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ON m-SEMIGROUPS*

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In this paper, we discuss semigroups S with the property that every subsemigroup is an ideal of some ideal of S, or m-semigroups. We obtain that m-semigroups are periodic semigroups with zero and have index less than or equal to 5. It follows that commutative m-semigroups are archimedean semigroups with zero. Those commutative m-semigroups whose index is less than or equal to 3 are characterized.

1. PRELIMINARY RESULTS

Lemma 1.1. *Let S be a semigroup and let T be a subsemigroup of S. Then there exists an ideal J of S such that T is an ideal of J if and only if T is an ideal of S^1TS^1.*

Proof. Let S be a semigroup. Let T be a subsemigroup of S. Suppose there exists an ideal J of S such that T is an ideal of J. Then $J^1TJ^1 \subseteq T$. Since S^1TS^1 is the smallest ideal of S containing T, we have that $S^1TS^1 \subseteq J$. Therefore, we have that

$$(S^1TS^1)^1 \cdot T \cdot (S^1TS^1)^1 \subseteq J^1TJ^1 \subseteq T.$$

Hence, T is an ideal of S^1TS^1. The converse is immediate. □

We say that a semigroup S is an *m-semigroup* provided that for every subsemigroup T of S, there exists an ideal J of S such that T is an ideal of J, or equivalently, T is an ideal of S^1TS^1. Thus, for every subsemingroup T of S, there exists an ideal J that "mediates" between T and S, i.e., there exists J such that $T \lhd J \lhd S$ (where $\lhd$ indicates ideal).

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Lemma 1.2. *If S is a m-semigroup, then every subsemigroup of S is an m-semigroup.*

Proof. Let S be an m-semigroup. Let R be a subsemigroup of S, and let T be a subsemigroup of R. We claim that T is an ideal of R^1TR^1. To see this, we first notice that T is also a subsemigroup of S. Therefore, since S is an m-semigroup, T is an ideal of S^1TS^1. Thus,

$$(R^1TR^1)^1 \cdot T \cdot (R^1TR^1)^1 \subseteq (S^1TS^1)^1 \cdot T \cdot (S^1TS^1)^1 \subseteq T.$$

Hence, R is an m-semigroup. □

Lemma 1.3. *Let S be an m-semigroup. Let $\varphi\colon S \to \hat{S}$ be a homomorphism from S onto a semigroup $\hat{S}$. Then $\hat{S}$ is an m-semigroup.*

Proof. Let S be an m-semigroup. Let $\varphi\colon S \to \hat{S}$ be a homomorphism from S onto a semigroup $\hat{S}$. We claim that $\hat{S}$ is an m-semigroup. Let $\hat{T}$ be a subsemigroup of $\hat{S}$. Let $T = \varphi^{-1}[\hat{T}]$. Then T is a subsemigroup of S. Thus, T is an ideal of S^1TS^1, as S is an m-semigroup. Hence,

$$(S^1TS^1)^1 \cdot T \cdot (S^1TS^1)^1 \subseteq T.$$

Since φ is a homomorphism onto $\hat{S}$, we have that

$$\begin{aligned}(\hat{S}^1\hat{T}\hat{S}^1)^1 \cdot \hat{T} \cdot (\hat{S}^1\hat{T}\hat{S}^1)^1 &= (\varphi[S]^1\varphi[T]\varphi[S]^1)^1 \cdot \varphi[T] \cdot (\varphi[S]^1\varphi[T]\varphi[S]^1)^1 \\ &= \varphi[S^1TS^1]^1 \cdot \varphi[T] \cdot \varphi[S^1TS^1]^1 \\ &= \varphi[(S^1TS^1)^1 \cdot T \cdot (S^1TS^1)^1] \subseteq \varphi[T] = \hat{T}.\end{aligned}$$

Hence, we have the desired result. □

We note that Example 3.9 shows that the product of m-semigroups is not, in general, an m-semigroup. Proposition 3.10 shows that the product S of commutative semigroups S_α with $\operatorname{index}(S_\alpha) \leqslant 3$ is an m-semigroup if and only if each S_α is an m-semigroup.

Let S be a semigroup, and let $a \in S$. We let $\langle a \rangle$ denote the subsemigroup generated by the element a; that is, $\langle a \rangle = \{a^n : n \in \mathbb{N}\}$. The *order* of a is defined to be the order of the subsemigroup $\langle a \rangle$. The set $E(S)$ denotes the set of all idempotents of S; that is, $E(S) = \{x \in S : x^2 = x\}$. If a is an element of finite order, then it is well-known that $\langle a \rangle$ contains exactly one idempotent.

Let S be a semigroup, and let $a \in S$. If $a^m = a^n$ for some $m > n$, then the *index* of a is defined to be the least such $n \in \mathbb{N}$. If $a^m \neq a^n$ for all $m \neq n$, we say that a has infinite index. The index of a is denoted by $\operatorname{index}(a)$. We define $\operatorname{index}(S)$ to be the maximum over $a \in S$ of $\operatorname{index}(a)$, if this maximum exists. Otherwise, we say that S has infinite index, or $\operatorname{index}(S) = \infty$.

A semigroup S is said to be *periodic* provided each element has finite index. In particular, if $\operatorname{index}(S) < \infty$, then S is periodic. However, by our definitions, it is possible that S may have infinite index and be periodic.

Theorem 2.1. *If S is an* m*-semigroup, then* $\operatorname{index}(S) \leqslant 5$ *and* $E(S) = \{0\}$.

Proof. Let S be an m-semigroup, and let $a \in S$. We first claim that $\langle a \rangle$ is finite. Suppose that $\langle a \rangle$ is not finite. Then $\langle a \rangle = \{a^n : n \in \mathbb{N},\ a^{n_1} \neq a^{n_2} \text{ for } n_1 \neq n_2\}$ is a subsemigroup of S. Now, $\langle a^2 \rangle = \{a^{2k} : k \in \mathbb{N}\}$ is a subsemigroup of $\langle a \rangle$. By Lemma 1.2, $\langle a \rangle$ is an m-semigroup. Thus,

$$[\langle a \rangle^1 \langle a^2 \rangle \langle a \rangle^1]^1 \cdot \langle a^2 \rangle \cdot [\langle a \rangle^1 \langle a^2 \rangle \langle a \rangle^1]^1 \subseteq \langle a^2 \rangle .$$

Hence, $a^5 = aa^2a^2 \in [\langle a \rangle^1 \langle a^2 \rangle \langle a \rangle^1]^1 \cdot \langle a^2 \rangle \cdot [\langle a \rangle^1 \langle a^2 \rangle \langle a \rangle^1]^1 \subseteq \langle a^2 \rangle$, a contradiction. Therefore, $\langle a \rangle$ is finite and thus contains an idempotent.

We now claim that $E(S) = \{0\}$. Let $e \in E(S)$. Then $T = \{e\}$ is a subsemigroup of S. Since S is an m-semigroup, $(S^1TS^1)^1 \cdot T \cdot (S^1TS^1)^1 \subseteq T$. Hence, $(S^1eS^1)^1 e (S^1eS^1)^1 = e$. Therefore, for all $x \in S$, $xe = xe^2 \in (S^1eS^1)^1 e (S^1eS^1)^1 = e$,

$$ex = e^2x \in (S^1eS^1)^1 e (S^1eS^1)^1 = e,$$

and e is a zero for S. Thus, $E(S) = \{0\}$.

Let $a \in S$. Then $\langle a \rangle$ is finite and contains the idempotent 0. We claim that $\operatorname{index}(a) \leqslant 5$. Let p be the smallest positive integer such that $a^p = 0 \in E(S)$, and suppose $p \geqslant 6$. Then $\langle a \rangle = \{a, a^2, a^3, \ldots, a^{p-1}, a^p = 0\}$. Let

$$T = \begin{cases} \{a^2, a^4, a^6\}, & \text{if } p = 6, \\ \{a^2, a^4, a^6\} \cup \{a^n : 7 \leqslant n \leqslant p\}, & \text{if } p > 6. \end{cases}$$

Then T is a subsemigroup of $\langle a\rangle$, and

$$a^5 = aa^2a^2 \in [\langle a\rangle^1 T \langle a\rangle^1]^1 \cdot T \cdot [\langle a\rangle^1 T \langle a\rangle^1]^1 \subseteq T,$$

as $\langle a\rangle$ is an m-semigroup. This is clearly a contradiction as $a^5 \notin T$. Thus, $p \leqslant 5$, as desired. Therefore, index$(a) \leqslant 5$, for all $a \in S$. Whence, index$(S) \leqslant 5$. □

Example 2.2. This is an example to illustrate that the bound index$(S) \leqslant 5$ in Theorem 2.1 is the lowest possible upper bound. Let $S = \{0, a, b, c, d, e\}$ with multiplication given by the Cayley table:

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	a
0	0	0	a	a	b
0	0	0	a	a	b
0	0	a	b	b	c

Then S is a commutative m-semigroup whose index is 5. To see index$(S) = 5$, check the index of each element of S: index$(0) = 1$, index$(a) = 2$, index$(b) = 2$, index$(c) = 3$, index$(d) = 3$, and index$(e) = 5$. We exhibit the subsemigroups T_i of S and S^1T_i for $i = 1, \ldots, 12$:

i	T_i	S^1T_i
1	$\{0\}$	$\{0\}$
2	$\{0, b\}$	$\{0, a, b\}$
3	$\{0, a\}$	$\{0, a\}$
4	$\{0, a, d\}$	$\{0, a, b, d\}$
5	$\{0, a, c\}$	$\{0, a, b, c\}$
6	$\{0, a, c, d\}$	$\{0, a, b, c, d\}$
7	$\{0, a, b\}$	$\{0, a, b\}$
8	$\{0, a, b, d\}$	$\{0, a, b, d\}$
9	$\{0, a, b, c\}$	$\{0, a, b, c\}$
10	$\{0, a, b, c, e\}$	$\{0, a, b, c, e\}$
11	$\{0, a, b, c, d\}$	$\{0, a, b, c, d\}$
12	$\{0, a, b, c, d, e\}$	$\{0, a, b, c, d, e\}$

One may check by inspection that S is an m-semigroup.

Corollary 2.3. *Let S be an m-semigroup. Then S is periodic and $E(S) = 0$.*

Lemma 2.4. *Let S be a periodic semigroup with $E(S) = \{0\}$. For $a, b \in S$, $ab = b$ (dually, $ba = b$) if and only if $b = 0$.*

Proof. The proof is the same as that given in [3] for Lemma 3.1. □

Let S be an m-semigroup. Note that for all subsemigroups T of S, we have that $S^1T^2 \subseteq T$ and $T^2S^1 \subseteq T$. For a commutative semigroup S, S is an m-semigroup if and only if $S^1T^2 \subseteq T$ for all subsemigroups T of S.

Let S be a semigroup containing a zero element. The **annihilator** S is defined to be $A(S) = \{x \in S : xS = Sx = \{0\}\}$. We frequently denote the annihilator of a semigroup with zero by simply A.

Propostion 2.5. *Let S be an m-semigroup. Then for each $x \in S$ with $\operatorname{index}(x) > 2$, $x^{\operatorname{index}(x)-1} \in A$.*

Proof. Let S be an m-semigroup. By Theorem 2.1, $\operatorname{index}(S) \leqslant 5$.

Let $x \in S$ such that $\operatorname{index}(x) > 2$. Then $3 \leqslant \operatorname{index}(x) \leqslant 5$. Consider the subsemigroup $T = \langle x \rangle$ of S. Since S is an m-semigroup, $S^1 \langle x \rangle^2 \subseteq \langle x \rangle$ and $\langle x \rangle^2 S^1 \subseteq \langle x \rangle$.

Let $s \in S$. We wish to show that $sx^{\operatorname{index}(x)-1} = 0$ and $x^{\operatorname{index}(x)-1}s = 0$. We will show $sx^{\operatorname{index}(x)-1} = 0$ for the case when $\operatorname{index}(x) = 5$, and all other cases will follow analogously. Suppose, then, that $\operatorname{index}(x) = 5$. We claim that $sx^4 = 0$. Now, $T = \langle x \rangle = \{0, x, x^2, x^3, x^4\}$ and $sx^2 \in S^1 \langle x \rangle^2 \subseteq \langle x \rangle$. We consider cases for sx^2 equaling each element of $\langle x \rangle$.

Case 1. $sx^2 = 0$. If $sx^2 = 0$, then $sx^4 = (sx^2)x^2 = 0$, as desired.

Case 2. $sx^2 = x$. If $sx^2 = (sx)x = x$, then $x = 0$ by Lemma 2.4. Hence, $sx^4 = 0$.

Case 3. $sx^2 = x^2$. If $sx^2 = x^2$, then by Lemma 2.4 $x^2 = 0$. Hence, $sx^4 = 0$.

Case 4. $sx^2 = x^3$. If $sx^2 = x^3$, then $sx^4 = (sx^2)x^2 = x^3x^2 = x^5 = 0$.

Case 5. $sx^2 = x^4$. If $sx^2 = x^4$, then $sx^4 = (sx^2)x^2 = x^4x^2 = x^6 = 0$.

In each case, we have established that $sx^4 = 0$, as desired.

If $\operatorname{index}(x) = 4$, then $T = \langle x \rangle = \{0, x, x^2, x^3\}$. We claim that $sx^3 = 0$. Four cases analogous to Cases 1–4 above will establish this.

If $\operatorname{index}(x) = 3$, then $T = \{0, x, x^2\}$. Three cases analogous to Cases 1–3 will establish that $sx^2 = 0$.

Thus, for $3 \leqslant \operatorname{index}(x) \leqslant 5$, we have shown that $sx^{\operatorname{index}(x)-1} = 0$. Dually, we obtain that $x^{\operatorname{index}(x)-1}s = 0$. The proof is complete. □

Corollary 2.6. *Let S be an m-semigroup. Let n denote $\operatorname{index}(S)$, and suppose that $n > 2$. Then $x^{n-1} \in A$ for all $x \in S$.*

Proof. Let S be an m-semigroup with $2 < n = \operatorname{index}(S)$. Let $x \in S$. By Proposition 2.5, $x^{\operatorname{index}(x)-1} \in A$. Certainly, $\operatorname{index}(x) \leqslant n$. We may assume that index$(x) < n$ for otherwise the result is clear. Then $n - \text{index}(x) > 0$. Hence,

$$x^{n-1} = x^{\text{index}(x)-1} \cdot x^{n-\text{index}(x)} \in A \cdot S = 0.$$

Therefore, $x^{n-1} \in A$. □

Example 2.7. This is an example to illustrate that Proposition 2.5, and hence Corollary 2.6, does not hold if index$(S) = 2$. Let $S = \{0, a, b, c, d\}$ with multiplication given by:

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & a \\ 0 & 0 & a & a & 0 \end{matrix}$$

Then S is a commutative semigroup with zero such that index$(S) = 2$. We have that index$(b) = 2$, but $b \notin A$ as $b \cdot d = a \neq 0$. The semigroup S is an m-semigroup by Proposition 2.8.

Note that a semigroup S with zero satisfying the condition that $S^2 \subseteq A$ has index less than or equal to 3. Indeed, let S be such a semigroup, and let $x \in S$. Then we have that $x^3 = x(x^2) \in xA = \{0\}$. Hence, $x^3 = 0$, for all $x \in S$, and index$(S) \leqslant 3$.

Propostion 2.8. *If S is a semigroup with zero such that $S^2 \subseteq A$, then S is an* m-*semigroup.*

Proof. Let S be a semigroup with zero such that $S^2 \subseteq A$. Suppose T is a subsemigroup of S. Then $0 \in T$ since $0 = t^3 \in T$ for all $t \in T$. Let $x, y, z \in S$. Then since $xyz = 0$, we have that $(S^1TS^1)^1 \cdot T \cdot (S^1TS^1)^1 \subseteq T$ and S is an m-semigroup. □

Remark 2.9. Let S be a semigroup with zero. Then $S^2 \subseteq A$ if and only if $S^3 = 0$. To see this, suppose that $S^2 \subseteq A$. Let $x, y, z \in S$. Then we have that $xyz = x(yz) \in xA = \{0\}$. Hence, $S^3 = 0$. Conversely, suppose that $S^3 = 0$. Let $a, b \in S$. We claim that $ab \in A$. Indeed, let $c \in S$. Then $abc = 0$, since $S^3 = 0$. Therefore, $ab \in A$.

Corollary 2.10. *Let S be a semigroup with zero. If $S^3 = 0$, then* index$(S) \leqslant 3$ *and S is an* m-*semigroup.*

We recall that a commutative semigroup S is said to be **archimedean** provided that for any two elements of S, each divides some power of the other. We use "|" to denote "divides". If a relation η is defined on a commutative semigroup S by

$$(a,b) \in \eta \equiv a \mid b^n \text{ and } b \mid a^m \text{ for some } n, m \in \mathbb{N},$$

then we have the following two well-known results from [2]:

(1) The relation η on any commutative semigroup S is a congruence on S, and S/η is the maximal semilattice homomorphic image of S.

(2) Every commutative semigroup S can be uniquely expressed as a semilattice Y of archimedean semigroups C_α $(\alpha \in Y)$. The semilattice Y is isomorphic with the maximal semilattice homomorphic image S/η of S, and the C_α $(\alpha \in Y)$ are the equivalence classes of S mod η.

The next three results concern archimedean semigroups with zero.

Lemma 3.1. [3] *Let S be an archimedean semigroup with zero. Then for $a, b \in S$, $ab = b$ if and only if $b = 0$.*

Lemma 3.2. [4] *Let S be a nontrivial, finite, archimedean semigroup with zero. Then the annihilator of S contains a nonzero element.*

Let K be a semigroup. Let L be a semigroup with a zero element 0 having no element in common with K. Let $M = K \cup (L \setminus \{0\})$. If an associative binary operation $\circ$ is defined on M satisfying:

$$x \circ y \begin{cases} = xy, & \text{if } x, y \in K \text{ or if } x, y \in L \text{ and } xy \neq 0, \\ \in K, & \text{otherwise}, \end{cases}$$

then M is a semigroup with respect to $\circ$, and M is called an *extension* of K by L. If K and L are commutative, then M is a commutative semigroup and is called a *commutative extension* of K by L.

Lemma 3.3. [4] *A commutative extension of a null semigroup of order 2 by an archimedean semigroup with zero of order n is an archimedean semigroup with zero of order $n+1$, and moreover every archimedean semigroup with zero of order $n+1$ is a commutative extension of a null semigroup of order 2 by an archimedean semigroup with zero of order n.*

Corollary 3.4. *If S is a commutative* m-semigroup, *then S is an archimedean semigroup with zero such that* $\operatorname{index}(S) \leqslant 5$.

Proof. Let S be a commutative m-semigroup. Then by Corollary 2.3, S is periodic and $E(S) = \{0\}$. Thus, S is an archimedean semigroup with zero. That $\operatorname{index}(S) \leqslant 5$ was established in Theorem 2.1. □

Example 3.5. This is an example to show that the converse of Corollary 3.4 does not hold. In order to see this, we take $S = \{0, a, b, c, d, e, f\}$ with multiplication given by the following Cayley table:

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	a
0	0	0	0	0	a	b
0	0	0	0	a	a	b
0	0	0	a	a	0	0
0	0	a	b	b	0	e

Then S is an archimedean semigroup with zero such that $\operatorname{index}(S) = 3$, but S is not an m-semigroup. To see that S is not an m-semigroup, consider the subsemigroup $T = \{0, e, f\}$ of S. We see that $a = c \cdot f \cdot f \in S^1T^2$, but $a \notin T$.

Let S be a semigroup. Recall that

$$\mathscr{H} = \{(a, b) \in S \times S \colon aS^1 = bS^1 \text{ and } S^1a = S^1b\}.$$

If S is a commutative semigroup, then $\mathscr{H}$ is a congruence on S.

Propostion 3.6. *Suppose S is an archimedean semigroup containing an idempotent. Then S is $\mathscr{H}$-trivial if and only if $E(S) = \{0\}$.*

Proof. Let S be an archimedean semigroup with an idempotent e. Then $E(S) = \{e\}$. Suppose first that S is $\mathscr{H}$-trivial, i.e., $\mathscr{H} = \Delta_S$. Then $aS^1 = bS^1$ implies that $a = b$ for $a, b \in S$. Let $a \in S$. We claim that $ae = e$. Now, $aeS^1 = eaS^1 \subseteq eS^1$. Since S is archimedean with idempotent e, there is $a' \in S$ with $aa' = a'a = e$. Thus, for $x \in S^1$, $ex = eex = eaa'x$. Therefore, $eS^1 \subseteq eaS^1$. Hence, $aeS^1 = eS^1$ which implies that $ae = e$. Thus, e is a zero for S.

Conversely, let $E(S) = \{0\}$. Suppose that S is not $\mathscr{H}$-trivial. Then there are distinct $a, b \in S$ such that $(a, b) \in \mathscr{H}$. Then there exist $x, y \in S$ such that $a = bx$ and $b = ay$. Now, $(bx, b) = (a, b) \in \mathscr{H}$. Compatibility of $\mathscr{H}$ yields that $(bx^2, bx) = (bx, b) \cdot x \in \mathscr{H}$. Consequently, $(bx^{n-1}, bx^n) \in \mathscr{H}$ for all $n \in \mathbb{N}$. By transitivity of $\mathscr{H}$, we have that $(b, bx^n) \in \mathscr{H}$ for all $n \in \mathbb{N}$. Since, S is archimedean with zero, there exists $m \in \mathbb{N}$ such that $x^m = 0$. Hence, $(b, 0) = (b, bx^m) \in \mathscr{H}$. Thus, $aS^1 = bS^1 = 0S^1 = \{0\}$. Therefore, $a = bx = 0 = ay = b$, contrary to $a \neq b$. Thus, $\mathscr{H}$ is trivial. □

Lemma 3.7. *Suppose that S is a finite archimedean semigroup with zero such that* $\operatorname{index}(S) \leqslant 3$. *If $S^3 \neq 0$, then there exists $w \in S$ such that $w^2 \notin A$.*

Proof. Let S be a finite archimedean semigroup with zero such that $\operatorname{index}(S) \leqslant 3$. Suppose that $S^3 \neq 0$. Then there exists $x, y, z \in S$ such that $xyz \neq 0$. We may assume that $x, y,$ and z are distinct. Indeed, if not, by renaming elements we obtain $w, u \in S$ with $w^2u \neq 0$ or $w^2 \notin A$. We will show that there is $w \in \{x, y, z\}$ such that $w^2 \notin A$. We let n denote the order of S and use mathematical induction.

Case 1. $n = 3$. Suppose that the order of S is 3. We have distinct $x, y, z \in S$ such that $xyz \neq 0$. Therefore, $x, y, z \in S \setminus \{0\}$, contrary to $0 \in S$ and $|S| = 3$. Thus, $S^3 = 0$. This case is complete.

Case 2. $n = 4$. Suppose that the order of S is 4. We have distinct $x, y, z \in S$ such that $xyz \neq 0$. Now, $|S| = 4$ implies that $S = \{x, y, z, 0\}$. Therefore, we have $xyz \in \{0, x, y, z\}$. In any case, Lemma 3.1 yields that $xyz = 0$, a contradiction. Thus, $S^3 = 0$. This case is complete.

Case 3. $n = 5$. Suppose that the order of S is 5. We have distinct $x, y, z \in S$ such that $xyz \neq 0$. Then $x, y, z \in S \setminus A$. Since $|S| = 5$, we obtain that $S = \{0, x, y, z, xyz\}$. By Lemma 3.2, $xyz \in A$. Now, by Lemma 3.1 we have that $xy \notin \{x, y\}$ and by assumption we have that $xy \notin \{0, xyz\} \subseteq A$. Hence, $xy = z$. Likewise, $xz = y$ and $yz = x$. Thus, $x, y, z \in H_x$. However, $\mathscr{H} = \Delta_S$ by Proposition 3.6. Therefore, we have a contradiction. Hence, for any semigroup of order 5 with $\operatorname{index}(S) \leqslant 3$, $S^3 = 0$. This case is complete.

Case 4. $n = 6$. Suppose that the order of S is 6. We have distinct $x, y, z \in S$ such that $xyz \neq 0$. Then $x, y, z \in S \setminus A$. By Lemma 3.2, there exists a nonzero annihilator $u \in S$. By Lemma 3.3, S is an ideal extension of $Z \setminus \{0_Z\}$ by $N = \{0_Z, u\}$ where Z is an archimedean semigroup with zero of order 5 and N is a null (or zero) semigroup. Now, $|S| = 6$ implies that $S = \{0_S, x, y, x, u, v\}$. Thus, $Z = \{0_Z, x, y, z, v\}$. We consider the product $xyz \in Z$. By the preceding case, $xyz = 0_Z \in Z$. Thus, $xyz = 0_S \in S$, a contradiction. Hence, $x, y,$ and z cannot be distinct. Whence, by renaming elements, we obtain $w, u \in S$ with $w^2u \neq 0$, that is, $w^2 \notin A$. This case is complete.

Case 5. $n = k$. Suppose that the order of S is k. We have distinct $x, y, z \in S$ such that $xyz \neq 0$. Assume that there exists $w \in \{x, y, z\}$ such that $w^2 \notin A$. This is our inductive hypothesis.

Case 6. $n = k + 1$. Suppose that the order of S is $k + 1$. We have distinct $x, y, z \in S$ such that $xyz \neq 0$. Then $x, y, z \in S \setminus A$. By Lemma 3.2, there exists a nonzero annihilator $u \in S$. By Lemma 3.3, S is an ideal extension of $Z \setminus \{0_Z\}$ by $N = \{0_Z, u\}$ where Z is an archimedean semigroup with zero of order k and N is a null (or zero) semigroup. Then $x, y, z \notin A(S)$ implies that $x, y, z \in Z$. Now, $xyz \neq 0_S$ implies that $xyz \neq 0_Z$ as a product in Z. By inductive hypothesis, there exists $w \in \{x, y, z\}$ such that $w^2 \notin A(Z)$. Therefore, $w^2 \notin A(S)$. Hence, the general case is complete.

Therefore, the lemma is established for all finite archimedean semigroups. □

Theorem 3.8. *Let S be an archimedean semigroup with zero. Then $S^3 = 0$ if and only if S is an* m-*semigroup and* index$(S) \leqslant 3$.

Proof. Let S be an archimedean semigroup with zero. Suppose that S is an m-semigroup and index$(S) \leqslant 3$. Suppose that $S^3 \neq 0$. Then there exists $x, y, z \in S$ such that $xyz \neq 0$. We have that x, y, and z are distinct by Corollary 2.6. Consider the subsemigroup $T = \langle x, y, z\rangle = \{x, x^2, xy, xz, y, y^2, z, z^2, yz, 0\}$ of S. Then T is a finite archimedean semigroup with zero, index$(T) \leqslant 3$, and T is an m-semigroup. By Lemma 3.7, $T^3 = 0$. Then $xyz \in T^3$ implies that $xyz = 0$, a contradiction. Hence, $S^3 = 0$, as desired. The converse is immediate from Corollary 2.10. □

Example 3.9. This is an example to show that the product of m-semigroups is not an m-semigroup in general. Let $S = \{0, a, b, c, d, e\}$ with multiplication given by:

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	a
0	0	0	a	a	b
0	0	0	a	a	b
0	0	a	b	b	c

Then S is an archimedean m-semigroup. We consider the archimedean semigroup with zero $S \times S$. To see that $S \times S$ is not an m-semigroup, we consider $T = \Delta_{S\times S}$, the diagonal of S. Then T is a subsemigroup of $S \times S$. Now, $(c, e) \cdot (d, d) \cdot (e, e) = (c, e) \cdot (b, b) = (0, a) \notin T$. Hence, $(S \times S)^1 T^2 \nsubseteq T$ and $S \times S$ is not an m-semigroup.

Propostion 3.10. *Let $\{S_\alpha \colon \alpha \in I\}$ be a family of archimedean semigroups with zero such that* index$(S_\alpha) \leqslant 3$ *for all $\alpha \in I$. Let $S = \prod\{S_\alpha \colon \alpha \in I\}$ with coordinate-wise multiplication. Then* index$(S) \leqslant 3$, *and S is an* m-*semigroup if and only if S_α is an* m-*semigroup for each $\alpha \in I$.*

Proof. Let $\{S_\alpha \colon \alpha \in I\}$ be a family of archimedean semigroups with zero such that index$(S_\alpha) \leqslant 3$ for all $\alpha \in I$. Let $S = \prod\{S_\alpha \colon \alpha \in I\}$. Then for each $x \in S$,

$x^3 = 0$ since ${x_\alpha}^3 = 0_\alpha$ for each $\alpha \in I$. Hence, index$(S) \leqslant 3$. Suppose that S is an m-semigroup. Then by Lemma 1.3, $S_\alpha = \pi_\alpha[S]$ is an m-semigroup for each $\alpha \in I$.

Conversely, suppose each S_α is an m-semigroup. Therefore, for each $\alpha \in I$, ${S_\alpha}^3 = 0_\alpha$. Let T be a subsemigroup of S. Let $T_\alpha = \pi_\alpha[T]$ for each $\alpha \in I$. Then T_α is a subsemigroup of S_α for each $\alpha \in I$. Since each S_α is an m-semigroup, we have that ${S_\alpha}^1 {T_\alpha}^2 \subseteq T_\alpha$, for each $\alpha \in I$.

Let $x \in S^1$ and $y, z \in T$. Then $x = (x_\alpha), y = (y_\alpha)$, and $z = (z_\alpha)$, where $x_\alpha \in S_\alpha$ and $y_\alpha, z_\alpha \in T_\alpha$ for each $\alpha \in I$. Now, $xyz = (x_\alpha y_\alpha z_\alpha) = (0_\alpha) = 0 \in T$. Thus, $S^1 T^2 \subseteq T$, and S is an m-semigroup. □

4. Topological results

The following results are topological analogues of previous results.

We say that a topological semigroup S is an m-*semigroup* provided that for every closed subsemigroup T of S, there exists a closed ideal J of S such that T is a closed ideal of J, or equivalently (for a compact semigroup S), T is a closed ideal of $S^1 T S^1$.

Suppose S is a topological semigroup, and let $a \in S$. In the topological setting, the standard notation for the set of positive integral powers of a is $\theta(a) = \{a^n \colon n \in \mathbb{N}\}$. The topological closure of $\theta(a)$, $\Gamma(a) = \overline{\theta(a)}$, is called the *monothetic subsemigroup* of S generated by a. If $S = \Gamma(a)$ for some $a \in S$, then S is called a *monothetic semigroup*. If $\Gamma(a)$ is a compact monothetic semigroup, then its minimal ideal $M(\Gamma(a))$ is a compact abelian group and $\Gamma(a) = \theta(a) \cup M(\Gamma(a))$. Furthermore, $M(\Gamma(a))$ consists of the cluster points of $\Gamma(a)$. We define the *monothetic index* of the element a as follows:

$$\mathrm{mi}(a) = \begin{cases} \min\{n \in \mathbb{N} \colon a^n \in M(\Gamma(a))\}, & \text{if } \theta(a) \cap M(\Gamma(a)) \neq \emptyset, \\ \infty, & \text{otherwise.} \end{cases}$$

The monothetic index of a semigroup S is defined to be $\mathrm{mi}(S) = \max\{\mathrm{mi}(a) \colon a \in S\}$ if this maximum exists. Otherwise, $\mathrm{mi}(S) = \infty$.

Lemma 4.1. *Let S be a compact semigroup and let T be a closed subsemigroup of S. Then there exists a closed ideal J of S such that T is a closed ideal of J if and only if T is a closed ideal of $S^1 T S^1$.*

Lemma 4.2. *If S is a compact* m*-semigroup, then every closed subsemigroup of S is an* m*-semigroup.*

Lemma 4.3. *Let S be a compact* m*-semigroup. Let $\varphi \colon S \to \hat{S}$ be a continuous homomorphism from S onto a semigroup $\hat{S}$. Then $\hat{S}$ is a compact* m*-semigroup.*

Example 3.13 shows that arbitrary products need not preserve compact m-semigroups. Proposition 4.7 shows that the product of commutative topological semigroups S_α with $\mathrm{mi}(S_\alpha) \leqslant 3$ is a compact m-semigroup if and only if each S_α is a compact m-semigroup.

Theorem 4.4. *Let S be a compact* m*-semigroup. Then $mi(S) \leqslant 5$ and $E(S) = \{0\}$.*

P r o o f. Let S be a compact m-semigroup. Then S has a compact group minimal ideal $M(S)$. We claim that $M(S) = \{0\}$. We first show $E(S) = \{0\}$. Let $e \in E$. Then $T = \{e\}$ is a closed subsemigroup of S. Since S is a compact m-semigroup, $(S^1TS^1)^1 \cdot T \cdot (S^1TS^1)^1 \subseteq T$. Hence, $(S^1eS^1)^1e(S^1eS^1)^1 = e$. Thus,

$$xe = xe^2 \in (S^1eS^1)^1e(S^1eS^1)^1 = e,$$
$$ex = e^2x \in (S^1eS^1)^1e(S^1eS^1)^1 = e,$$

for all $x \in S$, and e is a zero for S. Thus, $E(S) = \{0\}$. Now, we have that $M(S)$ is a compact group containing a zero. Hence, $M(S) = \{0\}$.

Let $a \in S$. We claim that $\mathrm{mi}(a) \leqslant 5$. We have that $\theta(a) = \{a^n : n \in \mathbb{N}\}$ is a subsemigroup of S, and $\Gamma(a) = \overline{\theta(a)}$ is a closed and therefore compact subsemigroup of S. Certainly, $\theta(a^2) = \{a^{2k} : k \in \mathbb{N}\}$ is a subsemigroup of $\theta(a)$. Thus, $\Gamma(a^2)$ is a closed subsemigroup of $\Gamma(a)$. By Lemma 4.2, we have that $\Gamma(a)$ is a compact m-semigroup. Therefore, $[\Gamma(a)^1\Gamma(a^2)\Gamma(a)^1]^1 \cdot \Gamma(a^2) \cdot [\Gamma(a)^1\Gamma(a^2)\Gamma(a)^1]^1 \subseteq \Gamma(a^2)$. Hence,

$$\begin{aligned} a^5 = aa^2a^2 &\in [\Gamma(a)^1\Gamma(a^2)\Gamma(a)^1]^1 \cdot \Gamma(a^2) \cdot [\Gamma(a)^1\Gamma(a^2)\Gamma(a)^1]^1 \\ &\subseteq \Gamma(a^2) = \theta(a^2) \cup M(\Gamma(a)). \end{aligned}$$

Since $a^5 \notin \theta(a^2)$, we conclude that $a^5 \in M(\Gamma(a)) = \{0\}$. Thus, $\theta(a) = \{a, a^2, a^3, a^4, a^5 = 0\}$ and $\theta(a) \cap M(\Gamma(a)) \neq \emptyset$. We therefore obtain that

$$\mathrm{mi}(a) = \min\{n \in \mathbb{N} : a^n \in M(\Gamma(a))\} = \min\{n \in \mathbb{N} : a^n \in \{0\}\} \leqslant 5.$$

Since $\mathrm{mi}(a) \leqslant 5$ for all $a \in S$, we have that $\mathrm{mi}(S) \leqslant 5$. □

For a compact m-semigroup S, the concepts of index and monothetic index are equivalent. Indeed, let S be a compact m-semigroup. Then by Theorem 4.4, $E(S) = 0$ and $\mathrm{mi}(S) \leqslant 5$. Therefore, we have that $M(S) = 0$. Thus for $a \in S$, we see that $\mathrm{mi}(a) = \min\{n \in \mathbb{N} : a^n \in M(\Gamma(a))\} = \min\{n \in \mathbb{N} : a^n = 0\} = \mathrm{index}(a)$.

Corollary 4.5. *Suppose S is a compact* m*-semigroup. Then S is periodic and $E(S) = 0$.*

Theorem 4.6. *Let S be a compact archimedean semigroup with zero. Then $S^3 = 0$ if and only if S is an* m*-semigroup and mi$(S) \leqslant 3$.*

Propostion 4.7. *Suppose $\{S_\alpha : \alpha \in I\}$ is a family of compact archimedean semigroups with zero such that mi$(S_\alpha) \leqslant 3$ for all $\alpha \in I$. If $S = \prod\{S_\alpha : \alpha \in I\}$ with coordinate-wise multiplication, then S is compact and mi$(S) \leqslant 3$. Moreover, S is an* m*-semigroup if and only if S_α is an* m*-semigroup for each $\alpha \in I$.*

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