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SOME TOPOLOGICAL PROPERTIES PRESERVED BY NEARNESS BETWEEN OPERATORS AND APPLICATIONS TO P.D.E.

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0. INTRODUCTION

In this paper some topological properties which are preserved by *nearness* between operators and which have significant applications in the existence theory of solutions for differential equations are studied. Before explaining shortly the content of each part of the paper, we shall give the definition of *near operators* and the basic theorem on isomorphism (Cf. [C_1]).

Let $\mathcal{X}$ be a set and $\mathcal{B}$ a Banach space normed with $\| \, \|$. Let A, B be operators between $\mathcal{X}$ and $\mathcal{B}$.

Definition 0.1. We say that A is *near* B if there exist two positive constants α and k, with $k \in (0,1)$, such that we have

$$(0.1) \quad \|B(x_1) - B(x_2) - \alpha[A(x_1) - A(x_2)]\| \leqslant k\|B(x_1) - B(x_2)\|, \quad \forall x_1, x_2 \in \mathcal{X}.$$

Theorem 0.1. *Let A be near B. If B is a bijective operator between $\mathcal{X}$ and $\mathcal{B}$, then A is bijective between $\mathcal{X}$ and $\mathcal{B}$, too.*

We recall that the theory of *near* operators was introduced in [C_1] for studying the existence and regularity of solutions for nonlinear non-variational elliptic systems. Subsequently, it was also applied to nonlinear parabolic systems ([C_2], [T_1]). The theory of *near* operators is connected with the monotone operator theory (see for example [C_3]).

In §1 we shall prove some preliminary results about topological properties of operators Φ, between $\mathcal{B}$ and $\mathcal{B}$, that are *near* $I_{\mathcal{B}}$ (identity map on $\mathcal{B}$). Theorem 1.1 in §1 requires some further remarks. It establishes that if $\Phi\colon \Omega_1 \to \mathcal{B}$ is *near* $I_{\mathcal{B}}$ and if Ω_1 is open in $\mathcal{B}$ then $\Omega_2 = \Phi(\Omega_1)$ is open in $\mathcal{B}$. Consequently, Theorem 1.1, when $\mathcal{B}$ has a finite demension, is a particular case of the following topology theorem.

Theorem 0.2. *Let $\mathcal{B}$ be an n-dimensional manifold, Ω_1 open in $\mathcal{B}$, $\Phi\colon \Omega_1 \to \mathcal{B}$ a bijective continuous map between Ω_1 and $\Omega_2 = \Phi(\Omega_1)$. Then, if Ω_1 is open in $\mathcal{B}$, Ω_2 is open in $\mathcal{B}$, too.*

(The proof of this Theorem is not easy and requires homology theory, see §18 of [G].)

Theorem 0.2 fails in finite dimension, even if we ask for stronger hypothesis on Φ, for example Lipschitzianity (see Remark 1.1). Then Theorem 1.1 is one of possible generalizations of Theorem 0.2 to infinite dimension. It is easy to prove that if Φ is *near* $I_{\mathcal{B}}$ then Φ is injective and Lipschitzian.

In §2 it is proved that if A is *near* B then

(i) If $B(\mathcal{X})$ is open in $\mathcal{B}$, $A(\mathcal{X})$ is open in $\mathcal{B}$, too.[1]

(ii) If $B(\mathcal{X})$ is dense in $\mathcal{B}$, $A(\mathcal{X})$ is dense in $\mathcal{B}$, too.

In §3 proposition (ii) is used to prove local existence of a solution for the following nonlinear elliptic system:

$$\sum_{i,j=1}^{n} A_{ij}(x,u)D_{ij}u(x) = f(x).$$

We observe that the results of §3 can be generalized, by easy modifications of hypotheses, to the systems

$$a(x,u,H(u)) = f(x),$$

where $H(u) = \{D_{ij}u\}_{i,j=1,\dots,n}$.

Consequently, the theory of *near* operators, up to now applied to systems in which, for every ξ, $a(x,u,\xi)$ is bounded in u (cf. [C_3]), can be extended, by Proposition (i), to systems in which $a(x,u,\xi)$ has a linear growth in u.

Up to now the theory of *near* operators has met with some difficulties to be applied to hyperbolic problems, perhaps owing to spaces on which these problems work. Proposition (ii) can be a useful tool to overcome these difficulties. In fact in §4 proposition (ii) is used to prove an existence theorem of a periodic solution of the following second order non linear equation in Hilbert spaces:

$$Au(t) + u''(t) + F(t,u'(t)) + cu(t) = g(t).$$

This equation has been also studied with other techniques by many authors, see for example [HA] (abstract case) or [P_1], [P_2], [R] (some concrete cases). The results obtained here can be compared with [P_1] (see Remark 4.1).

[1] $A(\mathcal{X}) = \{y \in \mathcal{B}\colon \exists x \in \mathcal{X}$ such that $y = A(x)\}$. The same definition can be given for $B(\mathcal{X})$.

Nevertheless, we observe that the above solved problem is only a first approach, for the theory of *near* operators, to hyperbolic problems. In fact, nonlinearity of the equation does not concern the principal part of the operator as it is the case instead when this theory is applied to elliptic or parabolic problems.

1. OPERATORS NEAR IDENTITY

Let $\mathcal{B}$ be a Banach space normed with $\| \ \|$, Ω_1 a open subset of $\mathcal{B}$ and $\Phi\colon \Omega_1 \to \mathcal{B}$. We set $\Omega_2 = \Phi(\Omega_1)$.

We say that Φ is *near* $I_{\mathcal{B}}$, identity on $\mathcal{B}$, if $\exists \alpha > 0$ and $K \in (0,1)$ such that

$$\|y_1 - y_2 - \alpha[\Phi(y_1) - \Phi(y_2)]\| \leqslant K\|y_1 - y_2\|, \quad \forall y_1, y_2 \in \Omega_1. \tag{1.1}$$

Lemma 1.1. *Let* (1.1) *hold. If* $0 \in \Omega_1$ *and* $\Phi(0) = 0$ *then there exists a* $\sigma_1 > 0$ *such that* $S(0,\sigma_1) \subseteq \Omega_2$.[2]

Proof. Proving this lemma is equivalent to prove that $\forall\, y \in S(0,\sigma_1)\ \exists x \in \Omega_1$ such that $\Phi(x) = y$. Let r be a positive number such that $S(0,r) \subseteq \Omega_1$. We choose $y \in S(0,\sigma)$, with $\sigma > 0$ and consider $\mathcal{T}\colon S(0,r) \to \mathcal{B}$, a map defined in the following way:

$$\mathcal{T}(x) = y - [\alpha\Phi(x) - x], \quad x \in S(0,r).$$

$\mathcal{T}$ possesses the following properties:

i) $\mathcal{T}(S(0,r)) \subseteq S(0,r)$. In fact

$$\|\mathcal{T}(x)\| = \|y - [\alpha\Phi(x) - x]\| \leqslant \|y\| + \|\alpha[\Phi(x) - \Phi(0)] - (x - 0)\|.$$

From this estimate and from (1.1), we obtain that there exists $K \in (0,1)$ such that $\|\mathcal{T}(x)\| \leqslant \|y\| + K\|x\| \leqslant \sigma + Kr \leqslant r$ for $\sigma \leqslant (1-K)r$.

ii) $\|\mathcal{T}(x_1) - \mathcal{T}(x_2)\| \leqslant K\|x_1 - x_2\|, \forall x_1, x_2 \in S(0,r)$. In fact, by means of (1.1) we obtain $\|\mathcal{T}(x_1) - \mathcal{T}(x_2)\| \leqslant \|x_1 - x_2 - \alpha[\Phi(x_1) - \Phi(x_2)]\| \leqslant K\|x_1 - x_2\|$ $(0 < K < 1)$.

□

It follows from i), ii) and the theorem of contractions that there exists a unique $x \in S(0,r)$ such that $\mathcal{T}(x) = x$, that is $y - \alpha\Phi(x) + x = x$.

Finally, $\forall y \in S(0,\sigma)$ (with $\sigma \leqslant (1-K)r$) $\exists_1\ x \in S(0,r)$ such that $y = \alpha\Phi(x)$.

We obtain that $\forall y \in S(0,\sigma_1)$ with $\sigma_1 = \sigma/\alpha$, there exists a unique $x \in S(0,r)$ such that $y = \Phi(x)$.

Theorem 1.1. *Let* (1.1) *hold. If* Ω_1 *is open in* $\mathcal{B}$, *then* Ω_2 *is open in* $\mathcal{B}$, *too.*

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[2] $S(0,\sigma_1) = \{y \in \mathcal{B}\colon \|y\| \leqslant \sigma_1\}$.

P r o o f. We prove that for every $y_0 \in \Omega_2$ $\exists S(y_0, \sigma_1) \subset \Omega_2$. Let $x_0 \in \Omega_1$ be such that $y_0 = \Phi(x_0)$. We set

$$\begin{aligned}\widetilde{\Omega}_1 &= \{z \in \mathcal{B}\colon z = x - x_0, x \in \Omega_1\} = \Omega_1 - x_0,\\ \widetilde{\Omega}_2 &= \{z \in \mathcal{B}\colon z = y - y_0, y \in \Omega_2\} = \Omega_2 - y_0,\\ \widetilde{\Phi}(Z) &= \Phi(z + x_0) - \Phi(x_0), \quad z \in \widetilde{\Omega}_1.\end{aligned}$$

It is obvious that $\widetilde{\Phi}\colon \widetilde{\Omega}_1 \longrightarrow \mathcal{B}, \widetilde{\Omega}_2 = \widetilde{\Phi}(\widetilde{\Omega}_1), \widetilde{\Omega}_1$ is open, $0 \in \widetilde{\Omega}_1, \widetilde{\Phi}(0) = 0$. Moreover, Φ verifies hypothesis (1.1) and the same is true for $\widetilde{\Phi}$. In fact, for every $z_1, z_2 \in \widetilde{\Omega}_1$ we have

$$\begin{aligned}&\|z_1 - z_2 - \alpha[\widetilde{\Phi}(z_1) - \widetilde{\Phi}(z_2)]\|\\ &= \|(z_1 + x_0) - (z_2 + x_0) - \alpha[\Phi(z_1 + x_0) - \Phi(z_2 + x_0)]\| \leqslant K\|z_1 - z_2\|.\end{aligned}$$

We get the hypothesis of Lemma 1.1 for $\widetilde{\Omega}_1$ and $\widetilde{\Phi}$, consequently $\exists S(0, \sigma_1) \subset \widetilde{\Omega}_2$ and so there exists a $S(y_0, \sigma_1) \subset \Omega_2$. □

Remark 1.1. Theorem 1.1 fails if we assume for Φ only injectivity and Lipschitzianity. Example:

$\mathcal{B} = \left\{x\colon x = \{x_n\}_{n\in\mathbb{N}}, x_n \in \mathbb{R}, \sum\limits_{n=1}^{\infty} x_n^2 < +\infty\right\}$ normed with $\|x\|_{\mathcal{B}} = \left(\sum\limits_{n=1}^{\infty} x_n^2\right)^{1/2}$;
$\Omega_1 = \mathcal{B}$;
$\Omega_2 = \{x\colon x \in \mathcal{B}, x_1 = 0\}$;
$\Phi(x) = \Phi(x_1, x_2, \ldots, x_n, \ldots) = (0, x_1, x_2, \ldots, x_n, \ldots)$.

Φ is a isometry, Ω_1 is open, while Ω_2 is not. On the other hand, Φ is not near $I_{\mathcal{B}}$ because estimate (1.1) fails at points $y_1 = (x_1, 0, 0, \ldots)$ and $y_2 = (0, 0, \ldots, 0, \ldots)$ for every α and k.

Theorem 1.2. *Let* (1.1) *hold. If* Ω_1 *is dense in* $\mathcal{B}$ *then* Ω_2 *is dense in* $\mathcal{B}$, *too.*

P r o o f. Fix $y \in \mathcal{B} - \Omega_1$, let $\{y_n\}_{n\in\mathbb{N}} \subset \Omega_1$ be such that $y_n \longrightarrow y$ in $\mathcal{B}$. In particular $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}$. We obtain from this and supposing Φ *near* $I_{\mathcal{B}}$ that $\{\Phi(y_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}$. In fact

$$\begin{aligned}\|\Phi(y_n) - \Phi(y_m)\| &\leqslant \frac{1}{\alpha}\|y_n - y_m - \alpha[\Phi(y_n) - \Phi(y_m)]\| + \frac{1}{\alpha}\|y_n - y_m\|\\ &\leqslant \frac{k+1}{\alpha}\|y_n - y_m\| \qquad \forall n, m \in \mathbb{N}.\end{aligned}$$

Then we set

$$\overline{\Phi}(y) = \begin{cases}\Phi(y) & \text{if } y \in \Omega_1,\\ \lim\limits_{n\to\infty} \Phi(y_n) & \text{if } y \notin \Omega_1.\end{cases} \tag{1.2}$$

$\bar{\Phi}\colon \mathcal{B} \to \mathcal{B}$ is a well defined map, because if $\{\bar{y}_n\}_{n\in\mathbb{N}}$ is another sequence in Ω_1 which converges to y, as Φ is *near* $I_{\mathcal{B}}$ we obtain

$$\begin{aligned}\|\Phi(y_n) - \Phi(\bar{y}_n)\| &\leqslant \frac{1}{\alpha}\|y_n - \bar{y}_n - \alpha[\Phi(y_n) - \Phi(\bar{y}_n)]\| + \frac{1}{\alpha}\|y_n - \bar{y}_n\| \\ &\leqslant \frac{k+1}{\alpha}\|y_n - \bar{y}_n\| \qquad \forall n \in \mathbb{N},\end{aligned}$$

so that $\lim\limits_{n\to\infty} \Phi(\bar{y}_n) = \lim\limits_{n\to\infty} \Phi(y_n) = \bar{\Phi}(y)$. □

We shall now prove that $\bar{\Phi}$ is near $I_{\mathcal{B}}$. $\forall y, z \in \mathcal{B}$ let $\{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ be two sequences in Ω_1 which converge respectively to y and z. As Φ is *near* $I_{\mathcal{B}}$, we obtain the followig estimate

$$\begin{aligned}\|y - z - \alpha[\bar{\Phi}(y) - \bar{\Phi}(z)]\| \leqslant{}& \|y - y_n\| + \|z - z_n\| + \|y_n - z_n - \alpha[\Phi(y_n) - \Phi(z_n)]\| \\ &+ \alpha\|\Phi(y_n) - \bar{\Phi}(y)\| + \alpha\|\Phi(z_n) - \bar{\Phi}(z)\|.\end{aligned}$$

We get from this and from the definition of $\bar{\Phi}$ the following estimate: $\forall \varepsilon > 0$

$$\|y - z - \alpha[\bar{\Phi}(y) - \bar{\Phi}(z)]\| \leqslant 2(1 + \alpha + k)\varepsilon + k\|y - z\|.$$

It follows that $\bar{\Phi}$ is near $I_{\mathcal{B}}$, so we get from Theorem 0.1 that $\bar{\Phi}$ is a bijective map between $\mathcal{B}$ and $\mathcal{B}$. If we fix $w \in \mathcal{B}$, we can find an $x \in \mathcal{B}$ such that $\bar{\Phi}(x) = w$. Let $\{x_n\}_{n\in\mathbb{N}} \subset \Omega_1$ be such that $\lim\limits_{n\to\infty} \Phi(x_n) = \bar{\Phi}(x)$. If we set $w_n = \Phi(x_n)$ we get that $\{w_n\}_{n\in\mathbb{N}} \subset \Omega_2$ and $w_n \to w$, so Ω_2 is dense in $\mathcal{B}$.

2. Some topological properties preserved by nearness between operators

Let $\mathcal{X}$ be a set and $\mathcal{B}$ a Banach space normed with $\|\ \|$. Let A, B be operators between $\mathcal{X}$ and $\mathcal{B}$.

Lemma 2.1. *If A is near B then*
$A(x_1) = A(x_2)$ *if and only if* $B(x_1) = B(x_2)$, $\forall x_1, x_2 \in \mathcal{X}$.

P r o o f. Let $B(x_1) = B(x_2)$. Then from hypothesis of vicinity between A and B (see definition (0.1)), we get

$$\|\alpha[A(x_1) - A(x_2)]\| = \|B(x_1) - B(x_2) - \alpha[A(x_1) - A(x_2)]\| \leqslant K\|B(x_1) - B(x_2)\| = 0,$$

hence $A(x_1) = A(x_2)$. Conversely, if $A(x_1) = A(x_2)$ then

$$\|B(x_1) - B(x_2)\| = \|B(x_1) - B(x_2) - \alpha[A(x_1) - A(x_2)]\| \leqslant K\|B(x_1) - B(x_2)\|$$

from which $\|B(x_1) - B(x_2)\|(1 - K) \leqslant 0$, hence $B(x_1) = B(x_2)$ because $k \in (0, 1)$. □

Consequently, from Lemma 2.1 we obtain the following corollary:

Corollary 2.1. *If A is near B, then A is injective if and only if B is injective.*

Now we define $\Phi\colon B(\mathcal{X}) \longrightarrow A(\mathcal{X})$ as follows:

$$\forall y \in B(\mathcal{X}) \quad \text{we set} \quad \Phi(y) = A(x) \quad \text{with} \quad x \in B^{-1}(y). \tag{2.1}$$

Φ is well defined by (2.1) because it does not depend of the choice of $x \in B^{-1}(y)$. In fact $\forall x_1, x_2 \in B^{-1}(y)$ one has $B(x_1) = B(x_2)$, so that by Lemma 2.1 it follows that $A(x_1) = A(x_2)$.

The map Φ, defined by (2.1), can be described equivalently as follows:

$$\Phi(y) = AB^{-1}(y) \qquad \forall y \in B(\mathcal{X}). \tag{2.2}$$

Lemma 2.2. *A is near B if and only if Φ is near to $I_{\mathcal{B}}$.*

Proof. Let $y_1, y_2 \in B(\mathcal{X})$ and $x_1, x_2 \in \mathcal{X}$ be such that $B(x_1) = y_1$, $B(x_2) = y_2$. We have

$$\begin{aligned}
&\|y_1 - y_2 - \alpha[\Phi(y_1) - \Phi(y_2)]\| \\
&\quad = \|B(x_1) - B(x_2) - \alpha[AB^{-1}(y_1) - AB^{-1}(y_2)]\| \\
&\quad = \|B(x_1) - B(x_2) - \alpha[A(x_1) - A(x_2)]\| \leqslant \quad \text{(by (0.1))} \\
&\quad \leqslant k\|B(x_1) - B(x_2)\| = k\|y_1 - y_2\|.
\end{aligned}$$

Then Φ is *near* $I_{\mathcal{B}}$. Conversely, if Φ is *near* $I_{\mathcal{B}}$ we obtain the following estimate:

$$\begin{aligned}
&\|B(x_1) - B(x_2) - \alpha[A(x_1) - A(x_2)]\| \\
&\quad = \|B(x_1) - B(x_2) - \alpha[AB^{-1}(y_1) - AB^{-1}(y_2)]\| \\
&\quad = \|y_1 - y_2 - \alpha[\Phi(y_1) - \Phi(y_2)]\| \\
&\quad \leqslant k\|y_1 - y_2\| = k\|B(x_1) - B(x_2)\|.
\end{aligned}$$

Then A is *near* B. □

Theorem 2.1. *Let A be near B.*
i) *If $B(\mathcal{X})$ is open in $\mathcal{B}$ then $A(\mathcal{X})$ is open in $\mathcal{B}$.*
ii) *If $B(\mathcal{X})$ is dense in $\mathcal{B}$ then $A(\mathcal{X})$ is dense in $\mathcal{B}$.*

Proof. We set

$$\begin{aligned}
&\Phi(y) = AB^{-1}(y), \quad \forall y \in B(\mathcal{X}), \\
&\Omega_1 = B(\mathcal{X}), \\
&\Omega_2 = A(\mathcal{X}).
\end{aligned}$$

We observe that $\Omega_2 = \Phi(\Omega_1)$ and that, by virtue of Lemma 2.2, Φ is *near* $I_{\mathcal{B}}$, because A is *near* B by the hypothesis. It follows that Φ satisfies the assumptions of Theorems 1.1 and 1.2.

Then we obtain the assertion of i), provided Ω_1 is open in $\mathcal{B}$ by virtue of Theorem 1.1, while we obtain the assertion of ii) provided Ω_1 is dense in $\mathcal{B}$ by virtue of Theorem 1.2. □

3. On the existence of a solution of a non linear elliptic system

In this part we shall give an example of application to elliptic systems of the topological properties we proved in the previous section, that is the open range property (Theorem 2.1 (i)).

Let Ω be a bounded convex open set in $\mathbb{R}^n$, with C^2 boundary. Let $x \in \Omega$, $u \in \mathbb{R}^n$, $N \geqslant 1$, $\xi = \{\xi_{ij}\}_{i,j=1,\dots,n}$, $\xi_{ij} \in \mathbb{R}^N$ for $i,j = 1,\dots,n$.

Let $a(x,u,H(u)) = \sum\limits_{i,j=1}^{n} A_{ij}(x,u)D_{ij}u$, with $A_{ij}(x,u)$ an $N \times N$ matrix, defined on $\Omega \times \mathbb{R}^N$, measurable in x, continuous in u, with the following properties:

Condition (A). There exist three positive constans α, γ, δ, with $\gamma + \delta < 1$, such that $\forall \xi \in \mathbb{R}^{n^2 N}$

$$\left\| \sum_{i=1}^{n} \xi_{ii} - \alpha \sum_{i,j=1}^{n} A_{ij}(x,0)\xi_{ij} \right\|_N \leqslant \gamma \|\xi\|_{n^2 N} + \delta \left\| \sum_{i=1}^{n} \xi_{ii} \right\|_N \tag{3.1}$$

$\forall x \in \Omega$.[3]

There exists $M > 0$ such that:

$$\begin{gathered} \|A_{ij}(x,u) - A_{ij}(x,v)\|_{N^2} \leqslant M\|u - v\|_N, \\ \forall i,j = 1,\dots,n,\ \forall x \in \Omega,\ \forall u,v \in \mathbb{R}^N. \end{gathered} \tag{3.2}$$

Theorem 3.1. *Let $n \leqslant 3$. If $\{A_{ij}\}_{i,j=1,\dots,n}$ satisfies* (3.1) *and* (3.2) *then there exists a $\varrho > 0$ such that, for every $f \in L^2(\Omega, \mathbb{R}^N)$ with $\|f\|_{L^2(\Omega,\mathbb{R}^N)} < \varrho$, the following system has a unique solution*

$$\begin{cases} u \in H^2 \cap H_0^1(\Omega, \mathbb{R}^N), \\ \sum\limits_{i,j=1}^{n} A_{ij}(x,u)D_{ij}u = f(x) \qquad \text{on } \Omega. \end{cases} \tag{3.3}$$

[3] It is known that if the operator $a(x,0,H(u))$ satisfies Condition (A) then it verifies the following ellipticity condition: $\exists \nu > 0$ such that $\sum_{i,j=1}^{n} \lambda_i \lambda_j (A_{ij}(x,0)\eta \mid \eta)_N \geqslant \nu \|\lambda\|_n^2 \|\eta\|_N^2$ $\forall \lambda \in \mathbb{R}^n$, $\forall \eta \in \mathbb{R}^N$, $\forall x \in \mathbb{R}^n$. Moreover, Condition (A) is equivalent to the Cordes condition when the operator is linear, cf. [C_4].

We prove the theorem by using of Theorem 2.1, by setting

$$\begin{aligned}
\mathcal{X}_\varrho &= \{u \in H^2 \cap H_0^1(\Omega\mathbb{R}^N)\colon \|u\|_{H^2} < \sigma\},^4\\
\mathcal{B} &= L^2(\Omega, \mathbb{R}^N),\\
A(u) &= \sum_{i,j=1}^{n} A_{ij}(x,0)D_{ij}u,\\
B(u) &= \Delta u,\\
C(u) &= \sum_{i,j=1}^{n} A_{ij}(x,u)D_{ij}u.
\end{aligned}$$

Before passing to the proof proper we state the following lemmas.

Lemma 3.1. *A is near B (as operators between $\mathcal{X}_\sigma$ and $\mathcal{B}$, $\forall \sigma > 0$).*

Lemma 3.2. *$B(\mathcal{X}_\sigma)$ is open in $\mathcal{B}$, $\forall \sigma > 0$.*

Lemma 3.3. *C is near A (as an operator between $\mathcal{X}_\sigma$ and $\mathcal{B}$, $\forall \sigma \leqslant \bar\sigma = \frac{1-(\gamma+\delta)}{2Mc_1c_2\sigma\alpha\sqrt{n}}$, where c_1 is Sobolev's constant, c_2 is such that $\|u\|_{H^2} \leqslant c_2\|\Delta u\|_{L^2}$, $\forall u \in H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$.)*

P r o o f of Theorem 3.1. If follows from Lemmas 3.1 and 3.2, that A is *near* B and $B(\mathcal{X}_\sigma)$ is open in $\mathcal{B}$, $\forall\sigma > 0$. Therefore, by Theorem 2.1 (i), we also obtain that $A(\mathcal{X}_\sigma)$ is open in $\mathcal{B}$, $\forall \sigma > 0$. Moreover, by Lemma 3.3, C is *near* A (as an operator between $\mathcal{X}_\sigma$ and $\mathcal{B}$, $\forall \sigma < \bar\sigma$) and so $C(\mathcal{X}_\sigma)$ is open in $\mathcal{B}$, $\forall \sigma < \bar\sigma$, by Theorem 2.1(i). Consequently, we observe that $C(0) = 0$ and so we have $\forall\sigma < \bar\sigma\ \exists S(0,\varrho) \subseteq L^2(\Omega, \mathbb{R}^N)$ such that $\forall f \in S(0,\varrho)$ there exists a unique[5] $u \in \mathcal{X}_\sigma$ such that $C(u) = f$, that there exists a unique $u \in H^1 \cap H_0^2(\Omega, \mathbb{R}^N)$ such that $\sum_{i,j=1}^{n} A_{ij}(x,u)D_{ij}u = f(x)$, $x \in \Omega$. □

Now we shall prove the above lemmas.

P r o o f of Lemma 3.1. It is not difficult to prove that $B\colon \mathcal{X}_\sigma \to \mathcal{B}$. Moreover, $A\colon \mathcal{X}_\sigma \to \mathcal{B}$ because (3.1) implies

$$\begin{aligned}
\|Au\|_{\mathcal{B}}^2 &\leqslant \frac{2}{\alpha^2}\int_\Omega \Big\|\Delta u - \alpha\sum_{i,j=1}^{n} A_{ij}(x,0)D_{ij}u\Big\|_N^2\,\mathrm{d}x + \frac{2}{\alpha^2}\int_\Omega \|\Delta u\|_N^2\,\mathrm{d}x\\
&\leqslant \frac{2}{\alpha^2}\gamma\int_\Omega \Big\|\sum_{i,j=1}^{n} D_{ij}u\Big\|_N^2\,\mathrm{d}x + \frac{2}{\alpha^2}(\delta+1)\int_\Omega \|\Delta u\|_N^2\,\mathrm{d}x,
\end{aligned}$$

[4] In particular, it is clear that $u \in C^0(\Omega, \mathbb{R}^N)$ and $\|u\|_\infty \leqslant c_1\|u\|_{H^2}$ by virtue of the Sobolev immersion theorem, since $n \leqslant 3$.

[5] Uniqueness of the solution follows from the injectivity of A and B and from Corollary 2.1.

$\forall u \in H^2 \cup H_0^1(\Omega, \mathbb{R}^N)$, in particular $\forall u \in \mathcal{X}_\sigma$. Finally it is easy to prove that A is *near* B, as an operator between $H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$ and $\mathcal{B}$, and consequently also as an operator between $\mathcal{X}_\sigma$ and $\mathcal{B}$, because $\sum_{i,j=1}^n A_{ij}(x,0)\xi_{ij}$ satisfies condition 3.1.[6] $\square$

Proof of Lemma 3.2. We fix a $\varphi \in B(\mathcal{X}_\sigma)$ and prove that there exists a $\tau > 0$ such that, $\forall g \in L^2(\Omega, \mathbb{R}^N)$ satisfying the estimate $\|\varphi - g\|_{L^2(\Omega,\mathbb{R}^N)} < \tau$, there esixts a $v \in \mathcal{X}_\sigma$ such that $\Delta v = g$. Let $v \in H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$ and $w \in \mathcal{X}_\sigma$ be such that $\Delta v = g$ and $\Delta w = \varphi$. The following estimates hold:

$$\begin{aligned}\|v\|_{H^2} &\leqslant \|w - v\|_{H^2} + \|w\|_{H^2} \leqslant \|\Delta(w - v)\|_{L^2} + \|w\|_{H^2}\\ &= \|\varphi - g\|_{L^2} + \|w\|_{H^2} < \tau + \|w\|_{H^2},\end{aligned}$$

and hence $v \in \mathcal{X}_\sigma$ for every $\tau > 0$ such that $\tau + \|w\|_{H^2} < \sigma$. $\square$

Proof of Lemma 3.3. For every $u, v \in \mathcal{X}_\sigma$ the following estimates hold:

$$\tag{3.5}\begin{aligned}&\|A(u) - A(v) - [C(u) - C(v)]\|_{\mathcal{B}}^2\\ &= \int_\Omega \Bigg\| \sum_{i,j=1}^n A_{ij}(x,0)D_{ij}(u - v)\\ &\qquad - \Bigg[\sum_{i,j=1}^n A_{ij}(x,u)D_{ij}u - A_{ij}(x,v)D_{ij}v\Bigg]\Bigg\|_N^2 \,\mathrm{d}x\\ &\leqslant \int_\Omega \Bigg(\Bigg\| \sum_{i,j=1}^n A_{ij}(x,0)D_{ij}(u - v) - \sum_{i,j=1}^n A_{ij}(x,u)D_{ij}(u - v)\Bigg\|_N\\ &\qquad + \Bigg\|\sum_{i,j=1}^n \Big[A_{ij}(x,u) - A_{ij}(x,v)\Big]D_{ij}v\Bigg\|_N\Bigg)^2 \,\mathrm{d}x\end{aligned}$$

(by hypothesis (3.2))

$$\begin{aligned}&\leqslant 2M^2 \int_\Omega \Bigg(\|u\|_N^2 \Bigg\|\sum_{i,j=1}^n D_{ij}(u - v)\Bigg\|_N^2 + \|u - v\|_N^2 \Bigg\|\sum_{i,j=1}^n D_{ij}v\Bigg\|_N^2\Bigg) \,\mathrm{d}x\\ &\leqslant 2M^2 n\Bigg(\|u\|_\infty^2 \int_\Omega \sum_{i,j=1}^n \|D_{ij}(u - v)\|_N^2 \,\mathrm{d}x + \|u - v\|_\infty^2 \int_\Omega \sum_{i,j=1}^n \|D_{ij}v\|_N^2 \,\mathrm{d}x\Bigg)\\ &\leqslant 2M^2 nc_1\Bigg(\|u\|_{H^2}^2 \int_\Omega \sum_{i,j=1}^n \|D_{ij}(u - v)\|_N^2 \,\mathrm{d}x + \|u - v\|_{H^2}^2 \int_\Omega \sum_{i,j=1}^n \|D_{ij}v\Bigg)\|_N^2 \,\mathrm{d}x\Bigg).\end{aligned}$$

[6] Since $\int_\Omega \|\Delta(u - v) - \alpha \sum_{i,j=1}^n A_{ij}(x,0)D_{ij}(u - v)\|_N^2 \,\mathrm{d}x \leqslant (\gamma + \delta)^2 \int_\Omega \|\Delta(u - v)\|_N^2 \,\mathrm{d}x$, cf. [C2] or [C3].

From the above estimates and considering that

$$\|u - v\|_{H^2} \leqslant c_2 \|\Delta(u - v)\|_{L^2}$$

we obtain that $\forall u, v \in \mathcal{X}$

$$\|A(u) - A(v) - [C(u) - C(v)]\|_{\mathcal{B}}^2 \leqslant 4m^2 n c_1^2 c_2^2 \sigma^2 \int_\Omega \|\Delta(u - v)\|_N^2 \, \mathrm{d}x. \tag{3.6}$$

The second member of (3.6) verifies the inequality[7]

$$\int_\Omega \|\Delta(u - v)\|_N^2 \, \mathrm{d}x \leqslant \frac{\alpha^2}{[1 - (\gamma + \delta)]^2} \int_\Omega \left\| \sum_{i,j=1}^{n} A_{ij}(x, 0) D_{ij}(u - v) \right\|_N^2 \mathrm{d}x. \tag{3.7}$$

From (3.6) and (3.7) we conclude

$$\begin{aligned} \|A(u) - A(v) - [C(u) - C(v)]\|_{\mathcal{B}}^2 &= k_\sigma^2 \int_\Omega \left\| \sum_{i,j=1}^{n} A_{ij}(x, 0) D_{ij}(u - v) \right\|_N^2 \mathrm{d}x \\ &= k_\sigma^2 \|A(u) - A(v)\|_{\mathcal{B}}^2. \end{aligned}$$

If $k_\sigma = 2Mc_1c_2\sigma\sqrt{n}\frac{\alpha}{1-(\gamma+\delta)} < 1$, this estimate implies the assertion of the lemma. □

4. On periodic solutions of a non linear abstract differential equation of second order in a Hilbert space

Let V, H be real and separable Hilbert spaces with $V \subset H$, V dense in H, with a continuous immersion map, and $\|u\|_H \leqslant c_1\|u\|_V$. We denote respectively by V^* and H^* the dual spaces of V and H. We identify H with its dual H^*, then $V \subset H \subset V^*$. We denote by $\langle , \rangle$ duality between V and V^*, while $(,)_V$, $(,)_H$, $\| \ \|_V$ and $\| \ \|_H$ are respectively the scalar products and norms in V and H.

Let $\mathcal{A}\colon V \to V^*$ be a linear and symmetric operator with the property

$$\exists \nu > 0 \quad \text{such that} \quad \langle \mathcal{A}v, v\rangle \geqslant \nu\|v\|_V^2 \qquad \forall v \in V. \tag{4.1}$$

$L_T^2(X)$ is the space of measurable and T-periodic functions that are defined on $\mathbb{R}$ with values in the Hilbert space X, normed by

$$\|u\|_{L_T^2(X)} = \left(\int_0^T \|u(t)\|_X^2 \, \mathrm{d}t\right)^{1/2}.$$

[7] This follows in the usual way from Lemma (3.1), see for example [C$_1$].

$H_T^m(X)$ is the space of functions between $\mathbb{R}$ and X whose distribution derivatives up to the order m belong to $L_T^2(X)$ normed by

$$\|u\|_{H_T^m}(X) = \left(\int_0^T \sum_{j=0}^{m} \|D^j u(t)\|_X^2 \, \mathrm{d}t\right)^{1/2}.$$

In this section we shall use Theorem 2.1(ii) to solve the following problem.

Problem 4.1. Let $g \in L_T^2(H)$ be given. We look for $u \in L_T^2(V) \cap H_T^1(H)$ such that[8]

$$\mathcal{A} + u'' + F(t, u') + cu = g, \tag{4.2}$$

where $F(t, u)$ is a map between $\mathbb{R} \times H$ and H, measurable in t, continuous in u and satisfying the following hypotheses:

(4.3) $t \to F(t, u)$ is T-periodic $\forall u \in H$.

(4.4) $t \to F(t, 0) \in L_T^2(H)$.

(4.5) There exist real numbers $\alpha > 0$, $k \in (0, 1)$, $a \neq 0$ such that:
$\|a(u_1 - u_2) - \alpha[F(t, u_1) - F(t, u_2)]\|_H \leqslant k\|a(u_1 - u_2)\|_H$, $\forall u_1, u_2 \in H$, $\forall t \in \mathbb{R}$.

In order to solve this problem, let us first observe that, using the above results concerning *near* operators, for the equation

$$\mathcal{A}u(t) + u''(t) + a\, u'(t) + c\, u(t) = g(t), \quad t \in \mathbb{R}, \tag{4.6}$$

the following theorems hold:

Theorem 4.1. *Let $c \neq -\lambda_n$,[9] $\forall n \in \mathbb{N}$ and $a \neq 0$. If $g \in L_T^2(H)$ then equation* (4.6) *has a unique solution $u \in L_T^2(V) \cap H_T^1(H)$.*[10]

Theorem 4.2. *Let $c \neq -\lambda_n$ $\forall n \in \mathbb{N}$ and $a \neq 0$. If $g \in H_T^1(H)$, then the solution u of the equation* (4.6) *belongs to $L_T^2(\mathrm{Dom}\,\mathcal{A}) \cap H_T^2(H)$, where $\mathrm{Dom}\,\mathcal{A} = \{v \in V : \mathcal{A}v \in H\}$.*

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[8] That is the same as to say that $\forall \varphi \in L_T^2(V) \cap H_T^1(H)$ $\int_0^T \langle \mathcal{A}u, \varphi\rangle - (u', \varphi')_H + (F(u, u'), \varphi)_H + c(u, \varphi)_\varphi \, \mathrm{d}t = \int_0^T (g, \varphi)_H \, \mathrm{d}t$.

[9] $\{\lambda_n\}_{n \in N} \subset \mathbb{R}^+$ is the eigenvector sequence of $\mathcal{A}$.

[10] $u \in L_T^2(V) \cap H_T^1(H)$ is a solution of (2.2) if $\forall \varphi \in L_T^2(V) \cap H_T^1(H)$, $\int_0^T \langle \mathcal{A}u, \varphi\rangle - \langle u', \varphi'\rangle + a(u', \varphi)_H + c\,(u, \varphi)_H \, \mathrm{d}t = \int_0^T (g, \varphi)\, \mathrm{d}t$

The proof of these theorems can be found in [HL] (Theorem (4.2)(1) and (7.3)(1), respectively).

We use the notation of §2 and set:

$$\begin{aligned}
Au &= \mathcal{A}u + u'' + F(t,u') + cu,\\
Bu &= \alpha(\mathcal{A}u + u'') + a\ u' + \alpha cu,\\
\mathcal{X} &= \{u \colon u \in L^2_T(\operatorname{Dom}\mathcal{A}) \cap H^2_T(H)\},\\
\mathcal{B} &= L^2_T(H).
\end{aligned}$$

We first state some lemmas to be used for solving Problem 4.1.

Lemma 4.1. *$\forall u \in \mathcal{X}$, the following estimate holds:*

$$\int_0^T \|\alpha(\mathcal{A}u + u'') + au' + \alpha cu\|_H^2 \,\mathrm{d}t \geqslant a^2 \int_0^T \|u'\|_H^2 \,\mathrm{d}t.$$

P r o o f. $\forall u \in \mathcal{X}$ we obtain the estimate

$$\int_0^T \|\alpha(\mathcal{A}u + u'') + au' + \alpha cu\|_H^2 \,\mathrm{d}t \geqslant \int_0^T \|\alpha(\mathcal{A}u + u'' + cu)\|_H^2 + a^2\|u'\|_H^2 \,\mathrm{d}t,$$

because, as a consequence of T-periodicity of u, the following equations hold:

$$\int_0^T \langle \mathcal{A}u, u'\rangle \,\mathrm{d}t = 0; \int_0^T (u'', u')_H \,\mathrm{d}t = 0; \int_0^T (u, u')_H \,\mathrm{d}t = 0.$$

□

Lemma 4.2. *A is near B.*

P r o o f. We shall first prove that A maps $\mathcal{X}$ into $\mathcal{B}$. As a consequence of (4.3), Au is T-periodic $\forall u \in \mathcal{X}$. Moreover, $Au \in L^2_T(H)$ because $F(t,u') \in L^2_T(H)$, and from (4.4), (4.5) we obtain

$$\begin{aligned}
\int_0^T \|F(t,u'(t))\|_H^2 \,\mathrm{d}t &\leqslant 2\int_0^T \|F(t,u'(t) - F(t,0)\|_H^2 + \|F(t,0)_H^2 \,\mathrm{d}t\\
&\leqslant 2\Big(\frac{1+k}{\alpha}\Big)^2 \int_0^T \|u'(t)\|_H^2 + \|F(t,0)\|_H^2 \,\mathrm{d}t.
\end{aligned}$$

Finally we prove that A is *near* B. $\forall u, v \in \mathcal{X}$ we have

$$\begin{aligned}
&\|B(u) - B(v) - \alpha[A(u) - A(v)]\|^2 \\
&\quad = \int_0^T \|\alpha\mathcal{A}(u-v) + \alpha(u'' - v'') + a(u' - v') + \alpha c(u-v) \\
&\qquad - \alpha[\mathcal{A}(u-v) - (u'' - v'') + F(t,u') - F(t,v') + c(u-v)\|_H^2 \,\mathrm{d}t \\
&\quad = \int_0^T \|a(u' - v') - \alpha[F(u') - F(v')]\|_H^2 \,\mathrm{d}t
\end{aligned}$$

(by means of (4.5))

$$\leqslant k^2 \int_0^T \|a(u' - v')\|_H^2 \,\mathrm{d}t$$

(by means of Lemma 4.1)

$$\begin{aligned}
&\leqslant k^2 \int_0^T \|\alpha\mathcal{A}(u-v) + \alpha(u'' - v'') + a(u' - v') + \alpha c(u-v)\|_H^2 \,\mathrm{d}t \\
&= k^2 \|B(u) - B(v)\|_H^2.
\end{aligned}$$

□

Lemma 4.3. *$B(\mathcal{X})$ is dense in $\mathcal{B}$.*

Proof. We observe that $H_T^1(H)$ is dense in $L_T^2(H)$ and that $H_T^1(H) \subset B(\mathcal{X})$. In fact, from Theorem 4.2 we get that for every $g \in H_T^1(H)$ there exists a unique $u \in \mathcal{X}$ such that $B(u) = g$. □

Lemma 4.4. *$A(\mathcal{X})$ is dense in $\mathcal{B} = L_T^2(H)$.*

Proof. The result follows from Lemmas 4.2, 4.3 and Theorem 2.1 (ii). □

Theorem 4.3. *If F verifies hypotheses (4.3), (4.4), (4.5) and $c > -\nu/c_1$ then Problem (4.1) has a unique solution $u \in L_T^2(V) \cap H_T^1(H)$.*

Proof. Let $g \in L_T^2(H)$. We know, by virtue of Lemma 4.4, that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ such that $A(u_n) \longrightarrow g$ in $L_T^2(H)$. Then $\{A(u_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_T^2(H)$ and it follows that $\{B(u_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_T^2(H)$, too.

In fact, by Lemma 4.2, B is *near* A and hence we get the following estimate:

$$\|B(u_n) - B(u_m)\|_H \leqslant \frac{\alpha}{1-k}\|A(u_n) - A(u_m)\|_H. \tag{4.7}$$

This together with Lemma 4.1 implies that $\{u'_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2_T(H)$. Let $g_n, g_m \in L^2_T(H)$ be such that $B(u_m) = g_m$ and $B(u_n) = g_n$. From the equation

$$\mathcal{A}(u_n - u_m) + (u''_n - u''_m) + a(u'_n - u'_m) + c(u_n - u_m) = g_n - g_m,$$

multiplying by $(u_n - u_m)$ and integrating between 0 and T, we obtain

$$\begin{aligned}(4.8)\qquad &\int_0^T \langle \mathcal{A}(u_n - u_m), u_n - u_m\rangle + c\|u_n - u_m\|_H^2 \,\mathrm{d}t \\ &= \int_0^T (g_n - g_m, u_n - u_m)_H + \|u'_n - u'_m\|_H^2 \,\mathrm{d}t.\end{aligned}$$

Consequently

$$\begin{aligned}(4.9)\qquad \left(c + \frac{\nu}{c_1}\right)\int_0^T \|u_n - u_m\|_H^2 \,\mathrm{d}t &\leqslant \frac{c_1}{\nu + cc_1}\int_0^T \|g_n - g_m\|_H^2 \,\mathrm{d}t \\ &\quad + 2\int_0^T \|u'_n - u'_m\|_H^2 \,\mathrm{d}t.\end{aligned}$$

From (4.9), using (4.7) and the fact that $\{u'_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2_T(H)$, we get that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2_T(H)$, and then it is so in $H^1_T(H)$, too: let $u \in H^1_T(H)$ be its limit. From (4.8) and (4.9) it follows that $u_n \longrightarrow u$ in $L^2_T(V)$. Finally we observe that $u \in L^2_T(V) \cap H^1_T(H)$ and that it is a weak solution of the problem (4.1), that is, $\forall \varphi \in L^2_T(V) \cap H^1_T(H)$,[11]

$$\begin{aligned}(g, \varphi)_{\mathcal{B}} &= \lim_{n\to\infty} (A(u_n), \varphi)_{\mathcal{B}} \\ &= \lim_{n\to\infty} \int_0^T \langle \mathcal{A}u_n, \varphi\rangle - (u'_n, \varphi')_H + (F(t, u'_n), \varphi)_H + c(u_n, \varphi)_H \,\mathrm{d}t \\ &= \int_0^T \langle \mathcal{A}u, \varphi\rangle - (u', \varphi')_H + (F(t, u'), \varphi)_H + c(u, \varphi)_H \,\mathrm{d}t.\end{aligned}$$

□

[11] In fact this follows from hypothesis (4.3):

$$\int_0^T \|F(t, u'_n) - F(t, u')\|_H^2 \,\mathrm{d}t \leqslant \left(\frac{k+1}{\alpha}\right)^2 a^2 \int_0^T \|u'_n - u'\|_H^2 \,\mathrm{d}t.$$

Remark 4.1. Let Ω be a bounded open set in $\mathbb{R}^n$, let $\gamma(x,t,p)$ be a map defined between $\Omega \times \mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$ with the following properties:

(4.5)(a) $\quad t \to \gamma(x,t,p)$ is T-periodic $\forall p \in \mathbb{R}$, a.e. in Ω.

(4.5)(b) $\quad t \to \gamma(x,t,0) \in L^2([0,T] \times \Omega)$

(4.5)(c) $\quad \exists m,\ M > 0$ such that

$$m \leqslant \frac{\gamma(x,t,p) - \gamma(x,t,q)}{p-q} \leqslant M, \ \forall t \in \mathbb{R}, \forall p,q \in \mathbb{R}\ p \neq q, \quad \text{a.e. in } \Omega.$$

We shall consider the following problem:

$$\text{(4.5)(d)} \qquad \begin{cases} u \in L^2_T(H^1_0(\Omega)) \cap H^1_T(L^2(\Omega)), \\ \Delta u - \dfrac{\partial^2}{\partial t^2} u + \gamma\Big(x,t,\dfrac{\partial u}{\partial t}\Big) = g(x,t) \end{cases}$$

where $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is T-periodic in t a.e. in Ω, with $g \in L^2([0,T] \times \Omega)$, and Δ is the Laplace operator.

The problem (4.5)(d), which was studied in §2 of [P_1], is a particular case of Problem (4.1). In fact, we set $H = L^2(\Omega)$, $V = H^1_0(\Omega)$ with the usual norms, moreover $\mathcal{A}u = -\Delta u, F(t,u) = \gamma(x,t,u)$, and we observe that hypothesis (4.5) is satisfied by virtue of (4.5)(c) with $a = 1$, $\alpha = \frac{m}{M^2}$, $k = 1 - \frac{m^2}{M^2}$ because the following estimates hold $\forall u_1, u_2 \in L^2(\Omega)$ (cf. [C_3]):

$$\begin{aligned} &\|u_1 - u_2 - \frac{m}{M^2}[\gamma(x,t,u_1) - \gamma(x,t,u_2)]\|^2_H \\ &\quad = \int_\Omega |u_1 - u_2 - \frac{m}{M^2}[\gamma(x,t,u_1) - \gamma(x,t,u_2)]|^2\,dx \\ &\quad = \int_\Omega \Bigg\{(u_1-u_2)^2 + \frac{m^2}{M^4}[\gamma(x,t,u_1) - \gamma(x,t,u_2)]^2 \\ &\qquad\qquad - \frac{2m}{M^2}(u_1-u_2)[\gamma(x,t,u_1) - \gamma(x,t,u_2)]\Bigg\}\,dx \end{aligned}$$

(by (4.5)(c))

$$\begin{aligned} &\leqslant \int_\Omega \Bigg\{(u_1-u_2)^2 + \frac{m^2}{M^2}(u_1-u_2)^2 - \frac{2m^2}{M^2}(u_1-u_2)^2\Bigg\}\,dx \\ &= \Big(1 - \frac{m^2}{M^2}\Big) \int_\Omega (u_1-u_2)^2\,dx. \end{aligned}$$

We conclude the study of the equation (4.2) with the following results concerning regularity, uniqueness and continuous dependence on data.

Theorem 4.4. *Under the hypotheses of Theorem 4.3, if $u \in L^2_T(V) \cap H^1_T(H)$ is a T-periodic solution of the equation (4.2) then $u \in C^0(\mathbb{R}, V) \cap C^1(\mathbb{R}, H)$.*

Proof. Let us consider the following test functions:

$$\varphi(t) = \vartheta_i(t)\psi_i(t), \qquad i = 1,2, \tag{4.10}$$

where $\psi_i \in L^2(0,T,V) \cap H^1(0,T,H)$ are such that $\psi_1(T) = 0$ and $\psi_2(0) = 0$; $\vartheta_i \in C^2([0,T],\mathbb{R})$ are such that $0 \leqslant \vartheta_i(t) \leqslant 1$, $i = 1,2$. Moreover

$$\begin{cases} \vartheta_1(t) = 0 & \text{if} \quad t \in [0, T/4], \\ \vartheta_1(t) = 1 & \text{if} \quad t \in [T/2, T], \end{cases} \qquad \begin{cases} \vartheta_2(t) = 1 & \text{if} \quad t \in [0, T/2], \\ \vartheta_2(t) = 0 & \text{if} \quad t \in [3/4\ T, T]. \end{cases}$$

If we write the equation (4.2) in the weak form on $[0,T]$ and choose test functions as in (4.10), we obtain

$$\int_0^T \langle \mathcal{A}(\vartheta_i u), \psi_i(t)\rangle - ((\vartheta_i u)', \psi_i'(t))_H \,\mathrm{d}t = \int_0^T (\Phi_i(t), \psi_i(t))_H \,\mathrm{d}t, \tag{4.11}$$

$\forall \psi_i \in L^2(0,T,V) \cap H^1(0,T,H)$ such that $\psi_1(T) = 0$ and $\psi_2(0) = 0$, where we set $\Phi_i(t) = \vartheta_i(t)(g - cu - F(t,u')) + 2\vartheta_i' u' + \vartheta_i'' u$. From (4.11), setting $v_i(t) = \vartheta_i(t)u(t)$, it follows that $v_i \in L^2(0,T,V) \cap H^1(0,T,H)$ for $i = 1,2$. Moreover, v_1, v_2 solve the following Cauchy problems:

$$\begin{cases} \mathcal{A}v_1 + v_1'' = \Phi_1(t), \\ v_1(0) = 0, \\ v_1'(0) = 0, \end{cases} \qquad \begin{cases} \mathcal{A}v_2 + v_2'' = \Phi_2(t), \\ v_2(T) = 0, \\ v_2'(T) = 0. \end{cases} \tag{4.12}$$

Hence, from the assumptions on g and F, it follows that $\Phi_i \in L^2(0,T,H)$; then we obtain, owing to the results of [BA] (§4) applied to problem (4.12), that $v_i \in C^0([0,T],V) \cap C^1([0,T],H)$, $i = 1,2$. Then $u \in C^0([T/4,T,V) \cap C^1(T/4,T,H)$ and $u \in C^0([0,3/4T],V) \cap C^1([0,3/4T],H)$, and the assertion of the theorem is proved. □

Theorem 4.5. *Let the hypotheses of Theorem 4.3 concerning $\mathcal{A}$ and F be satisfied, let $g_1, g_2 \in L^2_T(H)$ be given and let respectively $u_1, u_2 \in C^0(\mathbb{R}, V) \cap C^1(\mathbb{R}, H)$ be the T-periodic solution of the equation (4.2). We have the following estimates:*

$$\int_0^T \|u_1' - u_2'\|_H^2 \,\mathrm{d}t \leqslant \frac{\alpha^2}{a^2(1-k)^2} \int_0^T \|g_1 - g_2\|_H^2 \,\mathrm{d}t, \tag{4.13}$$

$$\int_0^T \|u_1 - u_2\|_V^2 \,\mathrm{d}t \leqslant c_2(\nu, c, c_1, a, \alpha, k) \int_0^T \|g_1 - g_2\|_H^2 \,\mathrm{d}t. \tag{4.14}$$

Proof. $u_1 - u_2 \in C^0(\mathbb{R}, V) \cap C^1(\mathbb{R}, H)$ satisfies the equation

$$\mathcal{A}(u_1 - u_2) + (u_1 - u_2)'' + c\,(u_1 - u_2) = g_1 - g_2 - [F(t, u_1) - F(t, u_2)]. \tag{4.15}$$

If we make use of the classical energy equality on $[0, T]$ for the equation (4.5) (see for example [BA] or [LM]), and if $u_1 - u_2$ is T-periodic we obtain

$$\int_0^T (F(t, u_1') - F(t, u_2'), a(u_1' - u_2'))_H \,dt = \int_0^T (g_1 - g_2, a(u_1' - u_2'))_H \,dt. \tag{4.16}$$

The following estimate can be obtained by a standard procedure from hypothesis (4.5) on F (see for example [C_3] §3)

$$\int_0^T (F(t, u_1') - F(t, u_2'), a(u_1' - u_2'))_H \,dt \geqslant \frac{1-k}{\alpha} a^2 \int_0^T \|u_1' - u_2'\|_H^2 \,dt. \tag{4.17}$$

Now (4.16) and (4.17) imply (4.13).

To prove (4.14) we take a weak form of (4.15) with $u_1 - u_2$ as a test function obtaining the equation

$$\begin{aligned} \int_0^T & \langle \mathcal{A}(u_1 - u_2), u_1 - u_2 \rangle + c\|u_1 - u_2\|_H^2 \\ &= \int_0^T (g_1 - g_2, u_1 - u_2)_H + \|u_1' - u_2'\|_H^2 \,dt \\ &\quad - \int_0^T (F(t, u_1') - F(t, u_2'), (u_1 - u_2))_H \,dt. \end{aligned} \tag{4.18}$$

From this equality, taking into account (4.5) and (4.13), we obtain: $\forall \sigma \in (0, 1]$

$$\begin{aligned} \nu \int_0^T \|u_1 - u_2\|_V^2 \,dt + (c - |c|\sigma) \int_0^T \|u_1 - u_2\|_H^2 \,dt \\ \leqslant \left(\frac{1}{2|c|\sigma} + \frac{\alpha^2}{a^2(1-k)^2} + \left(\frac{1+k}{1-k}\right)^2 \frac{1}{2|c|\sigma} \right) \int_0^T \|g_1 - g_2\|_H^2 \,dt. \end{aligned}$$

This implies (4.14).

From (4.13) and (4.14) we obtain the following corollary: □

Corollary 4.1. *Under the hypotheses of Theorem 4.3, there exists a unique T-periodic solution $u \in C^0(\mathbb{R}, V) \cap C^1(\mathbb{R}, H)$ of equation (4.2).*

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