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ON A CERTAIN SINGULAR BOUNDARY VALUE PROBLEM FOR LINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his 70th birthday

In the interval $I = [a, b]$ we consider a vector linear differential equation

$$u^{(m)}(t) = \sum_{i=1}^{m} P_i(t)u^{(i-1)}(\tau_i(t)) + q(t) \tag{1}$$

with the following complementary conditions outside I

$$u^{(i-1)}(t) = 0 \quad \text{for} \quad t \notin I \ (i = 1, \dots, m) \tag{2_1}$$

and the boundary conditions

$$u^{(i-1)}(a) = 0 \quad (i = 1, \dots, m-1), \ u^{(m-1)}(b) = 0, \tag{2_2}$$

where $m \geqslant 2$, the functions $\tau_i \colon I \to \mathbb{R}$ $(i = 1, \dots, m)$ and the matrix functions $P_i \colon I \to \mathbb{R}^{n\times n}$ $(i = 1, \dots, m)$ are measurable, $n \geqslant 1$, and the vector function $q \colon I \to \mathbb{R}^n$ is summable.

A vector function $u \colon I \to \mathbb{R}^n$ is called a solution of the problem (1), (2_1), (2_2), if

(i) u is absolutely continuous along with its derivatives up to and including the order $m-1$;

(ii) the equation (1) holds almost everywhere in I, where $u^{(i-1)}(\tau_i(t)) = 0$ for $\tau_i(t) \notin I$;

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(iii) the boundary conditions (2_2) are satisfied.

Obviously, if $a \leqslant \tau_i(t) \leqslant b$ $(i = 1, \dots, m)$ holds almost everywhere in I, then the conditions (2_1) are redundant.

We do not exclude from our considerations the case that the matrix functions P_i $(i = 1, \dots, m)$ are not summable in I. In this sense, the problem (1), (2_1), (2_2) is singular. For $\tau_i(t) \equiv t$ $(i = 1, \dots, m)$, problems of this type are discussed in [1]–[7].

In this paper, the results of [8] are used to establish optimal, in a certain sense, conditions guaranteeing unique solvability of the problem (1), (2_1), (2_2) and continuous dependence of its solutions on P_i, τ_i $(i = 1, \dots, m)$ and q.

We use the following notation and definitions:

χ_I—the characteristic function of the interval I, i.e. $\chi_I(t) = 1$ if $t \in I$ and $\chi_I(t) = 0$ if $t \notin I$;

$\mathbb{R}$—the set of the real numbers;

$\mathbb{R}^n$—the space of the column vectors $x = (x_i)_{i=1}^n$ with the components $x_i \in \mathbb{R}$ and the norm

$$\|x\| = \sum_{i=1}^{n} |x_i|;$$

$\mathbb{R}^{n\times n}$—the space of the $n\times n$ matrices $X = (x_{ik})_{i,k=1}^n$ with the components $x_{ik} \in \mathbb{R}$ and the norm

$$\|X\| = \sum_{i,k=1}^{n} |x_{ik}|;$$

$r(X)$—the spectral radius of a matrix X;

if $x = (x_i)_{i=1}^n \in \mathbb{R}^n$ and $X = (x_{ik})_{i,k=1}^n \in \mathbb{R}^{n\times n}$, then

$$|x| = (|x_i|)_{i=1}^n, \ |X| = (|x_{ik}|)_{i,k=1}^n;$$

if $x = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n \in \mathbb{R}^n$, $X = (x_{ik})_{i,k=1}^n$ and $Y = (y_{ik})_{i,k=1}^n \in \mathbb{R}^{n\times n}$, then

$$x \leqslant y \Leftrightarrow x_i \leqslant y_i \ (i = 1, \dots, n), \ X \leqslant Y \Leftrightarrow x_{ik} \leqslant y_{ik} \ (i, k = 1, \dots, n);$$

a matrix or vector function is called continuous, summable, etc. if its components are such;

$C(I; \mathbb{R}^n)$—the space of continuous vector functions $x\colon I \to \mathbb{R}^n$ with the norm

$$\|x\|_C = \max\{\|x(t)\|\colon t \in I\};$$

$L(I; \mathbb{R}^n)$—the space of summable vector functions $x\colon I \to \mathbb{R}^n$ with the norm

$$\|x\|_L = \int_a^b \|x(t)\| \, dt.$$

For arbitrary $i \in \{1, \dots, m\}$ assume

$$\tau_{0i}(t) = \begin{cases} a & \text{for } \tau_i(t) < a, \\ \tau_i(t) & \text{for } a \leqslant \tau_i(t) \leqslant b, \\ b & \text{for } \tau_i(t) > b \end{cases} \tag{3}$$

and

$$P_{0i}(t) = \chi_I(\tau_i(t))P_i(t). \tag{4}$$

Theorem 1. *Let*

$$\int_a^b [\tau_{0i}(t) - a]^{m-i} \|P_{0i}(t)\| \, \mathrm{d}t < +\infty \quad (i = 1, \dots, m). \tag{5}$$

Then the problem (1), (2_1), (2_2) *is uniquely solvable if and only if the problem*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^{m-1} \frac{1}{(m-1-i)!} P_{0i}(t) \int_a^{\tau_{0i}(t)} (\tau_{0i}(t) - s)^{m-1-i} x(s) \, \mathrm{d}s + P_{0m}(t) x(\tau_{0m}(t)), \tag{6}$$

$$x(b) = 0 \tag{7}$$

has only the trivial solution.

Proof. By (3) and (4), a vector function u is a solution of the problem (1), (2_1), (2_2) if and only if it is a solution of the differential equation

$$u^{(m)}(t) = \sum_{i=1}^{m} P_{0i}(t) u^{(i-1)}(\tau_{0i}(t)) + q(t) \tag{8}$$

with the boundary conditions (2_2).

Let u be a solution of the problem (1), (2_1), (2_2). Further, let

$$x(t) = u^{(m-1)}(t) \tag{9}$$

and

$$p(x)(t) = \sum_{i=1}^{m-1} \frac{1}{(m-1-i)!} P_{0i}(t) \int_a^{\tau_{0i}(t)} (\tau_{0i}(t) - s)^{m-1-i} x(s) \, \mathrm{d}s + P_{0m}(t) x(\tau_{0m}(t)). \tag{10}$$

Then it follows from (2_2) and (8) that

(11) $$u(t) = \frac{1}{(m-2)!} \int_a^t (t-s)^{m-2} x(s)\,\mathrm{d}s$$

and x is a solution of the vector functional-differential equation

(12) $$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = p(x)(t) + q(t)$$

satisfying the condition (7). Obviously, the inverse assertion also holds: if x is a solution of the problem (12), (7), then the vector function u defined by (11) is a solution of the problem (1), (2_1), (2_2).

Therefore, the problem (1), (2_1), (2_2) has a unique solution if and only if the problem (12), (7) has a unique solution.

It follows from (5) and (10) that $p\colon C(I;\mathbb{R}^n) \to L(I,\mathbb{R}^n)$ is a linear operator satisfying, for any $x \in C(I,\mathbb{R}^n)$, almost everywhere in I the inequality

$$\|p(x)(t)\| \leqslant \eta(t)\|x\|_C,$$

where

$$\eta(t) = \sum_{i=1}^{m-1} \frac{1}{(m-i)!} \|P_{0i}(t)\| [\tau_{0i}(t) - a]^{m-i}$$

and

$$\int_a^b \eta(t)\,\mathrm{d}t < +\infty.$$

By Theorem 1.1 from [8], the problem (12), (7) has a unique solution if and only if the homogeneous problem (6), (7) has only the trivial solution. □

Theorem 2. *Let the conditions* (5) *hold and let*

(13) $$r\left(\sum_{i=1}^{m} \frac{1}{(m-i)!} \int_a^b [\tau_{0i}(t) - a]^{m-i} |P_{0i}(t)|\,\mathrm{d}t\right) < 1.$$

Then the problem (1), (2_1), (2_2) *has a unique solution.*

P r o o f. It is sufficient to prove, as follows from Theorem 1, that the problem (6), (7) has only the trivial solution. Let $x = (x_i)_{i=1}^n$ be a solution of that problem. Then

(14) $$x(t) = \int_t^b p(x)(s)\,\mathrm{d}s \quad \text{for } t \in I,$$

where p is the operator defined by (10).

Put

$$|x|_C = (\|x_i\|_C)_{i=1}^n$$

and

$$A = \sum_{i=1}^m \frac{1}{(m-i)!} \int_a^b [\tau_{0i}(t) - a]^{m-i} |P_{0i}(t)| \, dt. \tag{15}$$

Then (10) and (14) give

$$|x|_C \leqslant A|x|_C,$$

i.e.

$$(E - A)|x|_C \leqslant 0, \tag{16}$$

where E is the unit matrix. On the other hand, by (13),

$$r(A) < 1.$$

Thus the matrix $E - A$ is not singular and $(E - A)^{-1}$ is non-negative. Multiplying (16) by $(E - A)^{-1}$, we get

$$|x|_C \leqslant 0,$$

i.e. $x(t) \equiv 0$. □

The following example shows that the condition (13) is optimal in the sense that it cannot be replaced (without further assumptions on $\tau_1, \ldots, \tau_m$), for any $i_0 \in \{1, \ldots, m\}$, by the inequality

$$r\left(\sum_{i=1}^m \frac{\gamma_i}{(m-i)!} \int_a^b [\tau_{0i}(t) - a]^{m-i} |P_{0i}(t)| \, dt\right) \leqslant 1 \tag{17}$$

where $\gamma_{i_0} = 1$ and γ_i $(i \neq i_0, i = 1, \ldots, m)$ are arbitrary great positive numbers.

Example 1. Let $0 < \delta < b - a$,

$$\tau_{i_0}(t) = b - \delta, P_{i_0}(t) = \begin{cases} \Theta & \text{for } a \leqslant t \leqslant b - \delta, \\ (m - i_0)! \delta^{-1} (b - a - \delta)^{i_0 - m} E & \text{for } b - \delta < t \leqslant b, \end{cases} \tag{18}$$

$$\tau_i(t) \equiv t, P_i(t) \equiv \Theta \ (i \neq i_0, i = 1, \ldots, m) \text{ and } q(t) \equiv 0, \tag{19}$$

where Θ and E are the null and unit $n \times n$ matrices. Then

$$\frac{1}{(m-i)!}\int_a^b [\tau_{0i}(t)-a]^{m-i}|P_{0i}(t)|\,dt = \begin{cases} \Theta & \text{for } i \neq i_0, \\ E & \text{for } i = i_0. \end{cases}$$

Therefore, (13) is violated but (17) holds for $\gamma_{i_0} = 1$ and any $\gamma_i > 0$ $(i \neq i_0, i = 1,\dots,m)$. On the other hand, in the case considered the problem (1), (2_1), (2_2) has infinitely many solutions. This easily follows from (18) and (19): for any $c \in \mathbb{R}^n$, the vector function

$$u(t) = cw(t),$$

where

$$w(t) = (t-a)^{m-1} \text{ for } a \leqslant t \leqslant b-\delta,$$

(20) $$w(t) = \sum_{i=0}^{m-2} \frac{(t-b+\delta)^i}{i!} w^{(i)}(b-\delta) + \frac{m-1}{\delta}\int_{b-\delta}^t (t-s)^{m-2}(s-b)\,ds$$
$$\text{for } b-\delta < t \leqslant b,$$

is a solution of the problem (1), (2_1), (2_2).

Example 2. Let t_i $(i = 1,\dots,m)$ be fixed points in the interval $[a,b]$, ν_i $(i = 1,\dots,m)$ arbitrary great positive numbers,

$$\tau_i(t) = (b-a)^{1-\nu_i}|t-t_i|^{\nu_i} + a,$$
$$P_i(t) = \frac{(m-i)!}{2mn}(b-a)^{(m-i)(\nu_i-1)-1}|t-t_i|^{-(m-i)\nu_i} E \ (i = 1,\dots,m),$$

and A the matrix defined by (15). Then

$$A = \frac{1}{2n}E, \ r(A) = \frac{1}{2}$$

and, by Theorem 2, the problem (1), (2_1), (2_2) has a unique solution.

Example 2 shows that in Theorem 2, the matrix functions $P_1,\dots,P_{m-1}$ may have non-integrable singularities of any order at the points of the interval $[a,b]$.

Theorem 3. *Let* $n = 1$,

(21) $$\tau_i(t) \geqslant t \text{ for almost all } t \in [a,b] \ (i = 1,\dots,m),$$

and

(22) $$\sum_{i=1}^{m-1} \frac{1}{(m-i)!}\int_a^b [\tau_{0i}(t)-a]^{m-i}|P_{0i}(t)|\,dt \leqslant \exp\left(-\int_a^b |P_{0m}(t)|\,dt\right).$$

Then the problem (1), (2_1), (2_2) *has a unique solution.*

P r o o f. First of all, we notice that, by (3) and (21),

$$\tau_{0i}(t) \geqslant t \quad (i = 1, \ldots, m) \tag{23}$$

holds almost everywhere in I. By Theorem 1 it is sufficient to prove that the problem (6), (7) has only the trivial solution. Suppose the contrary, i.e. that the problem (6), (7) has a non-trivial solution x. Then there exists $t_0 \in [a, b]$ such that

$$|x(t_0)| = \max\{|x(s)| \colon a \leqslant s \leqslant b\}, \ |x(t)| < |x(t_0)| \quad \text{for } t_0 < t \leqslant b.$$

If we now assume

$$y(t) = \max\{|x(s)| \colon t \leqslant s \leqslant b\},$$

then we shall have

$$y(t) < y(t_0) \text{ if } t_0 < t \leqslant b, \quad y(t) = y(t_0) \text{ if } a \leqslant t \leqslant t_0. \tag{24}$$

On the other hand, with regard to the inequality $\tau_{0n}(t) \geqslant t$, from (6) and (7) we get

$$y(t) \leqslant z(t) + \int_t^b |P_{0m}(s)| y(s)\,\mathrm{d}s \quad \text{for } a \leqslant t \leqslant b, \tag{25}$$

where

$$z(t) = \sum_{i=1}^{m-1} \frac{1}{(m-1-i)!} \int_t^b \left[|P_{0i}(s)| \int_a^{\tau_{0i}(s)} (\tau_{0i}(s) - \xi)^{m-1-i} y(\xi)\,\mathrm{d}\xi \right] \mathrm{d}s. \tag{26}$$

The function z is non-increasing. Thus, using Gronwall's lemma, (25) yields the estimate

$$y(t_0) \leqslant z(t_0) \exp\left(\int_{t_0}^b |P_{0m}(t)|\,\mathrm{d}t \right). \tag{27}$$

Since $y(t_0)$ is positive, this estimate implies that there exists a set $I_0 \subset [t_0, b]$ of a positive measure such that

$$\sum_{i=1}^{m-1} \frac{1}{(m-1-i)!} |P_{0i}(t)| > 0 \quad \text{for } t \in I_0.$$

Using this inequality along with the inequalities (23) and (24), we get from (26)

$$z(t_0) < y(t_0) \sum_{i=1}^{m-1} \frac{1}{(m-i)!} \int_{t_0}^b [\tau_{0i}(t) - a]^{m-i} |P_{0i}(t)|\,\mathrm{d}t.$$

However, this estimation together with the conditions (22) and (27) leads to the contradiction

$$y(t_0) < y(t_0),$$

which proves the theorem. □

As the following example confirms, the condition (22) is optimal in the sense that it cannot be replaced, for any $i_0 \in \{1, \dots, m-1\}$ and $\varepsilon \in]0,1[$, by the inequality

$$\sum_{i=1}^{m-1} \frac{\gamma_i}{(m-i)!} \int_a^b [\tau_{0i}(t) - a]^{m-i} |P_{0i}(t)| \, dt \leqslant \exp\left(- \int_a^b |P_m(s)| \, ds \right), \tag{28}$$

where $\gamma_{i_0} = 1 - \varepsilon$ and γ_i $(i \neq i_0, i = 1, \dots, m)$ are arbitrary great positive numbers.

Example 3. Let $\varepsilon \in]0,1[$, $i_0 \in \{1, \dots, m-1\}$, $\gamma_{i_0} = 1 - \varepsilon$. Further, let δ be the numbers defined by the equality

$$\left(\frac{b - a - \delta}{b - a} \right)^{m - i_0} = 1 - \varepsilon,$$

and w the function defined by (20). Finally, let

$$P_i(t) \equiv 0 \ (i \neq i_0, i = 1, \dots, m),$$

$$P_{i_0}(t) = \begin{cases} 0 & \text{for } a \leqslant t \leqslant b - \delta, \\ (m-1)! \delta^{-1} [w^{(i_0 - 1)}(b - \delta)]^{-1} & \text{for } b - \delta < t \leqslant b, \end{cases}$$

$$\tau_i(t) \equiv t \ (i = 1, \dots, m), \ q(t) \equiv 0.$$

Then

$$w^{(i_0-1)}(b - \delta) > \frac{(m-1)!}{(m-i_0)!} (b - a - \delta)^{m-i},$$

$$0 < P_{0i_0}(t) = P_{i_0}(t) < (m - i_0)! \delta^{-1} (b - a - \delta)^{i_0 - m}$$

and

$$\frac{1}{(m-i_0)!} \int_{b-\delta}^b (t-a)^{m-i_0} P_{i_0}(t) \, dt < \left(\frac{b - a - \delta}{b - a} \right)^{i_0 - 1} = (1 - \varepsilon)^{-1}.$$

Therefore,

$$\frac{1}{(m-i)!} \int_a^b [\tau_{0i}(t) - a]^{m-i} |P_{0i}(t)| \, dt \leqslant \begin{cases} 0 & \text{for } i \neq i_0, \\ (1 - \varepsilon)^{-1} & \text{for } i = i_0. \end{cases}$$

The inequality (28) is thus satisfied for $\gamma_{i_0} = 1 - \varepsilon$ and any $\gamma_i > 0$ $(i \neq i_0, i = 1, \dots, m-1)$. The unique solvability of the problem (1), (2) is violated because it has a solution $u(t) = cw(t)$ for any $c \in \mathbb{R}$.

From the inequality

$$\exp(-s) \geqslant 1 - s \quad \text{for } 0 \leqslant s \leqslant 1$$

and from Theorem 3 we obtain

Corollary. *If $n = 1$, the condition (21) is satisfied, and*

$$\sum_{i=1}^{m} \frac{1}{(m-i)!} \int_a^b [\tau_{0i}(t) - a]^{m-i} |P_{0i}(t)| \, \mathrm{d}t \leqslant 1,$$

then the problem (1), (2) has a unique solution.

As example 1 shows, in the above corollary the condition (21) is essential and cannot be omitted.

Along with the equation (1), we consider, for every positive integer k, the perturbed equation

$$u^{(m)}(t) = \sum_{i=1}^{m} P_{ik}(t) u^{(i-1)}(\tau_{ik}(t)) + q_k(t), \tag{29}$$

where $\tau_{ik}\colon I \to \mathbb{R}$ and $P_{ik}\colon I \to \mathbb{R}^{n\times n}$ are measurable, and $q_k\colon I \to \mathbb{R}^n$ is summable.

For arbitrary $i \in \{1, \ldots, m\}$ put

$$\tau_{0ik}(t) = \begin{cases} a & \text{for } \tau_{ik}(t) < a, \\ \tau_{ik}(t) & \text{for } a \leqslant \tau_{ik}(t) \leqslant b, \\ b & \text{for } \tau_{ik}(t) > b \end{cases}$$

and

$$P_{0ik}(t) = \chi_I(\tau_{ik}(t)) P_{ik}(t).$$

Theorem 4. *Let the condition (5) be satisfied and let there exist a summable function $\eta\colon I \to \mathbb{R}_+$ such that the inequalities*

$$\sum_{i=1}^{m} (\tau_{0ik}(t) - a)^{m-i} \|P_{0ik}(t)\| \leqslant \eta(t) \ (k = 1, 2, \ldots) \tag{30}$$

hold almost everywhere in I. Further, let

$$\operatorname{ess\,sup}\{|\tau_{0ik}(t) - \tau_{0i}(t)| \colon t \in I\} \to 0 \text{ if } k \to +\infty \ (1, \ldots, m), \tag{31}$$

$$\lim_{k\to+\infty} \int_a^t (\tau_{0ik}(s) - a)^{m-i} P_{0ik}(s) \, \mathrm{d}s \tag{32}$$

$$= \int_a^t (\tau_{0i}(s) - a)^{m-i} P_{0i}(s) \, \mathrm{d}s \text{ uniformly on } I \ (i = 1, \ldots, m),$$

$$\lim_{k\to+\infty} \int_a^t q_k(s) \, \mathrm{d}s = \int_a^t q(s) \, \mathrm{d}s \text{ uniformly on } I.$$

Finally, suppose that the problem (1), (2_1), (2_2) *has a unique solution* u. *Then there exists a positive integer* k_0 *such that the problem* (29), (2_1), (2_2) *has a unique solution* u_k *for every* $k \geqslant k_0$, *and*

$$\lim_{k\to+\infty} u_k^{(i-1)}(t) = u^{(i-1)}(t) \text{ uniformly on } I \ (i = 1, \dots, m-1).$$

Proof. For arbitrary $y \in C(I;\mathbb{R}^n)$ and a positive integer k put

$$g_i(y)(t) = 0 \text{ for } \tau_{0i}(t) = a, \tag{33}$$

$$g_i(y)(t) = (m-i)(\tau_{0i}(t) - a)^{i-m} \int_a^{\tau_{0i}(t)} (\tau_{0i}(t) - s)^{m-1-i} y(s)\,\mathrm{d}s$$

$$\text{for } \tau_{0i}(t) > a \quad (i = 1, \dots, m-1)$$

$$g_{ik}(y)(t) = 0 \text{ for } \tau_{0ik}(t) = a, \tag{34}$$

$$g_{ik}(y)(t) = (m-i)(\tau_{0ik}(t) - a)^{i-m} \int_a^{\tau_{0ik}(t)} (\tau_{0ik}(t) - s)^{m-1-i} y(s)\,\mathrm{d}s$$

$$\text{for } \tau_{0ik}(t) > a \quad (i = 1, \dots, m-1),$$

$$g_m(y)(t) = y(\tau_{0m}(t)), g_{mk}(y)(t) = y(\tau_{0mk}(t)), \tag{35}$$

$$p(y)(t) = \sum_{i=1}^{m} \frac{1}{(m-i)!} (\tau_{0i}(t) - a)^{m-i} P_{0i}(t) g_i(y)(t), \tag{36}$$

$$p_k(y)(t) = \sum_{i=1}^{m} \frac{1}{(m-i)!} (\tau_{0ik}(t) - a)^{m-i} P_{0ik}(t) g_{ik}(y)(t). \tag{37}$$

We consider the equation (12) and

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = p_k(x)(t) + q_k(t) \tag{38}$$

with the initial condition (7).

As shown in the proof of Theorem 1, the vector function $x(t) = u^{(m-1)}(t)$ is the unique solution of the problem (12), (7). As for the problem (29), (2_1), (2_2), it is uniquely solvable if and only if the problem (38), (7) is uniquely solvable. The solutions of these problems are connected by the equality

$$u_k(t) = \frac{1}{(m-2)!} \int_a^t (t-s)^{m-2} x_k(s)\,\mathrm{d}s.$$

We thus have to prove that there exists a positive integer k_0 such that the problem (38), (7) has a unique solution x_k for $k \geqslant k_0$ and

$$\lim_{k\to+\infty} x_k(t) = x(t) \text{ uniformly on } I.$$

By Corollary 1.6 from [8], it is sufficient to prove that

$$\lim_{k\to+\infty}\int_a^t p_k(y)(s)\,\mathrm{d}s=\int_a^t p(y)(s)\,\mathrm{d}s \text{ uniformly on } I \tag{39}$$

for any absolutely continuous vector function $y\colon I\to\mathbb{R}^n$.

By (36) and (37),

$$\left\|\int_a^t[p_k(y)(s)-p(y)(s)]\,\mathrm{d}s\right\|\leqslant\delta_k(y)+\Delta_k(y)(t), \tag{40}$$

where

$$\delta_k=\sum_{i=1}^m\int_a^b(\tau_{0ik}(s)-a)^{m-i}\|P_{0ik}(s)\|\|g_{ik}(y)(s)-g_i(y)(s)\|\,\mathrm{d}s,$$

$$\Delta_k(y)(t)=\sum_{i=1}^m\left\|\int_a^t[\tau_{0ik}(s)-a)^{m-i}P_{0ik}(s)-(\tau_{0i}(s)-a)^{m-i}P_{0i}(s)]g_i(y)(s)\,\mathrm{d}s\right\|.$$

In view of (31)-(35), the following conditions hold almost everywhere in I:

$$\lim_{k\to+\infty}\|g_{ik}(y)(t)-g_i(y)(t)\|=0\ (i=1,\dots,m)$$

and

$$\|g_{ik}(y)(t)-g_i(t)\|\leqslant 2\|y\|_C\ (i=1,\dots,m;k=1,2,\dots).$$

Combining this with the condition (30) and applying Lebesgue's theorem about a limit in an integral, we get

$$\lim_{k\to+\infty}\delta_k(y)=0. \tag{41}$$

On the other hand, in view of Lemma 2.1 from [8], it follows from (30) and (32) that

$$\lim_{k\to+\infty}\Delta_k(y)(t)=0 \text{ uniformly on } I. \tag{42}$$

Now (39) immediately follows from (40), (41) and (42). □

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