Ján Haluška

On integration in complete bornological locally convex spaces

Czechoslovak Mathematical Journal, Vol. 47 (1997), No. 2, 205–219

Persistent URL: `http://dml.cz/dmlcz/127352`

Czechoslovak Mathematical Journal, 47 (122) 1997, Praha

ON INTEGRATION IN COMPLETE BORNOLOGICAL LOCALLY CONVEX SPACES

JÁN HALUŠKA, Košice

(Received September 20, 1993)

Summary. A generalization of I. Dobrakov's integral to complete bornological locally convex spaces is given.

Keywords: complete bornological locally convex spaces, Dobrakov's integral, σ-finite semivariation, sequential convergence

MSC 1991: 46G10, 06F20

INTRODUCTION

We can observe that theories containing a certain compatible collection of basic theorems, a calculus, lie in the focus of the present measure and integration investigations. This calculus makes possible and determines further applications of the integral in a particular branch of mathematics.

Integral of I. Dobrakov. Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces, Δ a δ-ring of subsets of a set $T \neq \emptyset$, $L(\mathbf{X},\mathbf{Y})$ the space of all continuous operators $L\colon \mathbf{X} \to \mathbf{Y}$, $\mathbf{m}\colon \Delta \to L(\mathbf{X},\mathbf{Y})$ a measure σ-additive in the strong operator topology. We say that a measurable function $\mathbf{f}\colon T \to \mathbf{X}$ is *integrable in Dobrakov's sense* if there exists a sequence $\mathbf{f}_n\colon T \to \mathbf{X}, n \in \mathbb{N}$, of simple functions converging $\mathbf{m}$-a.e. to $\mathbf{f}$, such that for every $E \in \sigma(\Delta)$ (the σ-algebra generated by Δ), the sequence $\int_E \mathbf{f}_n \,\mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, is convergent in $\mathbf{Y}$, cf. [7]. The integral of the function $\mathbf{f}$ on $E \in \sigma(\Delta)$ is defined by the equality $\int_E \mathbf{f}\,\mathrm{d}\mathbf{m} = \lim\limits_{n\to\infty} \int_E \mathbf{f}_n \,\mathrm{d}\mathbf{m}$, cf. [7], Definition 2.

In [7]–[14] I. Dobrakov developed a Lebesgue-type integration theory in the Banach spaces for an operator valued measure. This theory involves *convergence theorems* *(the Lebesgue dominated theorem), integration per substitution, Fubini theorems, L_p-spaces, mean-value theorem, etc.* In [25] *a Radon-Nikodým theorem* for Dobrakov's integral is given. Papers [29], [30] present Dobrakov's integral as *a weak-type integral.* Dobrakov's integral yields a greater class of integrable functions than the also well-known (Lebesgue-type) integral of R. G. Bartle, [1], considering the same measure and set systems, cf. [7].

Dobrakov's construction of the integral is based on the Egoroff theorem. Note that the Egoroff theorem does not hold for arbitrary nets of measurable functions without some restrictions on the measure, net convergence, or the class of measurable functions. A necessary and sufficient condition in locally convex setting for the assertion that everywhere (net) convergence of measurable functions implies convergence in semivariation has been given in [19], Th. 3.3.

Various generalizations of Dobrakov's integral. In [31], W. Smith and D. H. Tucker used the idea of the *decomposition of locally convex (topological vector) spaces* (L.C.S.) *into the projective limit of normed spaces* for a generalization of Dobrakov's integral. The class of integrable functions is built via a transfinite induction starting with the class of simple functions. A representation theorem for this integral is proved.

The second generalization of Dobrakov's integral to L.C.S. is represented by papers in which authors consider *measures satisfying the so called *-condition* (e.g. [27] by R. Rao Chivukula and A. S. Sastry).

The third direction of the enlargement of Dobrakov's integral to L.C.S. is based on the fact that Dobrakov's integral is also *a weak-type integral* (e.g. papers of C. Debieve, [15], and S. K. Roy and N. D. Charkaborty, [28]). Integrals deal with functions ranging in a Banach space and with measures in locally convex spaces of continuous operators acting from a class of Banach subspaces of one locally convex space into another.

The fourth way how to extend the theory of I. Dobrakov is to avoid problems with uniform convergence of functions, i.e. to deal with L.C.S. *of functions for which a Egoroff theorem holds*, cf. the papers of M. E. Balvé, R. Bravo, and P. J. Jiménez Guerra, [2], [3], [4], [5]).

Aim of the paper. The bornological character of the bilinear integration theory developed in [27] shows the fitness of developing a bilinear integration theory in the context of bornological convex vector spaces.

The Dobrakov integral is defined in Banach spaces. If both $\mathbf{X}, \mathbf{Y}$ are considered to be inductive limits of Banach spaces, i.e. *complete bornological locally convex spaces* (C.B.L.C.S.), a natural question arises whether an integral in C.B.L.C.S. can be defined as a finite sum of Dobrakov's integrals in various Banach spaces, the choice of which may depend on the function which we integrate.

In this paper we (1) introduce a notion of σ-additive bornological operator valued measure in C.B.L.C.S., and (2) present a construction of the integral with respect to such measure.

1. Preliminaries

C.B.L.C.S. The theory of C.B.L.C.S. can be found in [23], [24], and [26].

Let $\mathbf{X}$, $\mathbf{Y}$ be two C.B.L.C.S. over the field $\mathbb{K}$ of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers equipped with bornologies $\mathfrak{B}_{\mathbf{X}}$, $\mathfrak{B}_{\mathbf{Y}}$. The basis $\mathcal{U}$ of the bornology $\mathfrak{B}_{\mathbf{X}}$ has *a marked element* $U_0 \in \mathcal{U}$, if $U_0 \subset U$ for every $U \in \mathcal{U}$. Let bases $\mathcal{U}, \mathcal{W}$ be chosen to consist of all $\mathfrak{B}_{\mathbf{X}}$-, $\mathfrak{B}_{\mathbf{Y}}$-bounded Banach disks in $\mathbf{X}$, $\mathbf{Y}$, with marked elements $U_0 \in \mathcal{U}$, $U_0 \neq \{0\}$, and $W_0 \in \mathcal{W}$, $W_0 \neq \{0\}$, respectively. Recall that *a Banach disk* in $\mathbf{X}$ is a set which is closed, absolutely convex and the linear span of which is a Banach space. The space $\mathbf{X}$ is an inductive limit of Banach spaces $\mathbf{X}_U, U \in \mathcal{U}$, $\mathbf{X} = \operatorname{inj\,lim}_{U \in \mathcal{U}} \mathbf{X}_U$, cf. [24], where $\mathbf{X}_U$ is the linear span of $U \in \mathcal{U}$ and $\mathcal{U}$ is directed by inclusion (analogously for $\mathbf{Y}$ and $\mathcal{W}$). If a sequence of elements $\mathbf{x}_n \in \mathbf{X}$, $n \in \mathbb{N}$, converges bornologically to $\mathbf{x} \in \mathbf{X}$ (in the bornology $\mathfrak{B}_{\mathbf{X}}$ with the basis $\mathcal{U}$), then we write $\mathbf{x} = \mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{x}_n$.

On $\mathcal{U}$ *the lattice operations* are defined as follows. For $U_1, U_2 \in \mathcal{U}$ we have: $U_1 \wedge U_2 = U_1 \cap U_2$, $U_1 \vee U_2 = \operatorname{acs}(U_1 \cup U_2)$, where acs denotes the topological closure of the absolutely convex span of the set. Analogously for $\mathcal{W}$. For $(U_1, W_1), (U_2, W_2) \in \mathcal{U} \times \mathcal{W}$ we write $(U_1, W_1) \ll (U_2, W_2)$ if and only if $U_1 \subset U_2$ and $W_1 \supset W_2$.

A more detailed consideration of a lattice structure of C.B.L.C.S. has been given in [20], §1.

Operator structures. Denote by $L(\mathbf{X}, \mathbf{Y})$ the space of all continuous linear operators $L \colon \mathbf{X} \to \mathbf{Y}$. The lattice structure of $L(\mathbf{X}, \mathbf{Y})$ is considered in [21]. Note that in the terminology of [26], Chap. 4, §2, Th. 1, the space $L(\mathbf{X}, \mathbf{Y})$ (as an inductive limit of seminormed spaces) is a bornological convex vector space.

Set structures. Let $T \neq \emptyset$ be a set. Denote by Δ a δ-ring of subsets of T. If $\mathcal{A}$ is a system of subsets of the set T, then $\sigma(\mathcal{A})$ denotes the σ-algebra generated by the system $\mathcal{A}$. Denote $\Sigma = \sigma(\Delta)$, $\mathbb{N} = \{1, 2, \dots\}$. We use χ_E to denote the characteristic function of the set E. By $p_U : \mathbf{X} \to [0, \infty]$ we denote the Minkowski functional of the set $U \in \mathcal{U}$. (If U does not absorb $\mathbf{x} \in \mathbf{X}$, we put $p_U(\mathbf{x}) = \infty$.) Similarly, p_W denotes the Minkowski functional of the set $W \in \mathcal{W}$.

For $(U, W) \in \mathcal{U} \times \mathcal{W}$, denote by $\hat{\mathbf{m}}_{U,W}$ the (U, W)-*semivariation* of a charge (= finitely additive measure) $\mathbf{m}\colon \Delta \to L(\mathbf{X}, \mathbf{Y})$, where

$$\hat{\mathbf{m}}_{U,W}(E) = \sup p_W\left(\sum_{i=1}^{I} \mathbf{m}(E \cap E_i)\mathbf{x}_i\right), \quad E \in \Sigma,$$

and the supremum is taken over all finite sets $\{\mathbf{x}_i \in U : i = 1, 2, \dots, I\}$ and all disjoint sets $\{E_i \in \Delta;\, i = 1, 2, \dots, I\}$. It is well-known that $\hat{\mathbf{m}}_{U,W}$ is a submeasure, i.e. a monotone, subadditive set function, and $\hat{\mathbf{m}}_{U,W}(\emptyset) = 0$. Denote by $\Delta_{U,W} \subset \Delta$ the largest δ-ring of sets $E \in \Delta$, such that $\hat{\mathbf{m}}_{U,W}(E) < \infty$. Denote $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}} = \{\hat{\mathbf{m}}_{U,W} : (U, W) \in \mathcal{U} \times \mathcal{W}\}$.

For $W \in \mathcal{W}$, denote by $|\mu|_W$ the W-*semivariation* of a charge $\mu\colon \Sigma \to \mathbf{Y}$, where

$$|\mu|_W(E) = \sup p_W\left(\sum_{i=1}^{I} \lambda_i \mu(E \cap E_i)\right), \quad E \in \Sigma,$$

and the supremum is taken over all finite sets of scalars $\{\lambda_i \in \mathbb{K};\, |\lambda_i| \leqslant 1,\ i = 1, 2, \dots, I\}$ and all disjoint sets $\{E_i \in \Delta;\, i = 1, 2, \dots, I\}$. The W-semivariation $|\mu|_W$ is a submeasure. Denote $\mu_{\mathcal{W}} = \{\mu_W;\, W \in \mathcal{W}\}$.

Various lattices of set functions (among them $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}, \mu_{\mathcal{W}}$) related to $L(\mathbf{X}, \mathbf{Y})$-valued measures have been studied in [20], §2, the lattices of set systems (and null sets) in [20], §3.

Convergences of functions. We assume that the generalizations of the classical notions (such as almost uniform convergence, almost everywhere convergence, and convergence in measure of measurable functions and relations among them) to integration in Banach spaces are commonly well-understood, cf. [7]. All this theory can be generalized to C.B.L.C.S. as follows.

Let $\beta_{\mathcal{U},\mathcal{W}}$ be a lattice of submeasures $\beta_{U,W} : \Sigma \to [0, \infty], (U, W) \in \mathcal{U} \times \mathcal{W}$, such that $\beta_{U_2,W_2} \wedge \beta_{U_3,W_3} = \beta_{U_2 \wedge U_3, W_2 \vee W_3}$, $\beta_{U_2,W_2} \vee \beta_{U_3,W_3} = \beta_{U_2 \vee U_3, W_2 \wedge W_3}$, $(U_2, W_2), (U_3, W_3) \in \mathcal{U} \times \mathcal{W}$, e.g. $\beta_{\mathcal{U},\mathcal{W}} = \hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$.

Denote by $\mathcal{O}(\beta_{U,W}) = \{N \in \Sigma;\, \beta_{U,W}(N) = 0\}$, $(U, W) \in \mathcal{U} \times \mathcal{W}$. The set $N \in \Sigma$ is called $\beta_{\mathcal{U},\mathcal{W}}$-*null* if there exists a couple $(U, W) \in \mathcal{U} \times \mathcal{W}$ such that $\beta_{U,W}(N) = 0$. We say that an assertion holds $\beta_{\mathcal{U},\mathcal{W}}$-*almost everywhere*, shortly $\beta_{\mathcal{U},\mathcal{W}}$-a.e., if it holds everywhere except in a $\beta_{\mathcal{U},\mathcal{W}}$-null set. A set $E \in \Sigma$ is said to be of *finite submeasure* $\beta_{\mathcal{U},\mathcal{W}}$ if there exists a couple $(U, W) \in \mathcal{U} \times \mathcal{W}$ such that $\beta_{U,W}(E) < \infty$.

For $E \in \Sigma, R \in \mathcal{U}$, $(U, W) \in \mathcal{U} \times \mathcal{W}$, we say that a sequence $\mathbf{f}_n : T \to \mathbf{X}$, $n \in \mathbb{N}$, of functions (R, E)-*converges* $\beta_{U,W}$-a.e. to a function $\mathbf{f}\colon T \to \mathbf{X}$ if $\lim_{n\to\infty} p_R(\mathbf{f}_n(t) - \mathbf{f}(t)) = 0$ for every $t \in E \setminus N$, where $N \in \mathcal{O}(\beta_{U,W})$. We say that a sequence $\mathbf{f}_n$:

$T \to \mathbf{X}$, $n \in \mathbb{N}$, of functions $(\mathcal{U}, E)$-*converges* $\beta_{\mathcal{U},\mathcal{W}}$-a.e. to a function $\mathbf{f}\colon T \to \mathbf{X}$ if there exist $R \in \mathcal{U}$, $(U, W) \in \mathcal{U} \times \mathcal{W}$ such that the sequence $\mathbf{f}_n$, $n \in \mathbb{N}$, of functions (R, E)-converges $\beta_{U,W}$-a.e. to $\mathbf{f}$. We write $\mathbf{f} = \mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{f}_n$ $\beta_{\mathcal{U},\mathcal{W}}$-a.e. If $E = T$, then we will simply say that the sequence R-converges $\beta_{U,W}$-a.e., or $\mathcal{U}$-converges $\beta_{\mathcal{U},\mathcal{W}}$-a.e.

For $E \in \Sigma$, $R \in \mathcal{U}$, $(U, W) \in \mathcal{U} \times \mathcal{W}$ we say that a sequence $\mathbf{f}_n\colon T \to \mathbf{X}$, $n \in \mathbb{N}$, of functions (R, E)-*converges uniformly* to a function $\mathbf{f}\colon T \to \mathbf{X}$ if $\lim_{n\to\infty} \|\mathbf{f}_n - \mathbf{f}\|_{E,R} = 0$, where $\|\mathbf{f}\|_{E,R} = \sup_{t\in E} p_R(\mathbf{f}(t))$. We say that a sequence $\mathbf{f}_n\colon T \to \mathbf{X}, n \in \mathbb{N}$, of functions (R, E)-*converges* $\beta_{U,W}$-*almost uniformly* to a function $\mathbf{f}\colon T \to \mathbf{X}$ if for every $\varepsilon > 0$ there exists a set $N \in \Sigma$ such that $\beta_{U,W}(N) < \varepsilon$ and the sequence $\mathbf{f}_n, n \in \mathbb{N}$, of functions $(R, E \setminus N)$-converges uniformly to $\mathbf{f}$. We say that a sequence $\mathbf{f}_n\colon T \to \mathbf{X}, n \in \mathbb{N}$, of functions $(\mathcal{U}, E)$-*converges* $\beta_{\mathcal{U},\mathcal{W}}$-*almost uniformly* to a function $\mathbf{f}\colon T \to \mathbf{X}$ if there exist $R \in \mathcal{U}, (U, W) \in \mathcal{U} \times \mathcal{W}$ such that the sequence $\mathbf{f}_n, n \in \mathbb{N}$, of functions (R, E)-converges $\beta_{U,W}$-almost uniformly to $\mathbf{f}$. If $E = T$, then we will simply say that the sequence of functions R-converges uniformly, or R-converges $\beta_{U,W}$-almost uniformly, or $\mathcal{U}$-converges $\beta_{\mathcal{U},\mathcal{W}}$-almost uniformly.

Convergences in measure, almost everywhere, almost uniform and relations between them have been studied in the context of $L(\mathbf{X}, \mathbf{Y})$-valued measures in C.B.L.C.S. in [21], where a Egoroff theorem has been proved, too.

2. Measures in C.B.L.C.S.

Charges of σ-finite $(\mathcal{U}, \mathcal{W})$-semivariation. We use Φ to denote the class of all functions $\mathcal{U} \to \mathcal{W}$ with an order $<$ defined as follows: for $\varphi, \psi \in \Phi$ we write $\varphi < \psi$ whenever $\varphi(U) \subset \psi(U)$ for every $U \in \mathcal{U}$.

For $(U, W) \in \mathcal{U} \times \mathcal{W}$ we say that a charge $\mathbf{m}$ is *of σ-finite (U, W)-semivariation* if there exist sets $E_i \in \Delta_{U,W}, i \in \mathbb{N}$, such that $T = \bigcup_{i=1}^{\infty} E_i$. For $\varphi \in \Phi$ we say that a charge $\mathbf{m}$ is of *σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation* if for every $U \in \mathcal{U}$ the charge $\mathbf{m}$ is of σ-finite $(U, \varphi(U))$-semivariation.

Definition 2.1. We say that a charge $\mathbf{m}$ is *of σ-finite $(\mathcal{U}, \mathcal{W})$-semivariation* if there exists a function $\varphi \in \Phi$ such that $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation.

Lemma 2.2. *Let $\varphi, \psi \in \Phi$ and $\varphi \leqslant \psi$. If a charge $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation, then $\mathbf{m}$ is also of σ_ψ-finite $(\mathcal{U}, \mathcal{W})$-semivariation.*

Proof. By the assumption, for each $U \in \mathcal{U}$ there exists a sequence $E_i(U, W) \in \Delta_{U,W}$, $i \in \mathbb{N}$, $W = \varphi(U)$, of sets such that $\bigcup_{i=1}^{\infty} E_i(U, W) = T$. From the implication $\hat{\mathbf{m}}_{U,W}(E_i(U, W)) < \infty$, $i \in \mathbb{N}$, $W \subset W_1$, $W_1 \in \mathcal{W} \Rightarrow \hat{\mathbf{m}}_{U,W_1}(E_i(U, W)) \leqslant$

$\hat{\mathbf{m}}_{U,W}(E_i(U,W))$ we see that we can put $E_i(U,\psi(U)) = E_i(U,\varphi(U))$, $i \in \mathbb{N}$. Hence $\mathbf{m}$ is of σ_ψ-finite $(\mathcal{U},\mathcal{W})$-semivariation. □

If $U \in U, \varphi \in \Phi$, and $\sigma_F(\Delta_{U,\varphi(U)})$ is the smallest local σ-ring of all sets of σ-finite $(U,\varphi(U))$-semivariation (i.e. the following implication is true: if $A \in \Delta_{U,\varphi(U)}, B \in \sigma_F(\Delta_{U,\varphi(U)})$, then $A \cap B \in \Delta_{U,\varphi(U)})$, then $\mathcal{O}_F(\hat{\mathbf{m}}_{U,\varphi(U)}) = \mathcal{O}(\hat{\mathbf{m}}_{U,\varphi(U)})$, where $\mathcal{O}_F(\hat{\mathbf{m}}_{U,\varphi(U)}) = \{N \in \sigma_F(\Delta_{U,\varphi(U)}); \hat{\mathbf{m}}_{U,\varphi(U)}(N) = 0\}$.

Lemma 2.3. *Let $\varphi \in \Phi$. If a charge $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U},\mathcal{W})$-semivariation, then $\Sigma = \sigma_F(\Delta_{U,\varphi(U)})$ for every $U \in \mathcal{U}$.*

Proof. Let $U \in \mathcal{U}$. The inclusion $\sigma_F(\Delta_{U,\varphi(U)}) \subset \Sigma$ is trivial.

Let us show that $\sigma_F(\Delta_{U,\varphi(U)}) \supset \Sigma$. Let $G \in \Sigma$. By the construction of Σ, there exist sets $G_j \in \Delta$, $j \in \mathbb{N}$, such that $\bigcup\limits_{j=1}^{\infty} G_j = G$. By the definition of the σ-finiteness of the $(U,\varphi(U))$-semivariation, there exist $T_i \in \Delta_{U,\varphi(U)}$, $i \in \mathbb{N}$, such that $T = \bigcup\limits_{i=1}^{\infty} T_i$. Clearly $T_i \cap G_j \in \Delta_{U,\varphi(U)}$. We have $G = T \cap G = \Big(\bigcup\limits_{i=1}^{\infty} T_i\Big) \cap \Big(\bigcup\limits_{j=1}^{\infty} G_j\Big) = \bigcup\limits_{j=1}^{\infty} \Big(G_j \cap \bigcup\limits_{i=1}^{\infty} T_i\Big) = \bigcup\limits_{j=1}^{\infty} \bigcup\limits_{j=1}^{\infty} (T_i \cap G_j)$, i.e. $G \in \sigma_F(\Delta_{U,\varphi(U)})$ and, therefore, $\sigma_F(\Delta_{U,\varphi(U)}) \supset \Sigma$. □

σ-additivity of measures in C.B.L.C.S. Let $W \in \mathcal{W}$. We say that a charge $\mu\colon \Sigma \to \mathbf{Y}$ is a *(W,σ)-additive vector measure*, if μ is $\mathbf{Y}_W$-valued (countable additive)vector measure. Note that if $\mu\colon \Sigma \to \mathbf{Y}$ is a (W,σ)-additive vector measure and $W \subset W_1, W, W_1 \in \mathcal{W}$, then μ is a (W_1,σ)-additive vector measure.

Definition 2.4. We say that a charge $\mu\colon \Sigma \to \mathbf{Y}$ is a *$(\mathcal{W},\sigma)$-additive vector measure*, if there exists $W \in \mathcal{W}$ such that μ is a (W,σ)-additive vector measure.

Let $W \in \mathcal{W}$. Let $\nu_n\colon \Sigma \to \mathbf{Y}$, $n \in \mathbb{N}$, be a sequence of (W,σ)-additive vector measures. Recall the following notion. If for every $\varepsilon > 0$, $E \in \Sigma$, $p_W(\nu_n(E)) < \infty$ and $E_i \in \Sigma$, $E_i \cap E_j = \emptyset$, $i \neq j$, $i,j \in \mathbb{N}$, there exists $J_0 \in \mathbb{N}$, such that for every $J \geqslant J_0$, $p_W\Big(\nu_n\Big(\bigcup\limits_{i=J+1}^{\infty} E_i \cap E\Big)\Big) < \varepsilon$ uniformly for every $n \in \mathbb{N}$, then we say that the sequence of measures ν_n, $n \in \mathbb{N}$, is *uniformly (W,σ)-additive on* Σ, cf. [6], I.1, Definition 14. Note that if a sequence ν_n, $n \in \mathbb{N}$, of measures is uniformly (W,σ)-additive on Σ, $W \in \mathcal{W}$, then the sequence ν_n, $n \in \mathbb{N}$, of measures is uniformly (W_1,σ)-additive on Σ whenever $W_1 \supset W$, $W_1 \in \mathcal{W}$.

Definition 2.5. We say that the family of measures $\nu_n\colon \Sigma \to \mathbf{Y}$, $n \in \mathbb{N}$, is *uniformly $(\mathcal{W},\sigma)$-additive on* Σ if there exists $W \in \mathcal{W}$ such that the family ν_n, $n \in \mathbb{N}$, of measures is uniformly (W,σ)-additive on Σ.

Let $\varphi \in \Phi$. We say that a charge $\mathbf{m}\colon \Delta \to L(\mathbf{X}, \mathbf{Y})$ is *a σ_φ-additive measure* if $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation, and for every $A \in \Delta_{U,\varphi(U)}$, the charge $\mathbf{m}(A \cap \cdot)\mathbf{x}\colon \Sigma \to \mathbf{Y}$ is a $(\varphi(U), \sigma)$-additive measure for every $\mathbf{x} \in \mathbf{X}_U$, $U \in \mathcal{U}$.

If $\varphi \leqslant \psi$, $\varphi, \psi \in \Phi$, and a charge $\mathbf{m}\colon \Delta \to L(\mathbf{X}, \mathbf{Y})$ is a σ_φ-additive measure, then $\mathbf{m}$ is a σ_ψ-additive measure. Indeed, the fact that $\mathbf{m}$ is of σ_ψ-finite $(\mathcal{U}, \mathcal{W})$-semivariation follows from Lemma 2.2. The assertion that for every $A \in \Delta_{U,W}$, the charge $\mathbf{m}(A \cap \cdot)\mathbf{x}\colon \Sigma \to \mathbf{Y}$ is a $(\psi(U), \sigma)$-additive measure for every $\mathbf{x} \in \mathbf{X}_U$, follows from the inequality $p_{\psi(U)}(\mathbf{y}) \leqslant p_{\varphi(U)}(\mathbf{y})$, $\mathbf{y} \in \mathbf{Y}$.

Definition 2.6. We say that a charge $\mathbf{m}\colon \Delta \to L(\mathbf{X}, \mathbf{Y})$ is *a σ-additive bornological (operator valued) measure* if there exists $\varphi \in \Phi$ such that $\mathbf{m}$ is a σ_φ-additive measure.

In what follows the charge $\mathbf{m}$ is supposed to be a σ-additive bornological measure.

3. An integral in C.B.L.C.S.

Basic spaces of functions. We use $\mathcal{M}_\mathcal{U}$ to denote *the space of all $\mathcal{U}$-measurable functions*, the largest vector space of functions $\mathbf{f}\colon T \to \mathbf{X}$ with the property: there exists $R \in \mathcal{U}$ such that for every $U \supset R, U \in \mathcal{U}$ and $\delta > 0$ the set $\{t \in T; p_U(\mathbf{f}(t)) \geqslant \delta\} \in \Sigma$. In what follows we deal only with functions which are $\mathcal{U}$-measurable, cf. [22], Definition 2.5.

A function $\mathbf{f}\colon T \to \mathbf{X}$ is called *Δ-simple* if $\mathbf{f}(T)$ is a finite set and $\mathbf{f}^{-1}(\mathbf{x}) \in \Delta$ for every $\mathbf{x} \in \mathbf{X} \setminus \{0\}$. The space of all Δ-simple functions is denoted by $\mathcal{S}$. For $(U, W) \in \mathcal{U} \times \mathcal{W}$, a function $\mathbf{f}\colon T \to \mathbf{X}$ is said to be *$\Delta_{U,W}$-simple* if $\mathbf{f} = \sum\limits_{i=1}^{I} \mathbf{x}_i \chi_{E_i}$, where $\mathbf{x}_i \in \mathbf{X}_U, E_i \in \Delta_{U,W}, E_i \cap E_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots, I$. The space of all $\Delta_{U,W}$-simple functions is denoted by $\mathcal{S}_{U,W}$. A function $\mathbf{f} \in \mathcal{S}$ is said to be *$\Delta_{\mathcal{U},\mathcal{W}}$-simple* if there exists a couple $(U, W) \in \mathcal{U} \times \mathcal{W}$ such that $\mathbf{f} \in \mathcal{S}_{U,W}$. The space of all $\Delta_{\mathcal{U},\mathcal{W}}$-simple functions is denoted by $\mathcal{S}_{\mathcal{U},\mathcal{W}}$.

Two classical theorems.

Theorem 3.1. (R. G. Bartle – N. Dunford – J. T. Schwartz) *Let Γ be a σ-additive vector measure with values in a Banach space and defined on a σ-algebra Σ. Then there exists a nonnegative real-valued σ-additive measure $\gamma\colon \Sigma \to [0, \infty)$ such that $\gamma(E) \to 0$ if and only if $|\Gamma|(E) \to 0$; the measure γ can be chosen so that $0 \leqslant \gamma(E) \leqslant |\Gamma|(E)$ for all $E \in \Sigma$.*

Proof. [6], Chap. I.2, Corollary 6, p. 14. □

Note that the measure γ in Th. 3.1 can be chosen to be finite. Such a measure is constructed in [6], Chap. I.2, the proof of Th. 4., p. 11.

The following theorem is only a rewriting of the classical Egoroff theorem.

Theorem 3.2. (D. T. Egoroff) *Let $\gamma\colon \Sigma \to [0, \infty)$ be a σ-additive measure and $E \in \Sigma$ be a set of (σ-) finite measure. If a sequence $\mathbf{f}_n \in \mathcal{M}_{\mathcal{U}}, n \in \mathbb{N}$, of functions $(\mathcal{U}, E)$-converges to a function $\mathbf{f} \in \mathcal{M}_{\mathcal{U}}$, then the sequence $\mathbf{f}_n, n \in \mathbb{N}$, of functions $(\mathcal{U}, E)$-converges γ-almost uniformly to $\mathbf{f}$.*

Proof. Same as in [18], §21, Th. A, p. 88. For the case of E being of σ-finite measure, cf. [18], §21, Exercise (3), p. 90. □

Construction of the integral. For every $E \in \Sigma$ and $\mathbf{f} \in \mathcal{S}_{U,W}$, $(U, W) \in \mathcal{U} \times \mathcal{W}$, we define the integral by the formula $\int_E \mathbf{f}\,\mathrm{d}\mathbf{m} = \sum_{i=1}^{I} \mathbf{m}(E \cap E_i)\mathbf{x}_i$, where $\mathbf{f} = \sum_{i=1}^{I} \mathbf{x}_i \chi_{E_i}$, $\mathbf{x}_i \in \mathbf{X}_U$, $E_i \in \Delta_{U,W}$, $E_i \cap E_j = \emptyset$, $i \neq j$, $i, j = 1, 2, \ldots, I$. Note that for the function $\mathbf{f}$, the integral $\int \mathbf{f}\,\mathrm{d}\mathbf{m}$ is a (W, σ)-additive measure on Σ.

Theorem 3.3. *If a sequence $\mathbf{f}_n \in \mathcal{S}_{\mathcal{U},\mathcal{W}}, n \in \mathbb{N}$ of functions $\mathcal{U}$-converges to $\mathbf{f} \in \mathcal{M}_{\mathcal{U}}$, then there exists a real-valued σ-additive measure $\gamma\colon \Sigma \to [0, 1]$ such that*

(a) *the sequence $\mathbf{f}_n, n \in \mathbb{N}$ of functions $\mathcal{U}$-converges γ-almost uniformly to $\mathbf{f}$.*

(b) *for each γ-null set $N \in \Sigma$, $\int_N \mathbf{f}_n\,\mathrm{d}\mathbf{m} = 0$ for every $n \in \mathbb{N}$.*

Proof. There exists $R \in \mathcal{U}$ such that the sequence $\mathbf{f}_n, n \in \mathbb{N}$, of functions R-converges to the function $\mathbf{f}$.

Consider $\mathbf{f}_n \in \mathcal{S}_{\mathcal{U},\mathcal{W}}, n \in \mathbb{N}$, i.e. there exist $(U_n, W_n) \in \mathcal{U} \times \mathcal{W}$ such that $\mathbf{f}_n \in \mathcal{S}_{U_n,W_n}, n \in \mathbb{N}$. For each $n \in \mathbb{N}$, the integral $\int \mathbf{f}_n\,\mathrm{d}\mathbf{m}$ is a (W_n, σ)-additive measure on Σ. By Theorem 3.1 for every $n \in \mathbb{N}$ there exist nonnegative real-valued σ-additive finite measures $\alpha_{U_n,W_n,n}$ on Σ such that $\alpha_{U_n,W_n,n}(E) \to 0$ if and only if $\left|\int \mathbf{f}_n\,\mathrm{d}\mathbf{m}\right|_{U_n,W_n}(E) \to 0$, $E \in \Sigma$. Choose the measures $\alpha_{U_n,W_n,n}$, $n \in \mathbb{N}$, so that $0 \leqslant \alpha_{U_n,W_n,n}(E) \leqslant \left|\int \mathbf{f}_n\,\mathrm{d}\mathbf{m}\right|_{U_n,W_n}(E)$ for every $E \in \Sigma$.

Construct the following set function γ on Σ:

$$\gamma(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\alpha_{U_n,W_n,n}(E)}{1 + \alpha_{U_n,W_n,n}(E)}, \qquad E \in \Sigma. \tag{1}$$

It is easy to see that $\gamma\colon \Sigma \to [0, 1]$ is a σ-additive measure on Σ.

(a) By Theorem 3.2, the sequence $\mathbf{f}_n$, $n \in \mathbb{N}$, of functions R-converges γ-almost uniformly to $\mathbf{f}$. Hence, it $\mathcal{U}$-converges γ-almost uniformly to $\mathbf{f}$.

(b) The equality (1) implies that for each γ-null set $N \in \Sigma$, $\int_N \mathbf{f}_n\,\mathrm{d}\mathbf{m} = 0$ for every $n \in \mathbb{N}$. □

Definition 3.4. Let $\mathbf{f} \in \mathcal{M}_{\mathcal{U}}$. For every $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-null set M, the function $\mathbf{f} \cdot \chi_M$ is said to be $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-null*. The family of all $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-null functions will be denoted by $\mathcal{H}_{\mathcal{U},\mathcal{W}}$. For $\mathbf{f} \in \mathcal{M}_{\mathcal{U}}$ and each $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-null set $M \in \Sigma$, define $\int_E \mathbf{f}\chi_M \,\mathrm{d}\mathbf{m} = \int_{M\cap E} \mathbf{f}_n \,\mathrm{d}\mathbf{m} = 0$, $E \in \Sigma$.

It is easy to see that the family $\mathcal{H}_{\mathcal{U},\mathcal{W}}$ is a vector space.

Lemma 3.5. *Let* $(U,W) \in \mathcal{U} \times \mathcal{W}$. *If a sequence* $\nu_n \colon \sigma(\Delta_{U,W}) \to \mathbf{Y}$, $n \in \mathbb{N}$, *is a family of uniformly* (W,σ)*-additive measures, then the* W*-semivariations* $|\nu_n|_{U,W}$, $n \in \mathbb{N}$, *of these measures are uniformly continuous on* $\sigma(\Delta_{U,W})$, *i.e.*

$$\lim_{E\to\emptyset} |\nu_n|_{U,W}(E) = 0,$$

$E \in \Sigma$, *uniformly in* $n \in \mathbb{N}$.

Proof. Same as in [7], cf. the note after Th. 1 in this paper. $\square$

Lemma 3.6. *Let* $U_n \subset U$, $W_n \subset W$, $U, U_n \in \mathcal{U}$, $W, W_n \in \mathcal{W}$, $n \in \mathbb{N}$. *If* $A \in \Delta_{U,W}$, $\mathbf{f}_n \in \mathcal{S}_{U_n,W_n}$, *then* $\mathbf{f}_n\chi_A \in \mathcal{S}_{U,W}$ *for every* $n \in \mathbb{N}$.

Proof. Clearly, $\mathbf{f}_n\chi_A \colon \Delta_{U_n,W_n} \cap \Delta_{U,W} \to \mathbf{Y}_{W_n} \subset \mathbf{Y}_W$. Since $U_n \subset U \subset U_n \vee U = U$, $W_n \cap W = W_n \wedge W$, we have $\Delta_{U_n,W_n} \cap \Delta_{U,W} \subset \Delta_{U_n\cup U,W_n\cap W} \subset \Delta_{U_n\vee U,W_n\wedge W} \subset \Delta_{U,W}$, i.e., $\mathbf{f}_n\chi_A \colon \Delta_{U,W} \to \mathbf{Y}_W$. $\square$

The proof of the following lemma is trivial.

Lemma 3.7. *Let* $(U,W) \in \mathcal{U} \times \mathcal{W}$. *If* $\mathbf{g} \in \mathcal{S}_{U,W}$ *and* $G \in \sigma(\Delta_{U,W})$, *then*

$$p_W\left(\int_G \mathbf{g}\,\mathrm{d}\mathbf{m}\right) \leqslant \|\mathbf{g}\|_{G,U}\cdot\hat{\mathbf{m}}_{U,W}(G). \tag{2}$$

Theorem 3.8. *Let* $\mathbf{m}$ *be a* σ*-additive bornological measure and* $\mathbf{f} \in \mathcal{M}_{\mathcal{U}}$. *If there exists a sequence* $\mathbf{f}_n \in \mathcal{S}_{\mathcal{U},\mathcal{W}}$, $n \in \mathbb{N}$, *of functions such that*

(a) $\mathcal{U}\text{-}\lim\limits_{n\to\infty} \mathbf{f}_n = \mathbf{f}$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-a.e.*,

(b) $\int_\cdot \mathbf{f}_n \,\mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, *are uniformly* $(\mathcal{W},\sigma)$*-additive measures on* Σ,

then the limit $\nu(E,\mathbf{f}) = \mathcal{W}\text{-}\lim\limits_{n\to\infty} \int_E \mathbf{f}_n \,\mathrm{d}\mathbf{m}$ *exists uniformly in* $E \in \Sigma$.

Proof. Let $E \in \Sigma$, $\varepsilon > 0$.

By assumption, there exist $U \in \mathcal{U}$, $(R,S) \in \mathcal{U} \times \mathcal{W}$, and $M \in \Sigma$ such that $\hat{\mathbf{m}}_{R,S}(M) = 0$ and $\lim\limits_{n\to\infty} p_U(\mathbf{f}_n(t) - \mathbf{f}(t)) = 0$ for every $t \in T \setminus M$. By Definition 3.4, $\int_E \mathbf{f}\chi_M \,\mathrm{d}\mathbf{m} = 0$. Without loss of generality, suppose that the sequence $\mathbf{f}_n$, $n \in \mathbb{N}$, of functions U-converges to $\mathbf{f}$.

Since $\mathbf{m}$ is a σ_{φ_1}-additive measure for some $\varphi_1 \in \Phi$, for U there exists $W_1 \in \mathcal{W}$ such that $\varphi_1(U) = W_1$. By assumption, there exists $W_2 \in \mathcal{W}$ such that the integrals $\int_\cdot \mathbf{f}_n \,\mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, are uniformly (W_2, σ)-additive measures on Σ. Put $\varphi(U) = W = W_1 \vee W_2$. Then the integrals $\int_\cdot \mathbf{f}_n \,\mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, are uniformly (W, σ)-additive measures on Σ and the measure $\mathbf{m}$ is also of φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation by Lemma 2.2. By virtue of σ_φ-finiteness of the (U, W)-semivariation of $\mathbf{m}$, there exist disjoint sets $A_j \in \Delta_{U,W}$, such that $\bigcup\limits_{j=1}^{\infty} A_j = T$, $j \in \mathbb{N}$.

Applying Definition 2.5, the uniform (W, σ)-additivity of integrals $\int_\cdot \mathbf{f}_n \,\mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, on Σ implies that there exists $i_0 \in \mathbb{N}$ such that for every $i \geqslant i_0$, $i \in \mathbb{N}$,

$$q_W\left(\int_{E \setminus B_i} \mathbf{f}_n \,\mathrm{d}\mathbf{m}\right) \leqslant \varepsilon \tag{3}$$

uniformly for every $n \in \mathbb{N}$, where $B_i = \bigcup\limits_{j=1}^{i} A_j$. Put $A = B_{i_0}$. Further, by Lemma 3.6, $\mathbf{f}_n \chi_A \in \mathcal{S}_{U,W}$.

Let $p \in \mathbb{N}$. By Theorem 3.3 there exists a real-valued σ-additive finite measure $\gamma\colon \Sigma \to [0, 1]$, a nondecreasing sequence of sets $F_k \in \Sigma$, $F_k \subset A$, $k \in \mathbb{N}$, and a γ-null set $N \in \Sigma$ such that $\bigcup\limits_{k=1}^{\infty} F_k = A \setminus N$, $\int_N \mathbf{f}_n \,\mathrm{d}\mathbf{m} = 0$, $n \in \mathbb{N}$, and the sequence $\mathbf{f}_n$, $n \in \mathbb{N}$, of functions (U, F_k)-converges uniformly to $\mathbf{f}$ for every $k \in \mathbb{N}$. For a given ε, there exists $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$, $n \in \mathbb{N}$, we have

$$\|\mathbf{f}_n - \mathbf{f}_{n+p}\|_{F_k, U} \leqslant \frac{\varepsilon}{\hat{\mathbf{m}}_{U,W}(A)}. \tag{4}$$

By Lemma 3.5, for a given ε there exists $k_0 \in \mathbb{N}$ such that for every $k \geqslant k_0, k \in \mathbb{N}$,

$$\left|\int_\cdot \mathbf{f}_n \,\mathrm{d}\mathbf{m}\right|_{U,W} (A \setminus F_k \setminus N) \leqslant \varepsilon \tag{5}$$

holds uniformly in $n \in \mathbb{N}$.

Let $n \geqslant n_0$, $k \geqslant k_0$. We have

$$\begin{aligned} p_W\left(\int_E \mathbf{f}_n \,\mathrm{d}\mathbf{m} - \int_E \mathbf{f}_{n+p} \,\mathrm{d}\mathbf{m}\right) &\leqslant p_W\left(\int_{E \setminus A} \mathbf{f}_n \,\mathrm{d}\mathbf{m}\right) + p_W\left(\int_{E \setminus A} \mathbf{f}_{n+p} \,\mathrm{d}\mathbf{m}\right) \\ &\quad + p_W\left(\int_{E \cap A \cap N} (\mathbf{f}_n - \mathbf{f}_{n+p}) \,\mathrm{d}\mathbf{m}\right) + p_W\left(\int_{E \cap A \setminus N} (\mathbf{f}_n - \mathbf{f}_{n+p}) \,\mathrm{d}\mathbf{m}\right); \end{aligned}$$

by (3) and Theorem 3.3(b),

$$\leqslant 2\varepsilon + 0 + p_W\left(\int_{E \cap F_k} (\mathbf{f}_n - \mathbf{f}_{n+p}) \,\mathrm{d}\mathbf{m}\right) + p_W\left(\int_{E \cap A \setminus F_k \setminus N} (\mathbf{f}_n - \mathbf{f}_{n+p}) \,\mathrm{d}\mathbf{m}\right);$$

by Lemma 3.7,

$$\leqslant 2\cdot\varepsilon + \|\mathbf{f}_n - \mathbf{f}_{n+p}\|_{E\cap F_k,U}\cdot \hat{\mathbf{m}}_{U,W}(E\cap F_k) + p_W\Big(\int_{E\cap A\setminus F_k\setminus N}\mathbf{f}_n\,\mathrm{d}\mathbf{m}\Big) + p_W\Big(\int_{E\cap A\setminus F_k\setminus N}\mathbf{f}_{n+p}\,\mathrm{d}\mathbf{m}\Big);$$

by (4) and (5),

$$\leqslant 2\cdot\varepsilon + \|\mathbf{f}_n - \mathbf{f}_{n+p}\|_{F_k,U}\cdot \hat{\mathbf{m}}_{U,W}(A) + \Big|\int_{\cdot}\mathbf{f}_n\,\mathrm{d}\mathbf{m}\Big|_{U,W}(A\setminus F_k\setminus N) + \Big|\int_{\cdot}\mathbf{f}_{n+p}\,\mathrm{d}\mathbf{m}\Big|_{U,W}(A\setminus F_k\setminus N)\leqslant 5\varepsilon.$$

Since ε is an arbitrary positive number, E an arbitrary element in Σ, and $\mathbf{Y}_W$ a complete space, the existence and the uniformity in $E\in\Sigma$ of the limit is proved. By Lemma 2.3, $\Sigma=\sigma_F(\Delta_{U,W})$, $U\in\mathcal{U}$, $W=\varphi(U)$. The theorem is proved. □

Remark 3.9. From the proof of Theorem 3.8 we see that $\nu(E,\mathbf{f})=\nu(T,\mathbf{f}\chi_E)$, $E\in\Sigma$.

Definition 3.10. A function $\mathbf{f}\in\mathcal{M}_{\mathcal{U}}$ is said to be $\Delta_{\mathcal{U},\mathcal{W}}$-*integrable*, we write $\mathbf{f}\in\mathcal{I}_{\mathcal{U},\mathcal{W}}$, if there exists a sequence $\mathbf{f}_n\in\mathcal{S}_{\mathcal{U},\mathcal{W}}$, $n\in\mathbb{N}$, of functions such that

(a) $\mathcal{U}\text{-}\lim_{n\to\infty}\mathbf{f}_n=\mathbf{f}$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e.,

(b) $\int_E\mathbf{f}_n\,\mathrm{d}\mathbf{m}$, $n\in\mathbb{N}$, are uniformly $(\mathcal{W},\sigma)$-additive measures on Σ.

The integral of the function $\mathbf{f}\in\mathcal{I}_{\mathcal{U},\mathcal{W}}$ on a set $E\in\Sigma$ is defined by the equality

$$\int_E\mathbf{f}\,\mathrm{d}\mathbf{m}=\mathcal{W}\text{-}\lim_{n\to\infty}\int_E\mathbf{f}_n\,\mathrm{d}\mathbf{m}.$$

4. Some properties of the integral

Theorem 4.1. *Let* $\mathbf{h},\mathbf{g}\in\mathcal{I}_{\mathcal{U},\mathcal{W}}$ *and* $E\in\Sigma$.
If $\mathbf{h}+\mathbf{g}=0$, *then* $\int_E\mathbf{h}\,\mathrm{d}\mathbf{m}+\int_E\mathbf{g}\,\mathrm{d}\mathbf{m}=0$.

P r o o f. Let $\mathbf{h}(T)\subset\mathbf{X}_{U_1}$, $\mathbf{g}(T)\subset\mathbf{X}_{U_2}$, $\int_{\cdot}\mathbf{h}\,\mathrm{d}\mathbf{m}\subset\mathbf{Y}_{W_1}$, $\int_{\cdot}\mathbf{g}\,\mathrm{d}\mathbf{m}\subset\mathbf{Y}_{W_2}$ for some $U_1,U_2\in\mathcal{U}$, $W_1,W_2\in\mathcal{W}$. (1) If $U_1=U_2$, $W_1=W_2$, cf. [7]. (2) The case $U_1\neq U_2$ or $W_1\neq W_2$ is reduced to (1) as follows: take $U=U_1\vee U_2$ and $W=\varphi(U)$, where $\varphi(U)=\varphi_1(U)\vee W_1\vee W_2$, $\varphi\in\Phi$, where $\varphi_1\in\Phi$ is such that T is of σ_{φ_1}-finite $(U,\varphi_1(U))$-semivariation. □

Theorem 4.2. *Let $\nu(E, \mathbf{f}) = \int_E \mathbf{f}\,\mathrm{d}\mathbf{m}$, $E \in \Sigma$, $\mathbf{f} \in \mathcal{I}_{\mathcal{U},\mathcal{W}}$. Then $\nu(.,\mathbf{f})\colon \Sigma \to \mathbf{Y}$ is a $(\mathcal{W}, \sigma)$-additive measure.*

Proof. Let $E = \bigcup_{i=1}^{\infty} E_i$, $E_i \cap E_j = \emptyset$, $E_i, E_j \in \Sigma$, $i \neq j$, $i, j \in \mathbb{N}$. By Definition 3.10, there exists $W \in \mathcal{W}$ such that for every $I \in \mathbb{N}$ and $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$, $n \in \mathbb{N}$, we have

$$p_W\left(\int_E \mathbf{f}\,\mathrm{d}\mathbf{m} - \int_E \mathbf{f}_n\,\mathrm{d}\mathbf{m}\right) < \varepsilon,$$

$$p_W\left(\int_{\bigcup_{i=1}^{I} E_i} \mathbf{f}\,\mathrm{d}\mathbf{m} - \int_{\bigcup_{i=1}^{I} E_i} \mathbf{f}_n\,\mathrm{d}\mathbf{m}\right) < \varepsilon. \tag{10}$$

By the uniform (W, σ)-additivity of the integrals $\int \mathbf{f}_n\,\mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, for every $\varepsilon > 0$ there exists $I \in \mathbb{N}$ such that

$$p_W\left(\int_E \mathbf{f}_n\,\mathrm{d}\mathbf{m} - \int_{\bigcup_{i=1}^{I} E_i} \mathbf{f}_n\,\mathrm{d}\mathbf{m}\right) < \varepsilon \tag{11}$$

uniformly for every $n \in \mathbb{N}$. Thus (10) and (11) imply

$$\begin{aligned} p_W\left(\int_E \mathbf{f}\,\mathrm{d}\mathbf{m} - \int_{\bigcup_{i=1}^{I} E_i} \mathbf{f}\,\mathrm{d}\mathbf{m}\right) &\leqslant p_W\left(\int_E \mathbf{f}_n\,\mathrm{d}\mathbf{m} - \int_{\bigcup_{i=1}^{I} E_i} \mathbf{f}_n\,\mathrm{d}\mathbf{m}\right) \\ &+ p_W\left(\int_E \mathbf{f}\,\mathrm{d}\mathbf{m} - \int_E \mathbf{f}_n\,\mathrm{d}\mathbf{m}\right) + p_W\left(\int_{\bigcup_{i=1}^{I} E_i} \mathbf{f}_n\,\mathrm{d}\mathbf{m} - \int_{\bigcup_{i=1}^{I} E_i} \mathbf{f}\,\mathrm{d}\mathbf{m}\right) < 3\varepsilon. \end{aligned}$$

□

Theorem 4.3. *Let $\mathbf{f} \in \mathcal{M}_{\mathcal{U}}$. The function $\mathbf{f} \in \mathcal{I}_{\mathcal{U},\mathcal{W}}$ if and only if there exists a sequence $\mathbf{f}_n \in \mathcal{S}_{\mathcal{U},\mathcal{W}}$, $n \in \mathbb{N}$, of functions such that*

(a) *it $(\mathcal{U}, E)$-converges $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e. to $\mathbf{f}$,*
(b) *the limit $\mathcal{W}\text{-}\lim\limits_{n\to\infty} \int_E \mathbf{f}_n\,\mathrm{d}\mathbf{m} = \nu(E)$ exists*

for every $E \in \Sigma$. In this case $\int_E \mathbf{f}\,\mathrm{d}\mathbf{m} = \nu(E)$ for every set $E \in \Sigma$ and this limit is uniform on Σ.

Proof. According to Theorem 3.8, we have to prove that the existence of the limit $\mathcal{W}\text{-}\lim\limits_{n\to\infty} \int_E \mathbf{f}_n\,\mathrm{d}\mathbf{m}$ for every $E \in \Sigma$ implies the uniform $(\mathcal{W}, \sigma)$-additivity of the integrals $\int_E \mathbf{f}_n \,\mathrm{d}\mathbf{m} = \nu_n(E)$, $n \in \mathbb{N}$. Let $E = \bigcup_{i=1}^{\infty} E_i$, $E_i \cap E_k = \emptyset$, $E_i, E_k \in \Sigma, i \neq k$, $i, k \in \mathbb{N}$, then by the definition of the W-semivariation,

$$p_W\left(\nu_n\left(\bigcup_{i=I+1}^{\infty} E_i\right)\right) \leqslant |\nu_n|_W\left(\bigcup_{i=I+1}^{\infty} E_i\right). \tag{12}$$

If ν_n, $n \in \mathbb{N}$, is a given sequence of σ-additive $\mathbf{Y}_W$-valued vector measures on $\Sigma, W \in \mathcal{W}$, and $\lim_{n\to\infty} \nu_n(E) = \nu(E) \in \mathbf{Y}_W$ exists for every set $E \in \Sigma$, then the semivariations $|\nu_n|_W(.)$, $n \in \mathbb{N}$, are uniformly continuous on Σ. From this fact and (12) we obtain the asserted uniform (W,σ)-additivity of integrals $\nu_n(\cdot) = \int_\cdot \mathbf{f} \,\mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, as a corollary. $\square$

The proof of the following theorem is easy.

Theorem 4.4. (a) *The family $\mathcal{I}_{\mathcal{U},\mathcal{W}}$ is a vector space.*
(b) *For every $E \in \Sigma$, the map $\int_E(\cdot) \,\mathrm{d}\mathbf{m}\colon \mathcal{I}_{\mathcal{U},\mathcal{W}} \to \mathbf{Y}$ is a linear operator.*

We can observe (analogously to [7]) that Theorems 3.3 and 3.8 hold when we replace sequences $\mathbf{f}_n \in \mathcal{S}_{\mathcal{U},\mathcal{W}}$, $n \in \mathbb{N}$, of functions by $\mathbf{f}_n \in \mathcal{I}_{\mathcal{U},\mathcal{W}}$, $n \in \mathbb{N}$. So, we obtain the following theorems as corollaries.

Theorem 4.5 (Theorem 3.8a). *If a sequence of functions $\mathbf{f}_n \in \mathcal{I}_{\mathcal{U},\mathcal{W}}$, $n \in \mathbb{N}$,*

(a) *$\mathcal{U}$-converges to a function $\mathbf{f} \in \mathcal{M}_{\mathcal{U},\mathcal{W}}$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e., and*
(b) *$\int_\cdot \mathbf{f}_n \,\mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, are uniformly $(\mathcal{W},\sigma)$-additive measures on Σ,*

then $\mathbf{f} \in \mathcal{I}_{\mathcal{U},\mathcal{W}}$, $\int_E \mathbf{f} \,\mathrm{d}\mathbf{m} = \mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n \,\mathrm{d}\mathbf{m}$, $E \in \Sigma$, and this limit is uniform in $E \in \Sigma$.

Theorem 4.6 (Theorem 4.3a). *If a sequence of functions $\mathbf{f}_n \in \mathcal{I}_{\mathcal{U},\mathcal{W}}$, $n \in \mathbb{N}$,*

(a) *$\mathcal{U}$-converges $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e. to a function $\mathbf{f} \in \mathcal{M}_{\mathcal{U}}$, and*
(b) *the limit $\mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n \,\mathrm{d}\mathbf{m} = \nu(E)$ exists for every $E \in \Sigma$,*

then $\mathbf{f} \in \mathcal{I}_{\mathcal{U},\mathcal{W}}$, $\int_E \mathbf{f} \,\mathrm{d}\mathbf{m} = \nu(E)$, $E \in \Sigma$, and this limit is uniform in $E \in \Sigma$.

Theorem 4.7. *The set $\mathcal{I}_{\mathcal{U},\mathcal{W}}$ is the smallest class of functions which contains $\mathcal{S}_{\mathcal{U},\mathcal{W}}$ and Theorem 4.6 holds.*

References

[1] *R. G. Bartle*: A general bilinear vector integral. Studia Math. *15* (1956), 337–352.

[2] *M. E. Ballvé*: L^α spaces for operator valued measures. Ann. Sc. Math. Québec *14* (1990), 9–15.

[3] *M. E. Ballvé*: On the stability of the completeness of L^α spaces. Rend. Circ. Mat. Palermo *39* (1990), 58–64.

[4] *M. E. Ballvé*: Geometry and integration for operator valued measures. J. Math. Anal. Appl. *165* (1992), 62–70.

[5] *R. Bravo, P. Jiménez Guerra*: Linear operators and vector integrals. Math. Japonica *36* (1991), 255–262.

[6] *J. Diestel and J. J. Uhl, Jr.*: Vector Measures. Providence, Rhode Island, 1977.

[7] *I. Dobrakov*: On integration in Banach spaces, I. Czechoslovak Math. J. *20* (1970), 511–536.

[8] *I. Dobrakov*: On integration in Banach spaces, II. Czechoslovak Math. J. *20* (1970), 680–695.

[9] *I. Dobrakov*: On representation of linear operators on $C_0(T, \mathbf{X})$. Czechoslovak Math. J. *21* (1971), 13–30.

[10] *I. Dobrakov*: On integration in Banach spaces, III. Czechoslovak Math. J. *29* ((1979), 478–499.

[11] *I. Dobrakov*: On integration in Banach spaces, IV. Czechoslovak Math. J. *30* (1980), 259–279.

[12] *I. Dobrakov*: On integration in Banach spaces, V. Czechoslovak Math. J. *30* (1980), 610–622.

[13] *I. Dobrakov*: On integration in Banach spaces, VI. Czechoslovak Math. J. *35* (1985), 173–187.

[14] *I. Dobrakov*: On integration in Banach spaces, VII. Czechoslovak Math. J. *38* (1988), 434–449.

[15] *C. Debieve*: Integration of vector valued functions with respect to vector valued measures. Rev. Roumaine Math. Pures Appl. *26* (1981), 943–957.

[16] *N. Dinculeanu*: Vector Measures. VEB Deutscher Verlag der Wissenschaften, Berlin, 1966.

[17] *N. Dunford and J. T. Schwartz*: Linear Operators. Part I: General Theory. Interscience Publishers, New York, 1958.

[18] *P. R. Halmos*: Measure Theory. Springer-Verlag, Berlin-Heidelberg-New York, 1950.

[19] *J. Haluška*: On the continuity of the semivariation in locally convex spaces. Math. Slovaca *43* (1993), 185–192.

[20] *J. Haluška*: On lattices of set functions in complete bornological locally convex spaces. Simon Stevin *67* (1993), 27–48.

[21] *J. Haluška*: On a lattice structure of operator spaces in complete bornological locally convex spaces. Tatra Mt. Math. Publ. *2* (1993), 143–147.

[22] *J. Haluška*: On convergences of functions in complete bornological locally convex spaces. Rev. Roumaine Math. Pures Appl. *38* (1993), 327–337.

[23] *H. Hogbe-Nlend*: Bornologies and Functional Analysis. North-Holland, Amsterdam-New York-Oxford, 1977.

[24] *H. Jarchow*: Locally Convex Spaces. Teubner, Stuttgart, 1981.

[25] *H. B. Maynard*: A Radon-Nikodým theorem for operator valued measures. Trans. Amer. Math. Soc. *173* (1972), 449–463.

[26] *J. V. Radyno*: Linear Equations and Bornology. Izd. Beloruskogo gosudarstvennogo universiteta, Minsk, 1982. (In in Russian.)

[27] *R. Rao Chivukula and A. S. Sastry*: Product vector measures via Bartle integrals. J. Math. Anal. Appl. *96* (1983), 180–195.
[28] *S. K. Roy and N. D. Charkaborty*: Integration of vector valued functions with respect to an operator valued measure. Czechoslovak Math. J. *36* (1986), 198–209.
[29] *C. Schwartz*: Integration for the Dobrakov integral. Czechoslovak Math. J. *30* (1980), 640–646.
[30] *C. Schwartz*: Weak Fubini theorem for the Dobrakov integral. Czechoslovak Math. J. *30* (1980), 647–654.
[31] *W. Smith and D. H. Tucker*: Weak integral convergence theorem and operator measures. Pacific J. Math. *111* (1984), 243–256.

Author's address: Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, 040 01 Košice, Slovakia, e-mail: `jhaluska@linux1.saske.sk`.