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ON f-DOMINATION NUMBER OF A GRAPH

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1. INTRODUCTION

The domination number $\gamma(G)$ of a graph G is the smallest cardinality of a set D of vertices such that every vertex outside D has at least one neighbor in D. Extensive studies on domination number and domination-related topics have been done in the past thirty years. Recently, some new domination models have been proposed. For example, [4, 5] studied the k-domination number. For a positive integer k, a subset D of $V(G)$ is a k-dominating set of G if each vertex of $V(G)\setminus D$ is adjacent to at least k distinct vertices of D. A k-independent set T is a subset of $V(G)$ such that the maximum degree of the induced subgraph $G[T]$ of G is less than k. The k-domination number of G, denoted by $\gamma_k(G)$, is the cardinality of the smallest k-dominating set of G ([4, 5]). The k-independence number of G ([4, 5]), $\beta_k(G)$, is the cardinality of the largest k-independent set of G. Evidently, $\gamma_1(G)$ and $\beta_1(G)$ are, respectively, the ordinary domination number $\gamma(G)$ and the ordinary independence number $\beta(G)$.

The following result was conjectured by J.F. Fink and M.S. Jacobson ([4, 5]) and proved in [3].

Theorem 1. *For any simple graph G and positive integer k, we have $\gamma_k(G) \leqslant \beta_k(G)$.*

This theorem generalizes the inequality $\gamma \leqslant \beta$. Another upper bound for γ_k is the following

Theorem 2 ([1]). *Let n and k be positive integers, and G a graph with minimum degree $\delta(G) \geqslant \frac{n+1}{n}k - 1$. Then $\gamma_k(G) \leqslant \frac{np}{n+1}$, where $p = |V(G)|$.*

More upper bounds for γ_k can be found in [9]. In the same paper a general domination concept was introduced. For any integer-valued function f defined on

$V(G)$, a subset D of $V(G)$ is called an f-dominating set of G if $|N_G(x) \cap D| \geqslant f(x)$ for each $x \in V(G) \setminus D$, where $N_G(x)$ is the set of neighbors of x in G. Then the f-domination number of G, denoted by $\gamma_f(G)$, is defined to be the smallest cardinality of an f-dominating set of G. Obviously, if f is such that $f(x) = k$ for all $x \in V(G)$, then $\gamma_f(G)$ is exactly the k-domination number. If $T \subseteq V(G)$ satisfies $d_{G[T]}(x) < f(x)$ for all $x \in T$, then we call T an f-independent set of G. The maximum cardinality of f-independent sets of G is then defined to be the f-independence number, denoted by $\beta_f(G)$.

In this paper we initiate the study on γ_f and β_f. Basic results for these two invariants are discussed in the next section. Some upper bounds for γ_f are given in Section 3. In particular, Theorems 1–2 are generalized. In the last section some open problems are proposed. Throughout the paper G is a finite, undirected graph with no loops and multiedges, and f, $V(G) \to Z$ is an integer-valued function. For $D \subseteq V(G)$ and $x \in V(G)$, let $N_D(x) = N_G(x) \cap D$ and $d_D(x) = |N_D(x)|$. Let p and $\varepsilon(G)$ represent the number of vertices and the number of edges of G, respectively.

2. Basic results

A subset S of $V(G)$ is called an f-transversal of G if it intersects all non-f-independent sets of G. The minimum cardinality of f-transversals of G is then defined to be the f-transversal number of G, denoted by $\alpha_f(G)$. The following Gallai-type equality is in fact a consequence of a more general result of [6].

Theorem 3. $\alpha_f(G) + \beta_f(G) = p$.

Proof. It can be shown that $S \subseteq V(G)$ is an f-transversal iff $V(G) \setminus S$ is an f-independent set. Then the theorem follows. □

Proposition 1. (1) *If H is a spanning subgraph of G, then $\gamma_f(G) \leqslant \gamma_f(H)$;*

(2) *If f': $V(G) \to Z$ is another function satisfying $f(x) \leqslant f'(x)$ for all $x \in V(G)$, then $\gamma_f(G) \leqslant \gamma_{f'}(G)$ and $\beta_f(G) \leqslant \beta_{f'}(G)$.*

Proposition 2. (1) *If $f(x) > d(x)$ for some $x \in V(G)$, then x must belong to any f-dominating set of G;*

(2) *If $f(x) < 1$ for a vertex x, then x can not be in any minimal f-dominating set of G.*

Proposition 3. *Let $M = \max_{x \in V(G)} f(x)$, then*

$$\gamma_f(G) \geqslant \frac{1}{M}\Big(\sum_{x \in V(G)} f(x) - \varepsilon(G)\Big).$$

Proof. Let D be an f-dominating set of G with the smallest cardinality. Then

$$\sum_{x\in V(G)} f(x) - \sum_{x\in D} f(x) = \sum_{x\in V(G)\setminus D} f(x) \leqslant \varepsilon(G).$$

So $M \cdot \gamma_f(G) = M \cdot |D| \geqslant \sum_{x\in D} f(x) \geqslant \sum_{x\in V(G)} f(x) - \varepsilon(G)$. This completes the proof. □

For any function $f\colon V(G) \to Z$, let $f^*\colon V(G) \to Z$ be a companion function defined by $f^*(x) = d(x) - f(x) + 1$, $x \in V(G)$. Then we have

Proposition 4.
(1) $\gamma_f(G) + \beta_{f^*}(G) \leqslant p$;
(2) $\gamma_{f^*}(G) + \beta_f(G) \leqslant p$.

Proof. Let T be a maximum f^*-independent set of G. Then $d_{G[T]}(x) \leqslant f^*(x) - 1$ for each $x \in T$. So $d_{V(G)\setminus T}(x) \geqslant d(x) - f^*(x) + 1 = f(x)$ for each $x \in T$. Thus $V(G) \setminus T$ is an f-dominating set of G, and (1) is true. Since $(f^*)^* = f$, (2) follows from (1) immediately. □

Corollary 1. *If $f(x) \leqslant \frac{d(x)+1}{2}$ for all $x \in V(G)$, then $\gamma_f(G) + \beta_f(G) \leqslant p$.*

Proof. The given condition implies that $f(x) \leqslant f^*(x)$ for each $x \in V(G)$. Hence $\beta_f(G) \leqslant \beta_{f^*}(G)$ by Proposition 1(2). The corollary then follows from Proposition 4(1).

Corollary 1 generalizes a known result ([8]) that $\gamma(G) + \beta(G) \leqslant p$ if $p \geqslant 2$ and G contains no isolated vertices. □

3. Some upper bounds for γ_f

As shown in Theorem 1, $\gamma_k(G) \leqslant \beta_k(G)$ for any positive integer k. Then we may naturally ask if $\gamma_f(G) \leqslant \beta_f(G)$ for any function f. The answer is affirmative. In fact we have the following more general result.

Theorem 4. *For any function $f\colon V(G) \to Z$, every f-independent set D of G such that $\sum_{x\in D} f(x) - \varepsilon(D)$ is maximum is an f-dominating set of G, where $\varepsilon(D)$ is the number of edges of $G[D]$.*

Proof. The proof is similar to that used in [3].

Suppose otherwise; then there must exist $v \in V(G) \setminus D$ such that $d_D(v) < f(v)$. Let $B = N_D(v)$, then $0 \leqslant |B| < f(v)$. Let

$$A = \{x \in B\colon d_D(x) = f(x) - 1\}$$

and let S be a maximal independent set of $G[A]$. Then $\Phi \subseteq S \subseteq A \subseteq B \subseteq D$. Let $C = (D \setminus S) \cup \{v\}$. Then C must be an f-independent set of G.

In fact,

$$\begin{aligned}
&d_C(v) \leqslant |B| < f(v),\\
&d_C(x) \leqslant d_D(x) < f(x), \quad \forall x \in D \setminus B,\\
&d_C(x) \leqslant d_D(x) + 1 \leqslant \big(f(x) - 2\big) + 1 < f(x), \quad \forall x \in B \setminus A.
\end{aligned}$$

Noting that S is a maximal independent set of $G[A]$, each $x \in A \setminus S$ is adjacent to at least one vertex in S. Hence

$$d_C(x) \leqslant \big(d_D(x) - 1\big) + 1 < f(x), \quad \forall x \in A \setminus S.$$

Thus C is indeed an f-independent set of G. We have

$$\varepsilon(C) = \varepsilon(D) - \sum_{x \in S} \big(f(x) - 1\big) + |B| - |S| = \varepsilon(D) - \sum_{x \in S} f(x) + |B|.$$

Hence,

$$\begin{aligned}
\sum_{x \in C} f(x) - \varepsilon(C) &= \Big(\sum_{x \in D} f(x) - \sum_{x \in S} f(x) + f(v)\Big) - \Big(\varepsilon(D) - \sum_{x \in S} f(x) + |B|\Big)\\
&= \sum_{x \in D} f(x) - \varepsilon(D) + f(v) - |B| > \sum_{x \in D} f(x) - \varepsilon(D),
\end{aligned}$$

contradicting the choice of D. This completes the proof. □

Corollary 2. *For any graph G and any function $f\colon V(G) \to Z$, we have $\gamma_f(G) \leqslant \beta_f(G)$.*

Proof. By Theorem 4 there exists an f-dominating set D which is also an f-independent set. So $\gamma_f(G) \leqslant |D| \leqslant \beta_f(G)$. □

Let f^* be defined as in Section 2, then we have

Corollary 3.
(1) $\gamma_f(G) + \gamma_{f^*}(G) \leqslant p$;
(2) $\gamma_f(G) \cdot \gamma_{f^*}(G) \leqslant \left(\frac{p}{2}\right)^2$.

Proof. By Theorem 4 we can choose an f-dominating set of G which is also an f-independent set. Thus for any $x \in D$,

$$|N_G(x) \cap (V \setminus D)| \geqslant d_G(x) - \big(f(x) - 1\big) = f^*(x).$$

So $V \setminus D$ is an f^*-dominating set of G. This implies (1). (2) is a direct consequence of (1). □

Combining Corollaries 1–2, we have

Corollary 4. *Let $f\colon V(G) \to Z$ be such that $f(x) \leqslant \frac{1}{2}(d(x) + 1)$, $\forall x \in V(G)$, then $\gamma_f(G) \leqslant \frac{1}{2}p$.*

Corollary 4 generalizes an early result of Ore which states that $\gamma(G) \leqslant \frac{1}{2}p$ if G has no isolated vertices.

The idea used in [1] can be applied to prove the following result, which generalizes Theorem 2.

Theorem 5. *Let n be a positive integer and let $f\colon V(G) \to Z$ be such that $f(x) \leqslant \frac{n}{n+1}\big(d_G(x) + 1\big)$, $\forall x \in V(G)$. Then $\gamma_f(G) \leqslant \frac{np}{n+1}$.*

Proof. Let $V_1, V_2, \ldots, V_{n+1}$ be a partition of $V(G)$ such that $E' = E(G) \setminus \bigcup_{i=1}^{n+1} E(G[V_i])$ contains as many edges as possible. Then by a theorem of Erdös ([2]) $d_H(x) \geqslant \left\lceil \frac{n}{n+1}\, d_G(x) \right\rceil$, $\forall x \in V(G)$, where $H = \big(V(G), E'\big)$ and $\lceil a \rceil$ is the smallest integer not less that a. The condition $f(x) \leqslant \frac{n}{n+1}\big(d_G(x) + 1\big)$ implies $d_G(x) \geqslant \frac{n+1}{n} f(x) - 1$. This gives

$$d_H(x) \geqslant \left\lceil \frac{n}{n+1}\Big(\frac{n+1}{n} f(x) - 1\Big) \right\rceil = \left\lceil f(x) - \frac{n}{n+1} \right\rceil = f(x), \quad \forall x \in V(G).$$

Without loss of generality we may suppose $|V_1| = \max_{1 \leqslant i \leqslant n+1} |V_i|$. By the above discussion, $\bigcup_{i=2}^{n+1} V_i$ is an f-dominating set of G. Thus

$$\gamma_f(G) \leqslant p - |V_1| \leqslant p - \frac{p}{n+1} = \frac{np}{n+1}.$$

□

Corollary 5. *Let $n_0 = \max_{x \in V(G)} \left\lceil \frac{f(x)}{f^*(x)} \right\rceil \big(f(x) \neq d(x) + 1$ for all $x \in V(G)\big)$. Then $\gamma_f(G) \leqslant \frac{n_0 p}{n_0+1}$.*

Note that this corollary generalizes Corollary 4.

For any $f_i\colon V(G) \to Z$ with $1 \leqslant f(x) \leqslant d_G(x)$, $x \in V(G)$, define a function $f-1\colon V(G) \to Z$ such that

$$(f-1)(x) = \max\{1, f(x)-1\}, \quad x \in V(G).$$

Inductively define the function $f-(i+1) = (f-i)-1$ for any positive integer i. Then it is not difficult to see that $\gamma_{f-m} = \gamma(G)$, where $m = \max_{x \in V(G)} f(x) - 1$. To investigate the relation between γ_f and γ, we prove the following

Theorem 6. *For any function $f\colon V(G) \to Z$ satisfying $1 \leqslant f(x) \leqslant d(x)$, $x \in V(G)$, we have*

$$\gamma_f(G) \leqslant \frac{1}{2}(p + \gamma_{f-1}(G)).$$

Proof. Let D_1 be an $(f-1)$-dominating set of G with the cardinality $\gamma_{f-1}(G)$, and

$$S = \{x \in V(G) \setminus D_1\colon f(x) = 1\}.$$

Then

$$(f-1)(x) = \begin{cases} 1, & x \in S, \\ f(x)-1, & x \in V(G) \setminus (D_1 \cup S). \end{cases}$$

Let A, B be, respectively, the set of non-isolated vertices and the set of isolated vertices of $G[V(G) \setminus (D_1 \cup S)]$. Let T be a minimum dominating set of $G[A]$. Then by Ore's theorem (mentioned earlier), $|T| \leqslant \frac{1}{2}|A|$.

It is easy to see that $D_1 \cup B \cup T$ is an f-dominating set of G, so

$$\gamma_f(G) \leqslant \gamma_{f-1}(G) + |B| + |T| \leqslant \gamma_{f-1}(G) + |B| + \frac{|A|}{2}. \tag{1}$$

On the other hand $D_1 \cup S \cup T$ is also an f-dominating set of G. In fact for any $x \in B$, $d_{D_1}(x) \geqslant (f-1)(x) = f(x)-1$. If $d_{D_1}(x) = f(x)-1 \leqslant d_G(x)-1$, then x must be adjacent to a vertex of S. Thus $d_{D_1 \cup S \cup T}(x) \geqslant f(x)$. If $d_{D_1}(x) = f(x)$, then $d_{D_1 \cup S \cup T}(x) \geqslant f(x)$ as well. It is obvious that $d_{D_1 \cup S \cup T}(x) \geqslant f(x)$ for any $x \in A \setminus T$. So $D_1 \cup S \cup T$ is indeed an f-dominating set. Thus

$$\gamma_f(G) \leqslant \gamma_{f-1}(G) + |S| + |T| \leqslant \gamma_{f-1}(G) + |S| + \frac{|A|}{2}. \tag{2}$$

Combining (1) and (2) we get

$$\gamma_f(G) \leqslant \gamma_{f-1}(G) + \frac{1}{2}(|A| + |B| + |S|) = \frac{1}{2}(p + \gamma_{f-1}(G)).$$

□

Corollary 6. *For any i with $1 \leqslant i \leqslant m = \max_{x \in V(G)} f(x) - 1$,*

$$\gamma_f(G) \leqslant p - \frac{1}{2^i}\bigl(p - \gamma_{f-i}(G)\bigr).$$

In particular,

$$\gamma_f(G) \leqslant p - \frac{1}{2^m}\bigl(p - \gamma(G)\bigr).$$

For each k, $1 \leqslant k \leqslant \Delta(G)$, let

$$\mathscr{X}_k = \{X \subseteq V(G) \colon |X| = k\}.$$

Denote $\Gamma_G(X) = \bigcap_{x \in X} N_G(x)$ for each $X \in \mathscr{X}_k$, and $\Delta_k(G) = \max_{X \in \mathscr{X}_k} |\Gamma_G(X)|$. Then $\Delta_k(G) \geqslant 1$. Let $X \in \mathscr{X}_k$ be such that $|\Gamma_G(X)| = \Delta_k(G)$ and $S = V(G) \setminus \bigl(X \cup \Gamma_G(X)\bigr)$. Then $V(G) \setminus \Gamma_G(X)$ is a k-dominating set of G. Thus

$$\gamma_k(G) \leqslant p - \Delta_k(G)$$

or, equivalently,

$$|\Gamma_G(X)| = \Delta_k(G) \leqslant p - n, \tag{3}$$

where $n = \gamma_k(G)$. Suppose

$$|\Gamma_G(X)| = p - n - r, \quad 0 \leqslant r < p - n. \tag{4}$$

Then

$$|S| = p - |X| - |\Gamma_G(X)| = n + r - k. \tag{5}$$

Note that for any $x \in X$ and $y \in \Gamma_G(X)$, $\bigl(S \setminus N_G(y)\bigr) \cup \{x, y\}$ is a dominating set of G. Hence

$$|S| - |S \cap N_G(y)| + 2 \geqslant \gamma(G),$$

i.e.

$$|S \cap N_G(y)| \leqslant n + r - k - \gamma(G) + 2.$$

Similarly, we have

$$|S \cap N_G(x)| \leqslant n + r - k - \gamma(G) + 2.$$

Let h_k be the maximum number of edges in a subgraph of G with $|S| = p - \Delta_k(G) - k$ vertices. Then we have

$$\begin{aligned}
(6)\quad 2\varepsilon(G) &\leqslant 2\varepsilon(G[S]) + |X|(|X|-1) + |X|\,|\Gamma_G(X)| \\
&\quad + \sum_{y\in N_G(X)} (|N_G(y)\cap S| + |N_G(y)| + \sum_{x\in X} |N_G(x)\cap S| \\
&\leqslant 2h_k + k(k-1) + k\Delta_k + \Delta_k[(n+r-k-\gamma(G)+2)+\Delta] \\
&\quad + k(n+r-k-\gamma(G)+2) \qquad (\Delta_k = \Delta_k(G),\ \Delta = \Delta(G)) \\
&= 2h_k + k(k-1) + k\Delta_k + (\Delta_k + k)(p - \Delta_k - k - \gamma(G) + 2) + \Delta_k\Delta \\
&= 2h_k + k(p-\gamma(G)+1) - \Delta_k^2 + \Delta_k(p - k + \Delta(G) - \gamma(G) + 2) \\
&\leqslant 2h_k + k(p-\gamma(G)+1) - \Delta_k^2 + (p-n)\big(p - k + \Delta(G) - \gamma(G) + 2\big).
\end{aligned}$$

This leads to the following

Theorem 7. *For each k, $1 \leqslant k \leqslant \Delta(G)$, let Δ_k and h_k be as before. Then*

$$\gamma_k(G) \leqslant p - \left\lceil \frac{2(\varepsilon(G) - h_k) + \Delta_k^2 - k(p - \gamma(G) + 1)}{p - k + \Delta(G) - \gamma(G) + 2} \right\rceil.$$

Since $\gamma(G) \geqslant 1$ and $h_k \leqslant \frac{1}{2}(p - \Delta_k - k)(p - \Delta_k - k - 1)$, we obtain the following two corollaries.

Corollary 7. *For each k with $1 \leqslant k \leqslant \Delta(G)$,*

$$\gamma_k(G) \leqslant p - \left\lceil \frac{2(\varepsilon(G) - h_k) + \Delta_k^2 - kp}{p - k + \Delta(G) + 1} \right\rceil.$$

Corollary 8. $\gamma_k(G) \leqslant p - \left\lceil \frac{2\varepsilon(G) + 2(p-k)\Delta_k + k(p+\gamma(G)-k-2) - p(p-1) - \Delta_k}{p - k + \Delta(G) - \gamma(G) + 2} \right\rceil.$

Taking $k = 1$ in (6) we get

$$2\varepsilon(G) \leqslant 2h_1 + (p - \gamma(G) + 1) - \Delta^2 + \Delta\big(p + \Delta + 1 - \gamma(G)\big).$$

This implies a new upper bound for $\gamma(G)$.

Corollary 9. *Let h_1 be the maximum number of edges in a subgraph of G having $p - \Delta - 1$ vertices. Then*

$$\gamma(G) \leqslant p - \left\lceil \frac{2(\varepsilon(G) - h_1)}{\Delta(G) + 1} \right\rceil + 1. \tag{7}$$

Example 1. Let G be the graph obtained from the cycle of five edges by adding a chord. Then $p = 5$, $\varepsilon(G) = 6$, $\Delta(G) = 3$, $\gamma(G) = 2$, $\Delta_2(G) = 2$ and $h_2 = 0$. Theorem 7 gives $\gamma_2(G) \leqslant 3$. But it is easy to see $\gamma_2(G) \geqslant 3$. So $\gamma_2(G) = 3$. This shows that the upper bound in Theorem 7 is attainable.

Example 2. If G is the cycle with four edges, then it is easy to see that both sides of (7) equal 2. So the upper bound in Corollary 9 is attainable.

For any subgraph H of G, the restriction of the function $f\colon V(G) \to Z$ to $V(H)$ is also denoted briefly by f. Thus $\gamma_f(H)$ is well-defined. The technique used in the proof of Theorem 7 can be applied to prove the next result, which shows the connection of $\gamma_f(G)$ and $\gamma_f(H)$.

Theorem 8. *Let t_q be the maximum number of edges in a subgraph of G with $p - q$ vertices, $1 \leqslant q \leqslant p - 1$. Then for any subgraph H of G with q ($> \gamma_f(H)$) vertices and without isolated vertices,*

$$\gamma_f(G) \leqslant p - \left\lceil \frac{2(\varepsilon(G) - t_q) + (q - a)^2 - (p - \gamma(G))\gamma_f(H)}{p + \gamma_f(H) + \Delta(G) - \gamma(G) + 1} \right\rceil.$$

Proof. Suppose that X is an f-dominating set of H with $\gamma_f(H) = a$ vertices, and that $Y = V(H) \setminus X$, $S = V(G) \setminus V(H)$. Since $X \cup S = V(G) \setminus Y$ is an f-dominating set of G, we have

$$\gamma_f(G) \leqslant p - |Y|,$$

or equivalently, $|Y| \leqslant p - n$, where $n = \gamma_f(G)$. Suppose

$$|Y| = p - n - r, \quad 0 \leqslant r < p - n.$$

Then $|S| = p - a - |Y| = n + r - a$.

For any $y \in Y$, $\big(S \setminus N_G(y)\big) \cup X \cup \{y\}$ is a dominating set of G, hence

$$|S| - |S \cap N_G(y)| + |X| + 1 \geqslant \gamma(G),$$

i.e. $|S \cap N_G(y)| \leqslant n + r - \gamma(G) + 1$. Similarly, $|S \cap N_G(x)| \leqslant p - a - \gamma(G) + 1$ for each $x \in X$. We have

$$\begin{aligned}
2\varepsilon(G) &\leqslant 2\varepsilon(G[S]) + |X|(|X| - 1) + |X|\,|Y| \\
&\qquad + \sum_{y \in Y} \bigl(|S \cap N_G(y)| + |N_G(y)|\bigr) + \sum_{x \in X} |S \cap N_G(x)| \\
&\leqslant 2t_q + a(a-1) + a|Y| + |Y|\bigl(n + r - \gamma(G) + 1 + \Delta(G)\bigr) \\
&\qquad + a(p - a - \gamma(G) + 1) \\
&= 2t_q + a(p - \gamma(G)) + a|Y| + |Y|\bigl(p - |Y| - \gamma(G) + \Delta(G) + 1\bigr) \\
&= 2t_q + a(p - \gamma(G)) - |Y|^2 + |Y|\bigl(p + a + \Delta(G) - \gamma(G) + 1\bigr) \\
&\leqslant 2t_q + a(p - \gamma(G)) - (q-a)^2 + (p-n)\bigl(p + a + \Delta(G) - \gamma(G) + 1\bigr).
\end{aligned}$$

This gives

$$n \leqslant p - \frac{2(\varepsilon(G) - t_q) + (q-a)^2 - (p - \gamma(G))a}{p + a + \Delta(G) - \gamma(G) + 1}.$$

This completes the proof. □

4. Remarks

A lot of problems concerning the f-domination number and the f-independence number can be proposed. Perhaps the most attractive one is whether there exist the Nordhaus–Gaddum type inequalities for γ_f. Such inequalities for γ have been shown in [7]. Naturally we can define the upper f-domination number $\Gamma_f(G)$ of G to be the maximum cardinality of a minimal f-dominating set of G. Also we can define the f-domatic number, $d_f(G)$, to be the maximum order of a partition of $V(G)$ into f-dominating sets. Another interesting invariant is $i_f(G)$, which is defined to be the smallest non-negative integer i such that $\gamma_{f-i}(G) = \gamma(G)$. Studies on these invariants are necessary, as well as interesting. For example, relations among γ_f, Γ_f, β_f, d_f, i_f and other graphical invariants, e.g. the domination number, the independence number, are valuable research topics. The lower bounds and the upper bounds for γ_f and β_f deserve further study as well.

This work is on-going and results will be published later.

References

[1] *Y. Caro and Y. Roditty*: A note on the k-domination number of a graph. Internat. J. Math. & Math. Sci. *13* (1990), no. 1, 205–206.

[2] *P. Erdös*: On some extremal problems in graph theory. Israel J. of Mathematics *3* (1965), 113–116.

[3] *O. Favaron*: On a conjecture of Fink and Jacobson concerning k-domination and k-dependence. J. of Combinatorial Theory, Ser B *39* (1985), 101–102.

[4] *J.F. Fink and M.S. Jacobson*: n-domination in graphs. Graph Theory with Applications to Algorithms and Computer Science. John Wiley & Sons, New York, 1985, pp. 283–300.

[5] *J.F. Fink and M.S. Jacobson*: On n-domination, n-dependence and forbidden subgraphs. Graph Theory with Applications to Algorithms and Computer Science. John Wiley & Sons, New York, 1985, pp. 301–311.

[6] *S.T. Hedetniemi*: Hereditary properties of graphs. J. of Combinatorial Theory, Ser B *14* (1973), 94–99.

[7] *F. Jaeger and C. Payan*: Relations du type Nordhaus–Gaddum pour le nombre d'absorption d'un graphe simple. CR Acad. Sci., Ser. A *274* (1972), 728–730.

[8] *R. Laskar and H.B. Walikar*: On domination related concepts in graph theory. Combinatorics and Graph Theory (S.B. Rao, ed.). Lecture Notes in Math. 885, Springer-Verlag, Berlin, Heidelberg, New York, 1981, pp. 308–320.

[9] *C. Stracke and L. Volkmann*: A new domination conception. J. Graph Theory *17* (1993), no. 3, 315–323.

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