann identities [HM73] for congruence $(k+1)$-permutability:

$$x \approx q_1(x,y,y);\; q_j(x,x,y) \approx q_{j+1}(x,y,y) \text{ for } j = 1,\dots,k-1;\; q_k(x,x,y) \approx y.$$

□

Varieties that satisfy the conditions of Corollary 4(a) *for some* unary terms $g_1, \dots, g_n$ are characterized in several different ways in [DMS87, Theorem 2.2], e.g., they are just the varieties K such that for every $\mathbf{A} \in \mathscr{K}$ and any subalgebra $\mathbf{B}$ of $\mathbf{A}$, every congruence θ on $\mathbf{A}$ is determined by $\{b/\theta \colon b \in B\}$. (See [Dud87, Theorem 1] also.) Such varieties are called *subregular.*

Corollary 5. *For each positive k, a variety K is subregular and $(k+1)$-permutable if and only if for some positive n, m, there are unary terms $g_1, \dots, g_n$, ternary terms $p_1, \dots, p_m$, $(m+3)$-ary terms $t_1, \dots, t_k$ and a function $r \mapsto i_r$ from $\{1, \dots, m\}$ to*

$\{1,\ldots,n\}$ *such that for* $j = 1,\ldots,k-1$ *and* $r = 1,\ldots,m$, K *satisfies* $p_r(x,x,z) \approx g_{i_r}(z)$ *and the identities of* $(*)_n$.

Requiring $n = 1$ and $g_1(z) = z$ in Corollary 5 amounts, for each k, to a Mal'cev condition for $(k+1)$-permutable congruence regular varieties. If each $g_r(z)$ is required to be an equationally defined constant 0_r and we replace z by y and the $p_r(x,y,z)$ by binary $d_r(x,y)$ throughout the argument, we obtain a Mal'cev condition for weakly regular $(k+1)$-permutable varieties. The further requirement $n = 1$ refines Fichtner's first theorem [Fic68, Theorem 2] as follows: for each k, a variety K with constant 0 is 0-regular and $(k+1)$-permutable if and only if for suitable terms, it satisfies

$$\begin{aligned} &x \approx t_1(x,y,0,\ldots,0),\\ (*)\qquad &t_j(x,y,d_1(x,y),\ldots,d_m(x,y)) \approx t_{j+1}(x,y,0\ldots,0) \qquad (j = 1,\ldots,k-1),\\ &t_k(x,y,d_1(x,y),\ldots,d_m(x,y)) \approx y \end{aligned}$$

and $d_i(x,x) \approx 0$ $(i = 1,\ldots,m)$. Sharpening Fichtner's second theorem similarly in the presence of n-permutability (for some n) yields a scheme whose line $j-1$ is the more elegant $t_j(x,y,d_j(x,y)) \approx t_{j+1}(x,y,0)$ but there seems no reason to hope that such a scheme can always be found involving no more equations than the variety's degree of permutability. Point regular $(k+1)$-permutable varieties which are not k-permutable exist for each $k > 1$ [BR].

Enriched groups and quasigroups are standard examples of congruence permutable congruence regular varieties. Here are some examples of point regular varieties which illustrate $(*)$ less trivially.

A *BCK-algebra* [IT78] is an algebra $\mathbf{A} = \langle A; \dot{-}, 0\rangle$ of type $\langle 2,0\rangle$ satisfying the axioms

$$((x \dot{-} y) \dot{-} (x \dot{-} z)) \dot{-} (z \dot{-} y) \approx 0, \tag{1}$$

$$x \dot{-} 0 \approx x, \tag{2}$$

$$0 \dot{-} x \approx 0, \tag{3}$$

$$(x \dot{-} y \approx 0)\,\&\,(y \dot{-} x \approx 0) \rightarrow x \approx y. \tag{4}$$

The quasivariety of all BCK-algebras is not a variety [Wro83]. It satisfies $x \dot{-} x \approx 0$ and $(x \dot{-} y) \dot{-} z \approx (x \dot{-} z) \dot{-} y$; we shall omit brackets from expressions $(a \dot{-} b) \dot{-} c$. Any variety K of BCK-algebras is 0-regular, 3-permutable [Idz83], not congruence permutable (unless trivial) and satisfies

$$
(5) \qquad \begin{gathered}
x \approx x \dot{-} 0 \dot{-} \ldots \dot{-} 0, \\
x \dot{-} u_1(x,y) \dot{-} \ldots \dot{-} u_p(x,y) \approx y \dot{-} v_1(x,y) \dot{-} \ldots \dot{-} v_q(x,y), \\
y \dot{-} 0 \dot{-} \ldots \dot{-} 0 \approx y
\end{gathered}
$$

for some terms u_i, v_j such that all BCK-algebras satisfy $u_i(x,x) \approx 0 \approx v_j(x,x)$ [Idz83], [BR95]. We interpret (5) as a case of $(*)$ by setting $k = 2$, $m = p + q$, $d_i = u_i$ $(i \leqslant p)$, $d_{p+j} = v_j$ $(1 \leqslant j \leqslant q)$, $t_1(x, y, \vec{z}) = x \dot{-} z_1 \dot{-} \ldots \dot{-} z_p$ and $t_2(x, y, \vec{z}) = y \dot{-} [v_1(x,y) \dot{-} z_{p+1}] \dot{-} \ldots \dot{-} [v_q(x,y) \dot{-} z_{p+q}]$ (where $\vec{z} = z_1, \ldots, z_{p+q}$). A more complex realization of $(*)$ with $m = 2$, $d_1(x,y) = x \dot{-} y$ and $d_2(x,y) = y \dot{-} x$ is also possible. In the variety of *commutative BCK-algebras*, the central equation of (5) takes the form $x \dot{-} (x \dot{-} y) \approx y \dot{-} (y \dot{-} x)$.

An algebra $\mathbf{A} = \langle A; \dot{-}, \vee \text{ (resp. } \wedge), 0\rangle$ of type $\langle 2,2,0\rangle$ is called an *upper* (resp. *lower*) *BCK-semilattice* if $\langle A; \dot{-}, 0\rangle$ is a BCK-algebra whose underlying partially ordered set, defined by $a \leqslant b$ iff $a \dot{-} b = 0$, is an upper (resp. lower) semilattice with join (resp. meet) operation $\vee$ (resp. $\wedge$). The class $\mathrm{BCK}^{\vee}$ (resp. $\mathrm{BCK}^{\wedge}$) of all such algebras is a 0-regular variety [Idz84a], [Idz84b]. The most useful identities of $\mathrm{BCK}^{\vee}$ and $\mathrm{BCK}^{\wedge}$ arise from the fact that $x \dot{-} (x \dot{-} y)$ is a lower bound for both x and y in any BCK-algebra. The variety $\mathrm{BCK}^{\vee}$ is congruence permutable. A realization of $(*)$ is given by $x \dot{-} x \approx 0$ and

$$
\begin{gathered}
x \approx (x \dot{-} 0) \vee (y \dot{-} (y \dot{-} x)), \\
(x \dot{-} (x \dot{-} y)) \vee (y \dot{-} 0) \approx y.
\end{gathered}
$$

Here, $k = 1$, $m = 2$, $d_1(x,y) = x \dot{-} y$, $d_2(x,y) = y \dot{-} x$ and $t_1(x, y, z_1, z_2) = (x \dot{-} z_1) \vee (y \dot{-} (y \dot{-} x \dot{-} z_2))$.

The variety $\mathrm{BCK}^{\wedge}$ is 4-permutable and not 3-permutable [Raf94] so its realizations of $(*)$ necessarily involve at least four lines. One such realization is:

$$
\begin{gathered}
x \approx x \dot{-} 0, \\
x \dot{-} (x \dot{-} y) \approx [x \dot{-} (x \dot{-} y)] \wedge (y \dot{-} 0), \\
(x \dot{-} 0) \wedge [y \dot{-} (y \dot{-} x)] \approx y \dot{-} (y \dot{-} x), \\
y \dot{-} 0 \approx y.
\end{gathered}
$$

Here, $k = 3$, $m = 2$, $d_1(x,y) = x \dot{-} y$, $d_2(x,y) = y \dot{-} x$, $t_1(x,y,z_1,z_2) = x \dot{-} z_1$, $t_2(x,y,z_1,z_2) = [x \dot{-} (x \dot{-} y \dot{-} z_1)] \wedge (y \dot{-} z_2)$ and $t_3(x,y,z_1,z_2) = y \dot{-} (y \dot{-} x \dot{-} z_2)$.

We do not know whether corresponding realizations of Fichtner's second theorem for $\mathrm{BCK}^{\wedge}$ and for arbitrary varieties of BCK-algebras are achievable in the same numbers of lines.

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Authors' addresses: Graham D. Barbour, Department of Computer Science, University of Natal, Durban 4001, South Africa, **barbour@cs.und.ac.za**; James G. Raftery, Department of Mathematics and Applied Mathematics, University of Natal, Durban 4001, South Africa, **raftery@ph.und.ac.za**.