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Oldřich Kowalski

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AN EXPLICIT CLASSIFICATION OF 3-DIMENSIONAL RIEMANNIAN SPACES SATISFYING $R(X,Y)\cdot R=0$

Oldřich Kowalski, Praha[1]

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Introduction

The Riemannian spaces with the curvature tensor R satisfying the identity $R(X,Y)\cdot R=0$ were recognized already by E. Cartan (see [Ca1], p. 265), who noticed that all locally symmetric spaces and all 2-dimensional Riemannian spaces belong to this class. Various results have been obtained by A. Lichnerowicz, R.S. Couty and N.S. Sinjukov. The last author ([Si 1,2]) introduced the name "semi-symmetric spaces" for this class of Riemannian manifolds. (See [Sz 1] for the other references). In 1968, K. Nomizu [N] asked the question if there exist complete, irreducible and simply connected Riemannian manifolds in dimension $n\geqslant 3$ satisfying the identity $R(X,Y)\cdot R=0$ and not locally symmetric, i.e. such that $\nabla R\neq 0$. The first positive example was constructed by H. Takagi [T] in 1972 as a hypersurface $M_3\subset\mathbb{R}^4$ with the induced Riemannian metric. The full local classification of Riemannian spaces with the above property was given in 1982 by Z.I. Szabó [Sz 1]. The completeness of semi-symmetric spaces was studied in the subsequent paper [Sz 2] and the complete semi-symmetric hypersurfaces of Euclidean spaces in [Sz 3].

One possible interpretation of the work [Sz 1] says that all building stones for the semi-symmetric spaces (SSS) are divided into three classes:

(a) The "trivial" SSS: all locally symmetric spaces and all two-dimensional Riemannian spaces.

(b) The "exceptional" SSS: elliptic cones, hyperbolic cones, Euclidean cones and Kaehlerian cones.

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(c) The "typical" SSS: Riemannian spaces foliated by $(n-2)$-dimensional Euclidean spaces.

Whereas all the "trivial" SSS are well-known and the "exceptional" ones are described in [Sz 1], [Sz 2] by explicit constructions, the most ample family (c) has not been much explored until recently. In [Sz 1] all foliated SSS are described by a (nonlinear and rather complicated) system of partial differential equations. The local existence theorem in dimension n then says that all solutions depend on $\frac{1}{2}n^2+n+2$ arbitrary functions of 2 variables and $\frac{1}{2}n^2+n-6$ arbitrary functions of 1 variable).

Yet, *no explicit solutions* have been presented in [Sz 1]. In [Sz 3], some classes of solutions are described as hypersurfaces of $\mathbb{R}^{n+1}$; they depend on arbitrary functions of 1 variable only. To our knowledge, the first explicit class depending on one arbitrary function of 2 variables was constructed by F. Tricerri, L. Vanhecke and the present author in [KTV 1] and [KTV 2]. It was obtained as a generalization of two examples by K. Sekigawa, [Se].

In the present paper we study the 3-dimensional case and we try to *resolve explicitly* the partial differential equations by Z. Szabó (after deriving these PDE by a different method and in a different form). We make a geometric classification "in gross" of the 3-dimensional foliated SSS: we distinguish the elliptic, hyperbolic, parabolic and planar ones. Then we give (local) explicit formulas for all hyperbolic, parabolic and planar metrics (involving three, two or one arbitrary functions of 2 variables, respectively). In the elliptic case we obtain, in general, only a "quasi-explicit" formula. Yet, some families of explicit solutions are also constructed which depend on one arbitrary function of 2 variables. Moreover, we prove that *the local isometry classes* of metrics in the elliptic and the hyperbolic case still depend on 3 arbitrary functions of 2 variables.

Whereas Z.Szabó and other authors are primarily interested in the *complete* SSS, our results show that most of our solutions are inherently incomplete. For instance, all SSS of the hyperbolic type are incomplete. Thus, incompleteness is the price to pay for a full and systematic classification of the SSS.

Our computational method is not closely connected with the dimension $n=3$ and it works in the *arbitrary dimension*, as well. Very recently, E. Boeckx, L. Vanhecke and the present author have explicitly classified all nonhomogeneous semi-symmetric spaces with *constant scalar curvature* (see [BKV]). Moreover, a modification of our method enables, surprisingly enough, to get a local classification of nonhomogeneous 3-dimensional Riemannian manifolds with the *prescribed constant eigenvalues of the Ricci tensor* (see [K]).

The content of the paper is the following: In Section 1 we recall some basic facts from [Sz 1]. In Section 2 we derive a canonical form for our metrics involving three unknown functions of 3 variables and one unknown function of 2 variables. In Section

3 we write down the basic system of 9 PDE for these unknown functions. In Section 4 this basic system of PDE is reduced to only three PDE and to a system of (many) algebraic equations for new unknown functions of 2 variables which appeared in the first integrals of the basic PDE system.

In Section 5 we study the Riemannian invariants for our class of spaces. Section 6 introduces the useful concept of asymptotic foliation and the geometric classification "in gross". The asymptotic foliations enable to introduce new coordinates which simplify dramatically the further computations in the hyperbolic, parabolic and planar case. The explicit classification of these cases is given in Section 7. Then Section 8 is devoted to the (more difficult) elliptic case. Finally, in Section 9 we study foliated SSS with *prescribed scalar curvature* and we prove the main existence theorem.

1. The basic concepts and properties

To be more precise, let us repeat that a semi-symmetric space is a Riemannian manifold (M, g) satisfying the identity

$$R(X,Y) \cdot R = 0 \quad \text{for all} \quad X, Y \in T_pM, \ p \in M \tag{1.1}$$

where each curvature transformation $R(X,Y)_p$ acts as derivation on the tensor algebra of T_pM. We always assume (M, g) to be of class C^∞.

As we have mentioned in the Introduction, we shall limit ourselves to the foliated SSS. In accordance with [Sz 1], *a foliated SSS* is a Riemannian manifold (M, g) whose index of nullity $\nu(p)$ is constant along M and equal to $n-2$. This means that every tangent space T_pM can be decomposed in the form

$$T_pM = V_p^{(0)} + V_p^{(1)} \tag{1.2}$$

where $\dim V_p^{(0)} = n-2$, $\dim V_p^{(1)} = 2$ for all $p \in M$, and $V_p^{(0)}$ is the *null-space* of the Riemannian curvature tensor R_p, i.e.,

$$V_p^{(0)} = \{X \in T_pM \mid R_p(X,Y) = 0 \text{ for all } Y \in T_pM\}. \tag{1.3}$$

Hence we see that the curvature tensor R_p (of type (0,4)) at each point $p \in M$ is "the same" as the curvature tensor of the space $M' = S^2(\lambda^2) \times R$, or $M' = H^2(-\lambda^2) \times R$ respectively, where the sectional curvature $\pm\lambda^2$ depends on the point p, in general. More precisely, for each $p \in M$ there is a linear isometry of tangent spaces, φ: $T_pM \longrightarrow T_0M'$, where M' is one of the model spaces above and $o \in M'$ is an arbitrary base point, such that $\varphi^* R'_0 = R_p$ holds for the corresponding curvature tensors. Because we exclude points of flatness, each connected foliated SSS has either only points of the "spherical" type, or only points of the "hyperbolic" type.

Further, Z. Szabó shows that the $(n-2)$-dimensional distribution $V^{(0)}$ on M is completely integrable, and the integral manifolds of $V^{(0)}$ are totally geodesic and locally Euclidean. This is why M is said to be "foliated by $(n-2)$-dimensional Euclidean spaces".

We see that the Ricci curvature $\sigma(X, X)$ ($X \in TM, \|X\| = 1$) is zero along each Euclidean leaf, and it is a nonzero constant on the unit circle of each subspace $V_p^{(1)}$ (equal to the corresponding sectional curvature). If we double this constant, we obtain the scalar curvature at the point p. In particular, for $n = 3$, the Euclidean foliation is a family of geodesics which are lines of zero principal Ricci curvature. Equivalently, a 3-dimensional foliated SSS can be characterized as a Riemannian 3-manifold whose Ricci tensor has, at each point, one nonzero double eigenvalue and one zero eigenvalue.

We shall close this short section by a new simple characterization of semi-symmetric spaces which is based on an idea by Ü. Lumiste [Lu]. It was noticed already by E. Cartan that every Riemannian manifold M has, at each point, a first order approximation, which is a Euclidean space. The present author and L. Vanhecke have observed that any semi-symmetric space has, at each point, a second order approximation, which is a symmetric space. In other words, *each semi-symmetric space is a 2^{nd} order envelope of an n-parameter family of symmetric spaces.* The precise formulation and a very simple proof (using the normal coordinates) will appear elsewhere. In particular, each foliated SSS is the 2^{nd} order envelope of symmetric spaces which are all homothetic to $S^2 \times \mathbb{R}^{n-2}$, or to $H^2 \times \mathbb{R}^{n-2}$.

2. The canonical local form for the metrics

The aim of this section is to prove the following

Theorem 2.1. *Let (M, g) be a smooth foliated SSS of dimension 3. Then, in a normal neighborhood U of any point $p \in M$, there are local coordinates w, x, y such that $g = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2$, where*

$$(2.1) \qquad \begin{cases} \omega^1 = f(w, x, y)\, dw, \\ \omega^2 = A(w, x, y)\, dx + C(w, x, y)\, dw, \\ \omega^3 = dy + H(w, x)\, dw \end{cases}$$

and $fA \neq 0$. Further

a) *the equations* $\omega^1 = \omega^2 = 0$ *determine the principal directions of zero Ricci curvature and the corresponding integral curves in* (U, g) *are geodesics,*
b) *the variable* y *measures the arclength along any geodesic of this family,*
c) *there exists a function* $\sigma(w, x) \neq 0$ *such that the sectional curvature* $k = k(w, x, y)$ *in the 2-direction* $\omega^3 = 0$ *is given by*

$$k = \sigma(w, x)/fA. \tag{2.2}$$

Remark 2.2. For the scalar curvature we get $Sc(g) = 2k$.

Proof. According to Section 1, in a neighborhood $U' \ni p$ there is a unique system of geodesics which are lines of zero Ricci curvature (one geodesic through each point). We shall call these lines "principal geodesics". Choose an *oriented* surface $S\colon D^2 \longrightarrow U'$ through p which is transversal w.r. to the principal geodesics at all cross-points but not orthogonal at p. Choose any coordinate system (w, x) on S. Then there is a normal neighborhood $U \ni p$, $U \subset U'$, with the property that each point $m \in U$ is projected to exactly one point $\pi(m) \in S$ via a principal geodesic. Then we define a local coordinate system (w, x, y) in U by the formulas $w(m) = w(\pi(m))$, $x(m) = x(\pi(m))$, and

$$y(m) = d^+(\pi(m), m) = \text{the oriented distance of } m \text{ from } \pi(m). \tag{2.3}$$

Obviously, y measures the arclength along each principal geodesic in U, and the coordinate vector field $\partial/\partial y$ is a unit vector field generating these principal geodesics. Now, choose in U an orthonormal moving frame $\{E_1, E_2, E_3\}$ such that $E_3 = \partial/\partial y$, and let $(\omega^1, \omega^2, \omega^3)$ be the corresponding dual coframe.

Because $\omega^1(E_3) = \omega^2(E_3) = 0$, the coordinate expression of ω^1, ω^2 must be of the form

$$\omega^i = P^i \, \mathrm{d}x + Q^i \, \mathrm{d}w \quad (i = 1, 2) \tag{2.4}$$

whereas ω^3 has the form

$$\omega^3 = \mathrm{d}y + P^3 \, \mathrm{d}x + Q^3 \, \mathrm{d}w. \tag{2.5}$$

Recall that the components ω^i_j of the connection form are uniquely determined by the standard equations (see [KN])

$$\mathrm{d}\omega^i + \sum \omega^i_j \wedge \omega^j = 0, \quad \omega^i_j + \omega^j_i = 0 \quad (i, j = 1, 2, 3). \tag{2.6}$$

The components Ω^i_j of the curvature form then satisfy

$$\Omega^1_2 = k\omega^1 \wedge \omega^2, \quad \Omega^1_3 = \Omega^2_3 = 0, \quad \Omega^i_j + \Omega^j_i = 0, \tag{2.7}$$

or, equivalently,

$$\mathrm{d}\omega^1_2 + \omega^1_3 \wedge \omega^3_2 = k\omega^1 \wedge \omega^2, \tag{2.8}$$

$$\mathrm{d}\omega^1_3 + \omega^1_2 \wedge \omega^2_3 = 0, \tag{2.9}$$

$$\mathrm{d}\omega^2_3 + \omega^2_1 \wedge \omega^1_3 = 0, \tag{2.10}$$

where $k = k(w, x, y)$ is the sectional curvature in the 2-direction $\omega^3 = 0$.

Taking the exterior differentials of (2.8)–(2.10) and substituting into the new equations from (2.8)–(2.10) we obtain

$$\mathrm{d}(k\omega^1 \wedge \omega^2) = 0, \tag{2.11}$$

$$\omega^1_3 \wedge \omega^1 \wedge \omega^2 = 0, \omega^2_3 \wedge \omega^1 \wedge \omega^2 = 0. \tag{2.12}$$

The last equations mean that ω^1_3 and ω^2_3 are linear combinations of ω^1, ω^2 only. According to (2.4), ω^1, ω^2, ω^1_3 and ω^2_3 are linear combinations of $\mathrm{d}w$, $\mathrm{d}x$ only.

The third equation of (2.6) says that

$$\mathrm{d}\omega^3 + \omega^3_1 \wedge \omega^1 + \omega^3_2 \wedge \omega^2 = 0 \tag{2.13}$$

and hence

$$\mathrm{d}\omega^3 = q\,\mathrm{d}w \wedge \mathrm{d}x \tag{2.14}$$

where $q = q(w, x, y)$ is some function. After differentiating (2.5) and substituting here from (2.14) we see that P^3, Q^3 are independent of y. Because the vector field E_3 is not orthogonal to the surface S at p, some of the functions P^3, Q^3 is nonvanishing in a neighborhood of p.

Let us fix any *potential function* $\overline{w} = \overline{w}(w, x)$ of the Pfaffian equation

$$P^3(w, x)\,\mathrm{d}x + Q^3(w, x)\,\mathrm{d}w = 0 \tag{2.15}$$

and let $\bar{x}(w, x)$ be another function such that $(\overline{w}, \bar{x})$ form a local coordinate system in a neighborhood of p on S. Then (2.5) can be rewritten in the form

$$\omega^3 = \mathrm{d}y + H(\overline{w}, \bar{x})\,\mathrm{d}\overline{w}. \tag{2.16}$$

Substituting the new variables $\bar{w}, \bar{x}$ also into (2.4), and using suitable orthogonal combinations of ω^1 and ω^2 instead of ω^1, ω^2, we obtain the remaining formulas of (2.1) up to a notation. (Here we use a smaller neighborhood U of p in U' if need be).

Finally, the equation (2.11) means, due to (2.1),

$$\frac{\partial}{\partial y}(fAk) = 0 \tag{2.17}$$

and hence (2.2) follows. Here $fA \neq 0$ because (2.1) is a coframe. □

Convention. In the following, by a (foliated) semi-symmetric space (M, g) we shall mean, as a rule, a *local* space in the sense of Theorem 2.1. In other words, we shall suppose that M is a convex open domain in $\mathbb{R}^3(w, x, y)$ on which the metric g is defined by the formulas 2.1 (if not stated otherwise).

3. The basic system of PDE for the foliated spaces

First, we define the function ψ by

$$\psi(w, x, y) = 1/fA = k(w, x, y)/\sigma(w, x) \tag{3.1}$$

(cf. (2.2)). Then it is easy to check, using (2.6), that the components of the connection form are given by

$$\begin{cases} \omega_2^1 = -A\alpha\, dx + R\, dw + \beta\, dy, \\ \omega_3^1 = A\beta\, dx + S\, dw, \\ \omega_3^2 = A'_y\, dx + T\, dw \end{cases} \tag{3.2}$$

where

$$\alpha = \psi(A'_w - C'_x - HA'_y), \quad \beta = \frac{1}{2}\psi(H'_x + AC'_y - CA'_y), \tag{3.3}$$

and

$$\begin{cases} R = \psi f f'_x - C\alpha + H\beta, \\ S = f'_y + C\beta, \\ T = C'_y - f\beta. \end{cases} \tag{3.4}$$

Now, in the notation given by (2.1) and (3.1)–(3.4), the curvature conditions (2.8)–(2.10) take on the following form, respectively:

(A) $\quad (A\alpha)'_y + \beta'_x = 0, \quad R'_y - \beta'_w = 0, \quad (A\alpha)'_w + R'_x + SA'_y - A\beta T = -\sigma,$

(B) $\quad A''_{yy} - A\beta^2 = 0, \quad -A''_{yw} + T'_x + A(\beta R + \alpha S) = 0, \quad T'_y - S\beta = 0,$

(C) $\quad (A\beta)'_y + A'_y\beta = 0, \quad S'_x - (A\beta)'_w - (A\alpha T + A'_y R) = 0, \quad S'_y + T\beta = 0.$

4. The first integrals and the reduction of the basic PDE system

The aim of this Section is to replace the PDE's of the series (B) and (C) by a system of algebraic equations. First of all, we eliminate the equations (B2) and (C2).

Proposition 4.1. *The equation* (B2) *is a consequence of* (A1) *and* (B1).

P r o o f. Using (3.4) we obtain

$$\begin{aligned} &T'_x - A''_{yw} + A(\beta R + \alpha S) \\ &\quad = C''_{yx} - f'_x\beta - f\beta'_x - A''_{yw} + A(\psi\beta f f'_x + \alpha f'_y + H\beta^2). \end{aligned}$$

From (3.3) we get, using also (3.1),

$$A''_{yw} = (\alpha f A + C'_x + HA'_y)'_y$$

and after the substitution we obtain, using again (3.1).

$$\begin{aligned} &T'_x - A''_{yw} + A(\beta R + \alpha S) \\ &\quad = -f\beta'_x - Af\alpha'_y - \alpha(Af)'_y + A\alpha f'_y - A''_{yy}H + AH\beta^2 \\ &\quad = f(-\beta'_x - (A\alpha)'_y) + H(-A''_{yy} + A\beta^2). \end{aligned}$$

This is zero w.r. to (A1) and (B1). □

Proposition 4.2. *The equation* (C2) *is a consequence of* (A1), (A2), (C1).

P r o o f. First we have, using (3.4) and (3.1),

$$(AR)'_y = (f'_x - AC\alpha + AH\beta)'_y = f''_{xy} - A(C\alpha)'_y - C\alpha A'_y + H(A\beta)'_y.$$

Using also (C1), we get

$$(AR)'_y = f''_{xy} - A(C\alpha)'_y - C\alpha A'_y - H\beta A'_y \tag{4.1}$$

Further, using (A2) we get from (3.4)

$$\begin{aligned} &S'_x - (A\alpha T + A'_y R) - (A\beta)'_w \\ &\quad = f''_{xy} + C'_x\beta + C\beta'_x - A\alpha(C'_y - f\beta) - \beta A'_w - (AR)'_y. \end{aligned}$$

Substituting now from (4.1), we obtain

$$\begin{aligned} S'_x - (A\alpha T + A'_y R) - (A\beta)'_w \\ = \beta(C'_x - A'_w + A'_y H) + C(\beta' x + (A\alpha)'_y) + \beta(f A\alpha). \end{aligned}$$

Due to (3.3) and (3.1), this is equal to

$$\beta(-\alpha/\psi) + C(\beta'_x + (A\alpha)'_y) + \beta(\alpha/\psi),$$

which is zero due to (A1), q.e.d. □

Next, the equations (B1), (B3), (C1), (C3) will be resolved by finding some first integrals of the system (A), (B), (C).

Proposition 4.3. *If* (A1)–(A3), (B1), (C1) *and* (C3) *hold, then*

$$fA = Ky^2 + Ly + M, \tag{4.2}$$

where K, L, M *are functions of* w, x *only. Further,* (A3) *is reduced to the equation*

$$((A\alpha)'_w + R'_x)_{y=0} + K(w, x) = -\sigma(w, x). \tag{4.3}$$

Proof. From (C3) we obtain, using also (3.4),

$$(SA)'_y = SA'_y - A\beta T = f'_y A'_y + \beta(CA'_y - AC'_y) + Af\beta^2. \tag{4.4}$$

Due to (B1) we get hence

$$(SA)'_y = f'_y A'_y + A''_{yy} f + \beta(CA'_y - AC'_y) = (A'_y f)'_y + \beta(CA'_y - AC'_y). \tag{4.5}$$

On the other hand, using (3.4) first and (C1) later, we get

$$(SA)'_y = (f'_y A + (A\beta)C)'_y = (f'_y A)'_y - \beta(CA'_y - AC'_y). \tag{4.6}$$

As the arithmetic mean-value of (4.5) and (4.6) we obtain

$$(SA)'_y = \frac{1}{2}(fA)''_{yy}. \tag{4.7}$$

Using (A1) and (A2) we obtain

$$((A\alpha)'_w + R'_x)'_y = 0. \tag{4.8}$$

Due to (4.4), (A3) takes on the form

$$(A\alpha)'_w + R'_x + (SA)'_y = -\sigma(w, x). \tag{4.9}$$

Differentiating w.r. to y we get, due to (4.8), $(SA)''_{yy} = 0$. Then (4.7) implies $(fA)'''_{yyy} = 0$ and hence (4.2) follows. Finally, because $(A\alpha)'_w + R'_x$ does not depend on y, and $(SA)'_y = \frac{1}{2}(fA)''_{yy} = K(w, x)$, the formula (4.3) follows from (4.9). □

In addition, we obtain hence

$$SA = Ky + \varphi_4(w, x) \tag{4.10}$$

where φ_4 is arbitrary.

Proposition 4.4. *The equations* (B1) *and* (C1) *are satisfied if and only if*

$$\beta A^2 = a_0(w, x) \quad (a_0 \text{ arbitrary}) \tag{4.11}$$

and

$$A^2 = a_1 y^2 + a_2 y + a_3 \quad (a_i = a_i(w, x),\ i = 1, 2, 3) \tag{4.12}$$

where

$$(a_2)^2 - 4a_1 a_3 + 4(a_0)^2 = 0 \tag{4.13}$$

is the only relation between the arbitrary functions a_0, a_1, a_2, a_3.

P r o o f. (C1) is obviously equivalent to $(\beta A^2)'_y = 0$ and hence to (4.11). (B1) then takes on the form

$$2A'_y A''_{yy} = 2A'_y A\beta^2 = 2(a_0)^2 A'_y A^{-3}$$

and integrating w.r. to y we obtain

$$(A'_y)^2 = -(a_0)^2 A^{-2} + p,$$

where $p = p(w, x) \geqslant 0$ is arbitrary. This is equivalent to

$$(AA'_y)^2 = pA^2 - (a_0)^2. \tag{4.14}$$

Suppose first $p > 0$; then we have $A^2 \geqslant (a_0)^2/p$, and

$$(A^2)'_y = \pm 2\sqrt{pA^2 - (a_0)^2}.$$

By a new integration w.r. to y we get hence, after a re-arrangement,

$$pA^2 = (py + q)^2 + (a_0)^2, \quad \text{where} \quad q = q(w, x) \quad \text{is arbitary,}$$

which is exactly (4.12) and (4.13).

Suppose now $p = 0$. Then (4.14) implies $a_0 = 0$, $A'_y = 0$, and thus $a_1 = a_2 = 0$. (4.12) and (4.13) hold again.

Conversely, we see that (4.11)–(4.13) imply (B1) and (C1), q.e.d. □

Proposition 4.5. *The equations* (B3) *and* (C3) *are satisfied if and only if the following formulas hold:*

$$fT - CS = \varphi_0, \tag{4.15}$$

$$S^2 + T^2 = \varphi_1, \tag{4.16}$$

$$f^2 + C^2 = \varphi_1 y^2 + \varphi_2 y + \varphi_3, \quad fS + CT = \varphi_1 y + \frac{1}{2}\varphi_2, \tag{4.17}$$

where $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ *are arbitrary functions of* w, x *satisfying the single relation*

$$(\varphi_2)^2 - 4\varphi_1\varphi_3 + 4(\varphi_0)^2 = 0. \tag{4.18}$$

Proof. a) Suppose first that (B3) and (C3) hold. We get directly $(S^2+T^2)'_y = 0$, i.e., formula (4.16). On the other hand, from the definition of S and T in (3.4) we obtain

$$(f^2 + C^2)'_y = 2(ff'_y + CC'_y) = 2(fS + CT).$$

Using also (B3) and (C3), we derive hence

$$(f^2 + C^2)''_{yy} = 2(fS + CT)'_y = 2(S^2 + T^2).$$

Using (4.16) and integration, we obtain (4.17). From (B3), (C3) and (3.4) we also obtain $(fT - CS)'_y = 0$, i.e., formula (4.15). Finally, (4.18) follows from the identity

$$(fT - CS)^2 + (fS + CT)^2 = (f^2 + C^2)(S^2 + T^2).$$

b) Let us now assume that (4.15) and (4.16) hold. Differentiating w.r. to y we obtain a system of linear algebraic equations for S'_y and T'_y in the form

$$CS'_y - fT'_y = f'_yT - C'_yS = -\beta(CT + fS), \quad SS'_y + TT'_y = 0.$$

(Here we have substituted for f'_y and C'_y from (3.4)). If $CT + fS \neq 0$, we can solve this system by the Cramer's rule and we get the equations (B3), (C3).

Suppose now $CT + fS = 0$. Then (4.17) and (4.18) imply $\varphi_1 = \varphi_2 = \varphi_0 = 0$. (4.15) and (4.17) then give $fT - CS = 0$, $CT + fS = 0$, where $f^2 + C^2 > 0$. Hence $S = T = 0$ and (B3), (C3) hold once again.

Finally, let us observe that each of the equations (4.17) is equivalent to the other one (as follows from our calculations). □

Proposition 4.6. *The differential equations* (A), (B), (C) *imply*

$$AC = b_1y^2 + b_2y + b_3, \tag{4.19}$$

where $b_i = b_i(w,x)$ are arbitrary functions.

P r o o f. Subtracting the equations (4.5) and (4.6) we obtain

$$(fA'_y - f'_yA)'_y + 2\beta(CA'_y - AC''_y) = 0,$$

i.e., using also (4.11),

$$(fA'_y - Af'_y)'_y = 2a_0(CA^{-1})'_y. \tag{4.20}$$

Integrating w.r. to y and multiplying then by A^2 we get

$$2a_0AC = \varphi_5A^2 + (fA)(A^2)'_y - A^2(fA)'_y, \tag{4.21}$$

where $\varphi_5 = \varphi_5(w,x)$ is an arbitrary function. If we substitute into (4.21) from (4.2) and (4.12), we see that the right-hand side of (4.21) is a quadratic polynomial w.r. to y. Thus if $a_0 \neq 0$, then the formula (4.19) holds.

Assume now $a_0 = 0$. Then (4.11) gives $\beta = 0$ and the equation (B3) means $T'_y = 0$. From (3.4) we get $C''_{yy} = 0$, i.e., C is a linear polynomial in y. On the other hand, the equation (B1) means $A''_{yy} = 0$ and A is also a linear polynomial in y. Hence (4.19) follows. □

For the later use, we shall write (4.21) more explicitly, using the notation (4.19), (4.2) and (4.12):

$$\begin{cases} 2a_0b_1 = \varphi_5a_1 - a_2K + a_1L, \\ 2a_0b_2 = \varphi_5a_2 - 2a_3K + 2a_1M, \\ 2a_0b_3 = \varphi_5a_3 - a_3L + a_2M. \end{cases} \tag{4.22}$$

Next, we have

Proposition 4.7. *Put $h(w,x) = H'_x(w,x)$. Then the PDE system* (A), (B), (C) *implies*

$$\begin{cases} 2a_0K + a_1b_2 - a_2b_1 - a_1h = 0, \\ 2a_0L + 2a_1b_3 - 2b_1a_3 - a_2h = 0, \\ 2a_0M - a_3b_2 + a_2b_3 - a_3h = 0. \end{cases} \tag{4.23}$$

P r o o f. From $(3.3)_2$ we have $H'_x = 2\beta\psi^{-1} - (AC)'_y + 2CA'_y$. Using (4.11) and (3.1) we obtain hence

$$2a_0fA - A^2(AC)'_y + (AC)(A^2)'_y = hA^2.$$

Substituting from (4.2), (4.12) and (4.19) we get (4.23). □

Many more algebraic equations are now obtained by using the "first integrals" from Proposition 4.5. Using (4.10), (4.15) and (3.4), we obtain first

$$S = f\psi Q, T = C\psi Q + \varphi_0\psi A, \quad \text{where} \quad Q = Ky + \varphi_4. \tag{4.24}$$

Let us substitute (4.24) into the equation (C3). We obtain, using also (4.11),

$$Q'_y A^2 - QAA'_y + (a_0 AC)\psi Q + a_0\varphi_0\psi A^2 = 0. \tag{4.25}$$

Substituting from (4.21) into the third term of (4.25) and then dividing by ψA^2 we obtain (using also (3.1))

$$2fAQ'_y + \varphi_5 Q - Q(fA)'_y + 2\varphi_0 a_0 = 0.$$

This means, due to (4.2),

$$K(L + \varphi_5 - 2\varphi_4) = 0, \quad 2KM + \varphi_4(\varphi_5 - L) + 2\varphi_0 a_0 = 0. \tag{4.26}$$

Next, we use the equation $(4.17)_2$:

$$CT + fS = \varphi_1 y + \frac{1}{2}\varphi_2,$$

in which we substitute from (4.24). We obtain easily, using also $(4.17)_1$,

$$\begin{aligned} 2\varphi_0 AC = (2\varphi_1 L - \varphi_2 K - 2\varphi_1\varphi_4)y^2 + (2\varphi_1 M + \varphi_2 L - 2\varphi_3 K - 2\varphi_2\varphi_4)y \\ + (\varphi_2 M - 2\varphi_3\varphi_4). \end{aligned} \tag{4.27}$$

This is equivalent to the system

$$\begin{cases} 2\varphi_0 b_1 = 2\varphi_1 L - \varphi_2 K - 2\varphi_1\varphi_4, \\ 2\varphi_0 b_2 = 2\varphi_1 M + \varphi_2 L - 2\varphi_3 K - 2\varphi_2\varphi_4, \\ 2\varphi_0 b_3 = \varphi_2 M - 2\varphi_3\varphi_4. \end{cases} \tag{4.28}$$

Next we substitute from (4.24) into (4.16): $S^2 + T^2 = \varphi_1$. We get

$$(f^2 + C^2)Q^2 + (2\varphi_0 AC)Q + \varphi_0^2 A^2 = \varphi_1 (fA)^2.$$

Using (4.27) and then $(4.17)_1$, we obtain an explicit formula for $\varphi_0^2 A^2$ as a quadratic polynomial w.r. to y. At the coefficient level we have

$$\begin{cases} \varphi_0^2 a_1 = \varphi_3 K^2 - \varphi_2 KL + \varphi_1 L^2 + \varphi_2\varphi_4 K - 2\varphi_1\varphi_4 L + \varphi_1\varphi_4^2, \\ \varphi_0^2 a_2 = -\varphi_2 KM + 2\varphi_1 LM + 2\varphi_3\varphi_4 K - \varphi_2\varphi_4 L - 2\varphi_1\varphi_4 M + \varphi_2\varphi_4^2, \\ \varphi_0^2 a_3 = \varphi_1 M^2 - \varphi_2\varphi_4 M + \varphi_3\varphi_4^2. \end{cases} \tag{4.29}$$

Finally, the equation $(4.17)_1$: $f^2 + C^2 = \varphi_1 y^2 + \varphi_2 y + \varphi_3$ implies

$$(AC)^2 = A^2(\varphi_1 y^2 + \varphi_2 y + \varphi_3) - (Af)^2, \tag{4.30}$$

i.e.,

$$\begin{aligned}(b_1 y^2 + b_2 y + b_3)^2 = (\varphi_1 y^2 + \varphi_2 y + \varphi_3)(a_1 y^2 + a_2 y + a_3) \\ - (K y^2 + L y + M)^2.\end{aligned}$$

This is equivalent to the system of algebraic equations

$$\begin{aligned}(b_1)^2 &= \varphi_1 a_1 - K^2, \\ 2b_1 b_2 &= \varphi_1 a_2 + \varphi_2 a_1 - 2KL, \\ (b_2)^2 + 2b_1 b_3 &= \varphi_1 a_3 + \varphi_2 a_2 + \varphi_3 a_1 - (L^2 + 2KM), \\ 2b_2 b_3 &= \varphi_2 a_3 + \varphi_3 a_2 - 2LM, \\ (b_3)^2 &= \varphi_3 a_3 - M^2.\end{aligned} \tag{4.31}$$

We shall summarize the content of this Section in the basic

Theorem 4.8. *Let $\varphi_0, \varphi_1, \dots, \varphi_5$, a_0, a_1, a_2, a_3, b_1, b_2, b_3, K, L, M, h by functions of two variables, w, x, satisfying the whole set of algebraic equations* (4.13), (4.18), (4.22), (4.23), (4.26), (4.28), (4.29) *and* (4.31). *Let A, f, C, H, ψ be functions defined by*

$$\begin{cases} A^2 = a_1 y^2 + a_2 y + a_3, \quad AC = b_1 y^2 + b_2 y + b_3, \\ 1/\psi = Af = K y^2 + L y + M, \quad H'_y = 0, \quad H'_x = h(w, x), \end{cases} \tag{4.32}$$

and let a metric g be defined by (2.1). *Further, let α, β, R be defined as in* (3.3), (3.4). *Then the curvature conditions* (2.8)–(2.10) *are satisfied for some function $k = k(w, x, y)$ if and only if the differential equations* (A1) *and* (A2) *are satisfied.*

The proof follows from the whole series of propositions and formulas given before. Let us only point out that *the equation* (A3) (or, equivalently, (4.3)) *does not mean any new condition for the functions involved.* It gives, in fact, a *formula for the computation of the curvature* $k(w, x, y) = \sigma(w, x)\psi(w, x, y)$.

Let us also notice that the algebraic equations (4.13), ..., (4.31) from Theorem 4.8 are not all independent. Yet, all of them are useful when a detailed analysis is made.

Remark 4.9. Theorem 4.8 says that one should start with an *algebraic classification*, i.e., to work out the list of all classes of solutions of the algebraic system above. Each separate class of solutions can be given in the form where some of the functions $\varphi_0, \varphi_1, \ldots, M, h$ are chosen as arbitrary and the others are fixed algebraic functions of the previous ones. In order not to expand this paper too much, we shall limit ourselves here to the "generic" cases, and we shall put aside the singular cases of various level. The generic algebraic solutions will be calculated in the next sections (and the singular cases will be treated in a separate paper).

We shall conclude this section by proving additional algebraic equations between our basic functions of two variables.

Proposition 4.10. *The following algebraic formulas hold:*

$$a_2L - 2a_3K - 2a_1M = -2a_0h, \tag{4.33}$$

$$b_2L - 2b_3K - 2b_1M = -\varphi_5 h, \tag{4.34}$$

$$\varphi_2L - 2\varphi_3K - 2\varphi_1M = 2\varphi_0 h, \tag{4.35}$$

$$a_2b_2 - 2a_1b_3 - 2a_3b_1 = -2a_0\varphi_5. \tag{4.36}$$

Proof. From (4.29) we obtain

$$\varphi_0^2(a_2L - 2a_3K - 2a_1M) = -\varphi_0a_0(\varphi_2L - 2\varphi_1M - 2\varphi_3K), \tag{4.37}$$

from (4.28) we get

$$2\varphi_0(b_2L - 2b_3K - 2b_1M) = -\varphi_5(\varphi_2L - 2\varphi_1M - 2\varphi_3K), \tag{4.38}$$

from (4.23) we get

$$a_0(a_2L - 2a_3K - 2a_1M) = -2a_0^2h, \tag{4.39}$$

and (4.22) implies

$$2a_0(b_2L - 2b_3K - 2b_1M) = \varphi_5(a_2L - 2a_3K - 2a_1M). \tag{4.40}$$

If $a_0\varphi_0 \neq 0$, then all formulas (4.33)–(4.36) follow from (4.37)–(4.40). For $a_0\varphi_0 = 0$ we either use the continuity argument (or a rather lengthy direct check). □

Proposition 4.11. *The algebraic formula*

$$\varphi_0 A^2 - a_0(f^2 + C^2) + \varphi_5 AC + hAf = 0, \tag{4.41}$$

or, equivalently,

$$\varphi_0(a_1y^2 + a_2y + a_3) - a_0(\varphi_1 y^2 + \varphi_2 y + \varphi_3) + \varphi_5(b_1y^2 + b_2y + b_3) + h(Ky^2 + Ly + M) = 0 \tag{4.42}$$

holds.

Proof. First we shall prove

$$L + \varphi_5 - 2\varphi_4 = 0. \tag{4.43}$$

In fact, for $K \neq 0$ this follows from $(4.26)_1$. For $K = 0$ we use the continuity argument (or a rather lengthy direct check).

Suppose now $\varphi_0 \neq 0$ and multiply the equation (4.42) by φ_0. Then substitute for $\varphi_0^2 a_i$ and $\varphi_0 b_i$ from (4.29) and (4.28), respectively; further, substitute for $\varphi_0 a_0$ from $(4.26)_2$ and for $\varphi_0 h$ from (4.35). Then we see easily that (4.42) holds as a consequence of (4.43).

If $\varphi_0 = 0$, we use the continuity argument (or a lengthy direct check). □

5. Riemannian invariants and isometries

Let us rewrite the formulas (3.2) using the forms ω^i as a basis. Then we obtain

$$\begin{cases} \omega_2^1 = \psi f_x' \omega^1 - \alpha\omega^2 + \beta\omega^3, \\ \omega_3^1 = f^{-1} f_y' \omega^1 + \beta\omega^2, \\ \omega_3^2 = (\beta - h\psi)\omega^1 + A^{-1}A_y'\omega^2. \end{cases} \tag{5.1}$$

We shall also write, for the simplicity

$$\omega_3^1 = a\omega^1 + b\omega^2, \quad \omega_3^2 = c\omega^1 + e\omega^2. \tag{5.2}$$

Consider the orthonormal frame $\{E_1, E_2, E_3\}$ introduced in Section 2. As we see, E_3 is uniquely determined by the geometry of (M, g) up to a sign, and E_1, E_2 are uniquely determined up to an orthogonal transformation (with the coefficients which are functions of w, x, y).

Let E_1, E_2 be chosen in the standard way, i.e., so that (2.1) holds. Using the well-known formula for the covariant differentiation

$$\nabla_{E_j} E_i = \sum_k \omega_i^k(E_j) E_k \quad (i, j = 1, 2, 3) \tag{5.3}$$

we obtain

$$\begin{cases} \nabla_{E_1} E_1 = -\psi f'_x E_2 - aE_3, & \nabla_{E_1} E_2 = \psi f'_x E_1 - cE_3, \\ \nabla_{E_2} E_1 = \alpha E_2 - bE_3, & \nabla_{E_2} E_2 = -\alpha E_1 - eE_3, \\ \nabla_{E_1} E_3 = aE_1 + cE_2, & \nabla_{E_2} E_3 = bE_1 + eE_2, \\ \nabla_{E_3} E_1 = -bE_2, \quad \nabla_{E_3} E_2 = bE_1, & \nabla_{E_3} E_3 = 0. \end{cases} \tag{5.4}$$

Due to (2.7), the *Ricci form* ϱ is given by

$$\varrho = k(\omega^1 \odot \omega^1 + \omega^2 \odot \omega^2) \tag{5.5}$$

where k is the sectional curvature given by (2.8), $k = \frac{1}{2} Sc(g)$. Because ϱ is an invariant tensor, $\nabla \varrho$ is also an invariant tensor. Using the standard formula

$$\nabla_X \omega^i = -\sum \omega_j^i(X) \omega^j \tag{5.6}$$

we obtain easily

$$\begin{aligned} \nabla \varrho = {} & dk \odot (\omega^1 \odot \omega^1 + \omega^2 \odot \omega^2) \\ & - k\{(a\omega^1 + b\omega^2) \odot (\omega^1 \odot \omega^3 + \omega^3 \odot \omega^1) \\ & + (c\omega^1 + e\omega^2) \odot (\omega^2 \odot \omega^3 + \omega^3 \odot \omega^2)\}. \end{aligned} \tag{5.7}$$

Hence we see that the tensor

$$\begin{aligned} Q = {} & (a\omega^1 + b\omega^2) \odot (\omega^1 \odot \omega^3 + \omega^3 \odot \omega^1) \\ & + (c\omega^1 + e\omega^2) \odot (\omega^2 \odot \omega^3 + \omega^3 \odot \omega^2) \end{aligned} \tag{5.8}$$

is also invariant. Now, because $E_3 = \partial/\partial y$ is determined up to a sign, and E_1, E_2 are determined up to an orthogonal transformation (with functional coefficients), the functions

$$\begin{aligned} Q(E_1, E_1, E_3) + Q(E_2, E_2, E_3) &= a + e, \\ Q(E_2, E_1, E_3) - Q(E_1, E_2, E_3) &= b - c. \end{aligned} \tag{5.9}$$

are Riemannian invariants up to a sign.

The square of the norm, $||Q||^2 = 2(a^2 + b^2 + c^2 + e^2)$ is a Riemannian invariant and hence (equivalently) $ae - bc$ is a Riemannian invariant. We summarize:

Proposition 5.1. *Denote*

$$a = f^{-1} f'_y, \quad b = \beta = a_0/A^2, \quad c = \beta - h\psi, \quad e = A^{-1} A'_y. \tag{5.10}$$

Then $ae - bc$ is a Riemannian invariant, and $a + e$, $b - c$ are Riemannian invariants up to a sign (i.e., depending on the orientation of the principal geodesics). Further, the partial derivative of any Riemannian invariant with respect to y is a Riemannian invariant up to a sign, and conversely.

Using (3.1) we get, in addition

$$a + e = (\ln(fA))'_y = -(\ln k)'_y, b - c = h\psi = hk/\sigma. \tag{5.11}$$

Further, we have

$$ae - bc = K\psi. \tag{5.12}$$

The last formula is obtained by a lengthy calculation using (4.33) and the obvious identities

$$a_1 A^2 = (AA'_y)^2 + (a_0)^2, \quad A^3 f'_y = (Af)'_y A^2 - (Af)(AA'_y). \tag{5.13}$$

Using (5.11), (5.12) and differentiations w.r. to y we see that

$$\frac{1}{h\psi} = \frac{Ky^2 + Ly + M}{h}, \quad \frac{K}{h}, \quad \frac{2Ky + L}{K} \tag{5.14}$$

are Riemannian invariants up to a sign, and

$$\frac{1}{K\psi} = \frac{Ky^2 + Ly + M}{K}, \quad \frac{2Ky + L}{h}, \quad \frac{L^2 - 4KM}{K^2}, \quad \frac{L^2 - 4KM}{h^2} \tag{5.15}$$

are Riemannian invariants (assuming everywhere that $h \neq 0$, or $K \neq 0$, respectively).

Next, we shall prove some simple results concerning isometries of SSS to be used later. Let (M, g) be and SSS given by (2.1) and let $(\overline{M}, \bar{g})$ be another SSS with the metric $\bar{g}$ given by the orthonormal coframe

$$\bar{\omega}^1 = \bar{f}\, d\bar{w}, \quad \bar{\omega}^2 = \bar{A}\, d\bar{x} + \bar{C}\, d\bar{w}, \quad \bar{\omega}^3 = d\bar{y} + \bar{H}\, d\bar{w}. \tag{5.16}$$

Suppose that there is an isometry $F\colon (M,g) \longrightarrow (\overline{M},\bar{g})$ given by

$$\bar{w} = \bar{w}(w,x,y), \quad \bar{x} = \bar{x}(w,x,y), \quad \bar{y} = \bar{y}(w,x,y). \tag{5.17}$$

Proposition 5.2. *The equations* (5.17) *can be reduced to the form*

$$\bar{w} = \bar{w}(w,x), \quad \bar{x} = \bar{x}(w,x), \quad \bar{y} = \varepsilon y + \Phi(w,x), \quad \varepsilon = \pm 1. \tag{5.18}$$

Proof. According to the geometric meaning of the principal geodesics we have (writing always $\bar{\omega}^i$ instead of $F^*\bar{\omega}^i$)

$$\bar{\omega}^1 = (\cos\varphi)\omega^1 - (\sin\varphi)\omega^2, \quad \bar{\omega}^2 = \varepsilon'((\sin\varphi)\omega^1 + (\cos\varphi)\omega^2), \quad \bar{\omega}^3 = \varepsilon\omega^3, \tag{5.19}$$

where φ is a function of w,x,y and $\varepsilon,\varepsilon' = \pm 1$. Hence

$$(\bar{\omega}^1)^2 + (\bar{\omega}^2)^2 = (\omega^1)^2 + (\omega^2)^2 = f^2\,\mathrm{d}w^2 + (C\,\mathrm{d}w + A\,\mathrm{d}x)^2.$$

On the other hand, from (5.16) and (5.17) we get

$$(\bar{\omega}^1)^2 + (\bar{\omega}^2)^2 = [\bar{f}^2(\partial\bar{w}/\partial y)^2 + (\bar{C}(\partial\bar{w}/\partial y) + \bar{A}(\partial\bar{x}/\partial y))^2]\,\mathrm{d}y^2 + \text{other terms}.$$

Hence $\partial\bar{w}/\partial y = \partial\bar{x}/\partial y = 0$.

Finally, $\bar{\omega}^3 = \varepsilon\omega^3$ means $\mathrm{d}\bar{y} - \varepsilon\,\mathrm{d}y = \varepsilon H\,\mathrm{d}w - \overline{H}\,\mathrm{d}\bar{w} = P^\varepsilon\,\mathrm{d}x + Q^\varepsilon\,\mathrm{d}w$, where $P^\varepsilon, Q^\varepsilon$ are functions of w,x only (depending on ε). Hence we get $\bar{y} = \varepsilon y + \Phi^\varepsilon(w,x)$, where Φ^ε is a potential function of the (integrable!) form $P^\varepsilon\,\mathrm{d}x + Q^\varepsilon\,\mathrm{d}w$, q.e.d. □

Proposition 5.3. *Suppose that $a_0 = 0$ on (M,g) and $\bar{a}_0 = 0$ on $(\overline{M},\bar{g})$. Further, assume that $a - e \neq 0$, or $c \neq 0$ holds on (M,g). Then any isometry $F\colon (M,g) \longrightarrow (\overline{M},\bar{g})$ implies the equalities*

$$\bar{\omega}^i = \varepsilon_i\omega^i, \quad \varepsilon_i = \pm 1 \quad (i = 1,2,3). \tag{5.20}$$

Proof. We have $\bar{b} = b = 0$ and, according to Proposition 5.1, we obtain, via the isometry F,

$$\bar{a} = \varepsilon a, \quad \bar{e} = \varepsilon e, \quad \bar{c} = \tilde{\varepsilon} c. \tag{5.21}$$

Here ε is the sign from (5.18) (as follows from $(5.11)_1$) and $\tilde{\varepsilon}$ is another sign. (The other possibility $\bar{a} = \varepsilon e$, $\bar{e} = \varepsilon a$ can be eliminated by choosing new local coordinates and a new coframe.)

Suppose now that $\sin\varphi \neq 0$ holds in (5.19). Substituting (5.21) and (5.19) into the invariant equation $\overline{Q} = Q$ (cf. (5.8)), we obtain after a routine calculation that $a - e = 0$ and $c = 0$ along (M,g). This completes the proof. □

Let us recall that the principal geodesics are trajectories of the vector field E_3. We shall introduce two basic definitions.

Definition 6.1. A smooth surface $N \subset (M, g)$ is called *an asymptotic leaf* if it is generated by the principal geodesics and its tangent planes are parallel along these principal geodesics (w.r. to the Riemannian connection ∇ of (M, g)).

Definition 6.2. *An asymptotic distribution* on (M, g) is a 2-dimensional smooth distribution which projects into a 1-dimensional distribution via the map π: $(w, x, y) \longmapsto (w, x)$, and satisfies the equation

$$a_0 \, \mathrm{d}x^2 + \varphi_5 \, \mathrm{d}x \, \mathrm{d}w - \varphi_0 \, \mathrm{d}w^2 = 0. \tag{6.1}$$

The following Proposition is obvious:

Proposition 6.3. *Let $\Delta = \varphi_5^2 + 4a_0\varphi_0$ denote the discriminant of* (6.1).

a) *If $\Delta < 0$ on (M, g), then there is no real asymptotic distribution on M.*
b) *If $\Delta > 0$ on (M, g), then there are exactly two different asymptotic distributions on M.*
c) *If $\Delta = 0$ on (M, g) and some of the functions a_0, φ_0, φ_5 is nonzero at each point, then there is a unique asymptotic distribution on M.*
d) *If $a_0 = \varphi_0 = \varphi_5 = 0$ on M, then any π-projectable smooth 2-dimensional distribution on M is asymptotic.*

Definition 6.4. The space (M, g) is said to be *elliptic*, or *hyperbolic*, or *parabolic*, or *planar*, respectively, if the case a), or b), or c), or d) of Proposition 6.3 occurs on the whole M, respectively.

Because we are interested in the local classification only, we shall investigate only the "pure" cases and not the combined ones in the sequel. (For a global treatment of *some* of our geometric types see [Sz 2]).

Now, the following theorem gives the connection between the definitions 6.1 and 6.2 and thus ensures the *geometric meaning* of Definition 6.4:

Theorem 6.5. *Let (M, g) be hyperbolic, or parabolic, or planar. Then the corresponding asymptotic distributions are integrable and their integral manifolds are asymptotic leaves. Conversely, each asymptotic leaf is an integral manifold of some asymptotic distribution. Consequently, (M, g) admits two, or one, or infinitely many asymptotic foliations, respectively.*

P r o o f. In the hyperbolic case, the both asymptotic distributions are given for $\varphi_5 \geqslant 0$ by

$$2a_0\,\mathrm{d}x + (\varphi_5 + \sqrt{\Delta})\,\mathrm{d}w = 0, \quad (\varphi_5 + \sqrt{\Delta})\,\mathrm{d}x - 2\varphi_0\,\mathrm{d}w = 0, \tag{6.2}$$

and for $\varphi_5 \leqslant 0$ by

$$2a_0\,\mathrm{d}x + (\varphi_5 - \sqrt{\Delta})\,\mathrm{d}w = 0, \quad (\varphi_5 - \sqrt{\Delta})\,\mathrm{d}x - 2\varphi_0\,\mathrm{d}w = 0. \tag{6.3}$$

In the parabolic case, the unique asymptotic distribution is given by some (or both) of the equations

$$2a_0\,\mathrm{d}x + \varphi_5\,\mathrm{d}w = 0, \quad \varphi_5\,\mathrm{d}x - 2\varphi_0\,\mathrm{d}w = 0. \tag{6.4}$$

To each of these asymptotic distributions one can find (at least locally) a potential function $P(x, w)$, i.e., such function that the distribution is given by the equation $\mathrm{d}P = 0$, and hence the integrability follows. In both cases we see that our distributions contain the vector field E_3.

In the planar case we know that a 2-dimensional distribution on (M, g) is asymptotic if and only if it is π-projectable. Because a π-projection is 1-dimensional and hence integrable, all these distributions are integrable (and contain the vector field E_3). Let us notice that the projection π is not orthogonal w.r. to the metric g, in general.

Let now $N \subset M$ be an asymptotic leaf. Then the tangent planes along N are determined by the formula

$$\sin\varphi \cdot \omega^1 + \cos\varphi \cdot \omega^2 = 0, \tag{6.5}$$

where φ is a smooth function on N. This means

$$T_mN = \operatorname{span}(\cos\varphi \cdot E_1 - \sin\varphi \cdot E_2, E_3)_m, \quad m \in N. \tag{6.6}$$

Now, the integrability condition

$$[\cos\varphi \cdot E_1 - \sin\varphi \cdot E_2, E_3] \in \operatorname{span}(\cos\varphi \cdot E_1 - \sin\varphi \cdot E_2, E_3) \tag{6.7}$$

and the asymptoticity condition

$$\nabla_{E_3}(\cos\varphi \cdot E_1 - \sin\varphi \cdot E_2) \in \operatorname{span}(\cos\varphi \cdot E_1 - \sin\varphi \cdot E_2, E_3) \tag{6.8}$$

must be satisfied along N. Hence it follows that also

$$\nabla_{(\cos\varphi\cdot E_1-\sin\varphi\cdot E_2)}E_3 \in \operatorname{span}(\cos\varphi\cdot E_1-\sin\varphi\cdot E_2, E_3) \tag{6.9}$$

holds along N. From the formulas (5.4) we obtain that (6.9) is equivalent to

$$\sin^2\varphi\cdot b+\sin\varphi\cos\varphi(e-a)-\cos^2\varphi\cdot c=0. \tag{6.10}$$

Using (6.5) as a proportion formula, we see that the tangent distribution of N satisfies the equation

$$c(\omega^1)^2+(e-a)\omega^1\omega^2-b(\omega^2)^2=0. \tag{6.11}$$

Now, substituting for ω^1, ω^2 from (2.1) and form a, b, c, e from (5.10) we get hence

$$\begin{aligned} a_0A^2\,\mathrm{d}x^2 &+ (a_0C^2+AC(Af'_y-fA'_y)+hAF-a_0f^2)\,\mathrm{d}w^2 \\ &+ (2a_0AC+A^2(Af'_y-fA'_y))\,\mathrm{d}x\,\mathrm{d}w=0. \end{aligned} \tag{6.12}$$

According to (4.21) we see that

$$A^2(Af'_y-fA'_y)=A^2(fA)'_y-(Af)(A^2)'_y=\varphi_5A^2-2a_0AC. \tag{6.13}$$

Substituting from (6.13) into (6.12), and using the algebraic formula (4.41), we obtain the equation (6.1). Thus we have proved that the tangent distribution along N is a part of some asymptotic distribution on M, and hence N is an integral manifold of this distribution.

The assertion that the integral manifolds of any asymptotic distribution are asymptotic leaves can be now proved just by reversing our procedure. □

Let us point out that the nontrivial part of the proof is the algebraic formula (4.41), which involves most of the computations of Section 4.

Now, the following Theorem will be crucial for the explicit geometric classification of the non-elliptic SSS in Section 7.

Theorem 6.6. *In the hyperbolic and parabolic case, there exists a transformation of local coordinates preserving the form (2.1) of the metric and annihilating the function $a_0(w,x)$.*

Proof. Suppose that (M,g) is hyperbolic, i.e., $\Delta>0$. Then at the basic point $p\in M$ either $\varphi_5\neq 0$ or $\varphi_0\neq 0$. In each case, one of the equations (6.2), (6.3) has a nonzero coefficient at $\mathrm{d}w$ in a neighborhood of p. If we fix any potential function

$\overline{w} = \overline{w}(w, x)$ of such an equation, then $(\overline{w}(w,x), x, y)$ is a new local coordinate system in a neighborhood of p. Now, one can find an orthonormal coframe $(\bar{\omega}^1, \bar{\omega}^2, \omega^3)$ such that

$$\bar{\omega}^1 = \bar{f}\,\mathrm{d}\overline{w}, \quad \bar{\omega}^2 = \bar{A}\,\mathrm{d}x + \bar{C}\,\mathrm{d}\overline{w}, \quad \omega^3 = \mathrm{d}y + H\,\mathrm{d}w. \tag{6.14}$$

Further, we look for a substitution $\bar{y} = y + \Phi(w, x)$ such that

$$\mathrm{d}y + H\,\mathrm{d}w = \mathrm{d}\bar{y} + \overline{H}\,\mathrm{d}\overline{w}, \tag{6.15}$$

where $\overline{H} = \overline{H}(w, x)$ is the second unknown function. We get the conditions

$$\Phi'_w + \overline{H}\overline{w}'_w = H, \quad \Phi'_x + \overline{H}\overline{w}'_x = 0. \tag{6.16}$$

The corresponding (local) integrability condition is

$$\overline{H}'_x \overline{w}'_w - \overline{H}'_w \overline{w}'_x = H_x, \tag{6.17}$$

which is a PDE for $\overline{H}$.

Let us fix one solution $\overline{H}(w, x)$ of (6.17). Then the function Φ is determined by (6.16) up to a constant. As we see, (6.14) and (6.15) give the standard form (2.1) for our metric in the new local coordinates $\overline{w}$, $\bar{x} = x$, $\bar{y}$.

Now, the couple $\mathscr{F}_1$, $\mathscr{F}_2$ of the asymptotic foliation of (M, g) is described by the equation analogous to (6.1):

$$\bar{a}_0\,\mathrm{d}\bar{x}^2 + \bar{\varphi}_5\,\mathrm{d}\bar{x}\,\mathrm{d}\overline{w} - \bar{\varphi}_0\,\mathrm{d}\overline{w}^2 = 0. \tag{6.18}$$

Because one of these foliations, say $\mathscr{F}_1$, is given by the equation $\mathrm{d}\overline{w} = 0$, we get hence $\bar{a}_0 = 0$ in the whole neighborhood, q.e.d.

Suppose now that (M, g) is parabolic. If $\varphi_5 \neq 0$ or $\varphi_0 \neq 0$ at p, we make use of (6.4) and the proof is similar as in the hyperbolic case. Assume now $\varphi_5 = \varphi_0 = 0$ and thus $a_0 \neq 0$ at p. We first substitute $\overline{w} = x$, $\bar{x} = w$, then the formulas (2.1) take on the form

$$\omega^1 = f\,\mathrm{d}\bar{x}, \quad \omega^2 = A\,\mathrm{d}\overline{w} + C\,\mathrm{d}\bar{x}, \quad \omega^3 = \mathrm{d}y + H\,\mathrm{d}\bar{x}. \tag{6.19}$$

Now we construct a new orthonormal coframe

$$\bar{\omega}^1 = \cos\varphi \cdot \omega^2 - \sin\varphi \cdot \omega^1, \quad \bar{\omega}^2 = \sin\varphi \cdot \omega^2 + \cos\varphi \cdot \omega^1, \quad \bar{\omega}^3 = \omega^3, \tag{6.20}$$

such that

$$\bar{\omega}^1 = \bar{f}\,\mathrm{d}\bar{w}, \quad \bar{\omega}^2 = \bar{A}\,\mathrm{d}\bar{x} + \bar{C}\,\mathrm{d}\bar{w}, \tag{6.21}$$

and introduce a new variable $\bar{y} = y + \Phi(w, x)$ such that

$$\omega^3 = \mathrm{d}\bar{y} + \bar{H}\,\mathrm{d}\bar{w}. \tag{6.22}$$

We obtain easily

$$\cos\varphi \neq 0, \quad \tan\varphi = cf^{-1}, \quad \bar{A}\bar{f} = Af, \quad \bar{A}^2 = f^2 + C^2, \quad \bar{f}^2 + \bar{C}^2 = A^2. \tag{6.23}$$

We see that the coframe $(\bar{\omega}^i)$ given by (6.21), (6.22) is of the standard form (2.1). Then, in the notation of the proposition 4.4, 4.5, we have

$$\bar{\varphi}_1 = a_1, \quad \bar{\varphi}_2 = a_2 - 2\Phi a_1, \quad \bar{\varphi}_3 = a_1\Phi^2 - a_2\Phi + a_3, \tag{6.24}$$

and due to (4.13), (4.18) we obtain $\bar{\varphi}_0 = \pm a_0 \neq 0$. Then we continue as in the hyperbolic case. □

Remark. As concerns the planar case, we have $a_0 = 0$ (in a neighborhood of p) by definition. Thus for every non-elliptic space (M, g) we can assume $a_0 = 0$. Conversely, from (6.1) we see that $a_0 = 0$ always implies that (M, g) is non-elliptic.

In the second part of this Section we shall prove a number of geometric results on asymptotic foliations. We shall also compare our Definition 6.4 with the terminology of Z. Szabó.

First, let us notice that the discriminant of (6.1) is given alternatively by the formula

$$\Delta = L^2 - 4KM. \tag{6.25}$$

Indeed, combining (4.43) with $(4.26)_2$ we obtain at once

$$4KM - L^2 + (\varphi_5)^2 + 4\varphi_0 a_0 = 0. \tag{6.26}$$

Hence we get

Proposition 6.7. *The space (M, g) is elliptic if $L^2 - 4KM < 0$, hyperbolic if $L^2 - 4KM > 0$, and parabolic or planar if $L^2 - 4KM = 0$. In the nonplanar case and for $K \neq 0$, the number of asymptotic foliations is the same as the number of real* *roots of the quadratic equation $Ky^2 + Ly + M = 0$, i.e., the number of singularities of the scalar curvature $Sc(g)$ along a principal geodesic.*

On the other hand, from (6.11) we obtain

Proposition 6.8. *The space (M, g) is elliptic if $(a-e)^2 + 4bc < 0$, hyperbolic if $(a-e)^2 + 4bc > 0$, and parabolic or planar if $(a-e)^2 + 4bc = 0$. (M, g) is planar if and only if $a - e = b = c = 0$.*

Corollary 6.9. *The space (M, g) is planar if and only if $f = \lambda(w, x)A$, $C = \mu(w, x)A$ and $a_0 = 0$. Assuming $a_0 = 0, (M, g)$ is parabolic if and only if $f = \lambda(w, x)A$ and $h \neq 0$ (λ, μ are arbitrary functions).*

P r o o f. $a - e = 0$ means $(f/A)'_y = 0$, $b = 0$ means $a_0 = 0$, and $c = 0$ means $h = 0$ (see (5.10)). Due to $(3.3)_2$, $a_0 = h = 0$ is equivalent to $(C/A)'_y = 0$.

Next, we have □

Proposition 6.10. *If $h = 0$, then (M, g) is hyperbolic or planar. In the hyperbolic case, $h = 0$ means that the asymptotic foliations $\mathscr{F}_1$, $\mathscr{F}_2$ are mutually orthogonal.*

P r o o f. $h = 0$ means $b = c$ (cf. (5.11)) and hence $(a-e)^2 + 4bc \geqslant 0$. In the hyperbolic case, the equation (6.10) means that $2b \cos 2\varphi + (a - e) \sin 2\varphi = 0$. Hence if φ characterizes one of the asymptotic foliations, then $\varphi + \frac{\pi}{2}$ characterizes the second one. From (6.5) we see that both foliations are mutually orthogonal.

The asymptotic foliations are not totally geodesic, in general. Yet, we have the following □

Proposition 6.11. *Let the metric g be hyperbolic, or parabolic, or planar, and expressed in such a coordinate system that $a_0 = 0$. If $\alpha = 0$, then at least one of the asymptotic foliations is totally geodesic.*

P r o o f is straightforward: because $b = 0$, formulas (5.4) show that $\mathrm{span}(E_2, E_3)$ is an asymptotic distribution. But the corresponding asymptotic foliation is totally geodesic if and only if

$$\nabla_{E_2} E_2 \in \mathrm{span}(E_2, E_3), \quad \text{i.e.,} \quad \alpha = 0.$$

We shall conclude this Section with the following □

Remark 6.12. Proposition 6.8 enables to compare our geometric terminology with that used by Z. Szabó in this study of complete foliated manifolds satisfying $R(X, Y) \cdot R = 0$ (see [Sz 2]). For the 3-dimensional case, the definitions by Szabó can be presented in the following way: let B denote the endomorphism of TM over the identity $id\colon M \to M$ defined by the formula

$$B(X) = \nabla_X E_3, \quad X \in TM. \tag{6.27}$$

Then we have, according to (5.4),

$$B(E_1) = aE_1 + cE_2, \quad B(E_2) = bE_1 + eE_2, \quad B(E_3) = 0. \tag{6.28}$$

Now, the foliated space (M, g) is said to be *trivial* if $B = 0$, and *parabolic* if $B \neq 0$ but $B^2 = 0$ on M. We shall show later that the trivial case means the product case, and that the parabolic case in the above sense is a special subcase of our parabolic case (the only one in which the completeness can occur). Further, a space (M, g) is said in [Sz 2] to be *hyperbolic* if B has two imaginary eigenvalues along M. This condition means

$$(a+e)^2 - 4(ae - bc) < 0, \quad \text{i.e.,} \quad (a-e)^2 + 4bc < 0. \tag{6.29}$$

Hence the hyperbolic spaces by Szabó are just *elliptic* foliated SSS in our sense.

7. The explicit classification of generic nonelliptic spaces

In this section we shall explicitly classify all *generic* hyperbolic, parabolic and planar foliated SSS, and we shall also present some nongeneric but interesting examples. Moreover, we shall answer the question *how the distinct local isometry classes can be parametrized. In this section we always assume* $a_0 = 0$ (which is allowed by Theorem 6.6).

We shall start with some general results.

Proposition 7.1. *The coefficients A, C, f from (2.1) can be expressed in the form*

$$A = py + q, \quad C = ry + s, \quad f = ty + u, \tag{7.1}$$

where p, q, r, s, t, u are functions of w, x only, such that

$$ps - qr = h. \tag{7.2}$$

Moreover, if $h \neq 0$, we can assume $h = 1$, $H = x$; and if $h = 0$, we can assume $H = 0$.

Proof. Because $a_0 = 0$, i.e., $\beta = 0$, the equation (B1) implies $A''_{yy} = 0$, and the equations (B3), (C3) together with (3.4) imply $C''_{yy} = f''_{yy} = 0$. The formula (7.2) follows from $(3.3)_2$ and (7.1) because $h = H'_x$, $\beta = 0$.

It remains to prove the last part. If $h \neq 0$, then $H'_x \neq 0$ and one can introduce the new variable $\bar{x} = H(w, x)$ instead of x. Then we get our orthonormal coframe in the standard form

$$\omega^1 = f\,\mathrm{d}w, \quad \omega^2 = (A/h)\,\mathrm{d}\bar{x} + (C - H'_w/h)\,\mathrm{d}w, \quad \omega^3 = \mathrm{d}y + \bar{x}\,\mathrm{d}w.$$

Let now $h = 0$. Because H depends only on w, we get $\omega^3 = \mathrm{d}\bar{y}$, where $\bar{y} = y + \int H\,\mathrm{d}w$. This concludes the proof. □

Proposition 7.2. *The differential equation* (A1) *is satisfied if and only if the following equation holds:*

$$u\mathscr{D} - t\mathscr{E} = 0, \tag{7.3}$$

where

$$\mathscr{D} = p'_w - r'_x, \quad \mathscr{E} = q'_w - s'_x - pH. \tag{7.4}$$

Proof. Substituting from (7.1) into $(3.3)_1$ we get

$$A\alpha = f^{-1}(A'_w - C'_x - HA'_y) = \frac{\mathscr{D}y + \mathscr{E}}{ty + u}. \tag{7.5}$$

Because $\beta = 0$, the equation (A1) simply means that $A\alpha$ does not depend on y and hence (7.3) follows. □

Proposition 7.3. *Assuming that* (A1) *is satisfied and*

$$A\alpha = \mathscr{V}(w, x), \tag{7.6}$$

then (A2) *is satisfied if and only if*

$$pu'_x - qt'_x = h\mathscr{V}. \tag{7.7}$$

Proof. We have first, using (3.4) and (7.6),

$$R = \psi f f'_x - C\alpha = A^{-1}(f'_x - \mathscr{V}C) = \frac{(t'_x - r\mathscr{V})y + (u'_x - s\mathscr{V})}{py + q}.$$

But the equation (A2) only means that R does not depend on y, and (7.7) follows from (7.2). □

Now, we can state the "converse" of Proposition 7.1.

Proposition 7.4. *Let p, q, r, s, t, u be arbitrary functions of two variables w, x. Let us define the functions A, C, f by (7.1), and let $H = H(w,x)$ be any function satisfying*

$$H'_x = h = ps - qr. \tag{7.8}$$

If the equations (A1), (A2) are satisfied, then (2.1) defines a foliated semi-symmetric metric of nonelliptic type.

P r o o f. Define the functions K, L, M and a_i, b_i, φ_i ($i = 1,2,3$) by the formulas (4.2), (4.12), (4.17) and (4.19), respectively. Put, in addition,

$$\varphi_5 = tq - up, \quad \varphi_4 = \frac{1}{2}(L + \varphi_5), \quad \varphi_0 = ru - ts. \tag{7.9}$$

Then we check easily that all algebraic conditions of Theorem 4.8 are satisfied, and hence the result follows.

We can now prove our main results. □

Theorem 7.5. *The generic family of hyperbolic (foliated semi-symmetric) metrics in dimension 3 is given by*

$$\begin{cases} \omega^1 = (ty + u)\,\mathrm{d}w, \\ \omega^2 = (py + q)\,\mathrm{d}x + (ry + s)\,\mathrm{d}w, \\ \omega^3 = \mathrm{d}y + x\,\mathrm{d}w, \end{cases} \tag{7.10}$$

where p, q, r, s are arbitrary functions of w, x such that $ps - qr = 1$, and t, u are calculated from p, q, r, s as follows:

$$u = \exp\left(\frac{1}{2}\int P\,\mathrm{d}x\right)\left[\int Q \exp\left(-\int P\,\mathrm{d}x\right)\mathrm{d}x\right]^{1/2}, \quad t = u\mathscr{D}/\mathscr{E}, \tag{7.11}$$

where

$$P = \frac{2q(\mathscr{E}\mathscr{D}'_x - \mathscr{D}\mathscr{E}'_x)}{\mathscr{E}(p\mathscr{E} - q\mathscr{D})}, \quad Q = \frac{2\mathscr{E}^2}{p\mathscr{E} - q\mathscr{D}}. \tag{7.12}$$

$$\mathscr{D} = p'_w - r'_x, \quad \mathscr{E} = q'_w - s'_x - px, \quad \mathscr{E}(p\mathscr{E} - q\mathscr{D}) \neq 0. \tag{7.13}$$

The local isometry classes of the metrics (7.10) are parametrized by 3 arbitrary functions of 2 variables modulo 2 arbitrary functions of 1 variable.

Proof. In the generic case we can assume $h \neq 0$ and hence $H = x, h = 1$ (Proposition 7.1). One can also assume $\mathscr{E}(p\mathscr{E} - q\mathscr{D}) \neq 0$. We express (7.3) in the form $t = u\mathscr{D}/\mathscr{E}$ and substitute into (7.7), where $\mathscr{V} = \mathscr{E}/u$. We obtain easily the differential equation

$$(u^2)'_x - Pu^2 = Q, \tag{7.14}$$

which can be solved by the standard method of "variation of constants". Because $p\mathscr{E} - q\mathscr{D} \neq 0$, Formula (7.3) implies $pu - tq \neq 0$, i.e., $(f/A)'_y \neq 0$. According to Corollary 6.9, our metrics are neither parabolic nor planar. Because $a_0 = 0$, they must be hyperbolic. This proves the first part of Theorem 7.5.

We now prove the statement about the local isometry classes. Let $(M, g), (\overline{M}, \bar{g})$ be two spaces with the metrics of the form (7.10) and let $F\colon M \to \overline{M}$ be an isometry. We shall identify the forms $F^*\bar{\omega}^i$ with $\bar{\omega}^i$, as usual. Because $b = 0$ and $h \neq 0$, $(5.11)_2$ implies $c \neq 0$ and we can use Proposition 5.3. Let us assume, for the simplicity, $\varepsilon_2 = \varepsilon_3 = 1$ (for the other signs, the argument is similar). We get first $\bar{\omega}^1 = \varepsilon\omega^1$ and hence

$$\bar{w} = \varphi(w),\ \mathrm{d}\bar{w} = \varphi'(w)\,\mathrm{d}w, \quad \bar{t}\bar{y} + \bar{u} = (ty + u)/\varphi'(w). \tag{7.15}$$

The equation $\bar{\omega}^3 = \omega^3$ means $d(\bar{y} - y) = (x - \bar{x}\varphi'(w))\,\mathrm{d}w$, i.e.,

$$\bar{y} = y + \psi(w), \quad \bar{x} = (x - \psi'(w))/\varphi'(w). \tag{7.16}$$

Finally, $\bar{\omega}^2 = \omega^2$ implies easily

$$\begin{aligned}[\bar{p}(y + \psi(w)) + \bar{q}]&[(1/\varphi'(w))\,\mathrm{d}x + \bar{x}'_w\,\mathrm{d}w] \\ &+[\bar{r}(y + \psi(w)) + \bar{s}]\varphi'(w)\,\mathrm{d}w = (py + q)\,\mathrm{d}x + (ry + s)\,\mathrm{d}w.\end{aligned} \tag{7.17}$$

Comparing the coefficients of $\mathrm{d}x$ and $\mathrm{d}w$, respectively, and then the coefficients of y^1 and y^0 in each case, we obtain

$$\bar{p} = p\varphi'(w), \quad \bar{q} = \varphi'(w)(q - p\psi(w)), \quad \bar{r}\varphi'(w) = r - p\varphi'(w)\bar{x}'_w, \tag{7.18}$$

where $\bar{x}'_w$ can calculated from (7.16). Further, $\bar{s}$ obviously satisfies the relation $\bar{p}\bar{s} - \bar{r}\bar{q} = 1$.

The formulas (7.15)–(7.18) show that the function $\bar{w}$, $\bar{x}$, $\bar{y}$, $\bar{p}$, $\bar{q}$, $\bar{r}$, $\bar{s}$, $\bar{t}$, $\bar{u}$ can be expressed through w, x, y, p, q, r, s, t, u and the arbitrary functions $\varphi(w)$, $\psi(w)$ of one variable. Thus each local isometry class depends on 2 arbitrary functions of 1 variable. This completes the proof. □

Let us notice that the isometry part of Theorem 7.5 can be stated more precisely using the concept of germs (cf. [KTV 1]).

Next, we have

Theorem 7.6. *The generic family of parabolic metrics is given by*

$$(7.19)\qquad \begin{cases} \omega^1 = \lambda(py+q)\,\mathrm{d}w, \\ \omega^2 = (py+q)\,\mathrm{d}x + (ry+s)\,\mathrm{d}w, \\ \omega^3 = \mathrm{d}y + x\,\mathrm{d}w, \end{cases}$$

where p, q are arbitrary functions of 2 variables, $(p/q)'_x \neq 0$, and

$$(7.20)\qquad r = \frac{pE + p'_x}{D}, \quad s = \frac{qE + q'_x}{D},$$

$$(7.21)\qquad \lambda = [(p'_w - r'_x)/pD]^{1/2},$$

$$(7.22)\qquad D = pq'_x - qp'_x, \quad E = -p^2x + pq'_w - qp'_w.$$

The local isometry classes are parametrized by 2 arbitrary functions of 2 variables modulo 2 arbitrary functions of 1 variable.

Proof. According to Corollary 6.9 we only have to assume $h = 1$ and the parabolicity condition $f = \lambda(w,x)A$. Thus, the only algebraic relations for the basic functions are

$$(7.23)\qquad ps - qr = 1,$$

$$(7.24)\qquad t = \lambda p, \quad u = \lambda q.$$

Now, the equation (7.3) can be rewritten as

$$(7.25)\qquad q\mathscr{D} - p\mathscr{E} = 0,$$

which means

$$(7.26)\qquad ps'_x - qr'_x = E,$$

or, due to (7.23),

$$(7.27)\qquad rq'_x - sp'_x = E.$$

Here (7.23) and (7.27) form a system of linear algebraic equations for r, s with the coefficients depending on p, q only. Hence we get the expressions (7.20) by the Cramer's rule.

An concerns the equation (7.7), it can be written, via (7.24), in the form

$$\lambda(pq'_x - qp'_x) = \mathscr{V} = \mathscr{D}/t = \mathscr{D}/\lambda p. \tag{7.28}$$

Hence the formula (7.21) follows.

Let us notice that the functions p, q only have to satisfy a differential inequality $pD(p'_w - s'_x) \geqslant 0$; thus p, q can be still considered as arbitrary functions of 2 variables.

The last assertion about the local isometry classes can be proved exactly in the same way as we did for the hyperbolic case. □

Now, we shall recover a (nongeneric) family of parabolic metrics from [KTV 1].

Theorem 7.7. *Let (M, g) be 3-dimensional foliated SSS such that the scalar curvature $Sc(g)$ is constant along each principal geodesic. If (M, g) is not locally a direct product, then it is parabolic and the metric g is locally determined by the orthonormal coframe*

$$\omega^1 = f(w, x)\,\mathrm{d}w, \quad \omega^2 = \mathrm{d}x - y\,\mathrm{d}w, \quad \omega^3 = \mathrm{d}y + x\,\mathrm{d}w. \tag{7.29}$$

The asymptotic leaves are totally geodesic and Euclidean. The local isometry classes depend on 1 arbitrary function of 2 variables modulo some constants.

Proof. Because the scalar curvature is given by the formula

$$Sc(g) = 2\sigma(w, x)/(Ky^2 + Ly + M),$$

our condition means that $K = L = 0$, and hence the discriminant $\Delta = L^2 - 4KM$ is zero. Thus (M, g) is either parabolic, or planar. We can assume that A, C, f are given by formulas (7.1), (7.2). Here $h = 1$ if g is parabolic, and $h = H = 0$ if g is planar (cf. Corollary 6.9). Because $K = L = 0$, we get $Af = M(w, x)$, and hence $p = t = 0$.

Consider first the parabolic case. Here $C = ry + s$ and $rq = -1$. From (7.3) we get $r'_x = 0$ and hence $r = r(w), q = q(w)$. From (7.7) we get $\mathscr{V} = 0$, i.e., $\mathscr{E} = 0$, which means

$$q'_w - s'_x = 0. \tag{7.30}$$

Introducing a new variable $\bar{w} = -\int r(w)\,\mathrm{d}w$, we obtain our metric in the form

$$\omega^1 = \bar{u}\,\mathrm{d}\bar{w}, \quad \omega^2 = q\,\mathrm{d}x - (y + \bar{s})\,\mathrm{d}\bar{w}, \quad \omega^3 = \mathrm{d}y + \overline{H}\,\mathrm{d}\bar{w}, \tag{7.31}$$

where

$$\bar{s} = s/r(w), \quad \overline{H} = -x/r(w). \tag{7.32}$$

Hence we get $\overline{H}'_x = q$, and introducing the new variable $\bar{x} = \overline{H}$ we can write

$$\omega^2 = \mathrm{d}\bar{x} - (y + \bar{s} + H'_{\bar{w}})\,\mathrm{d}\bar{w}, \quad \omega^3 = \mathrm{d}y + \bar{x}\,\mathrm{d}\bar{w}. \tag{7.33}$$

Now, $(\bar{s} + \overline{H}'_{\bar{w}})'_x = \bar{s}'_x + q'_{\bar{w}} = (s'_x - q'_w)/r(w) = 0$ according to (7.30). We can write, denoting $\bar{w}$ once again as w,

$$\omega^1 = \bar{u}\,\mathrm{d}w, \quad \omega^2 = \mathrm{d}\bar{x} - (y + \varphi(w))\,\mathrm{d}w, \quad \omega^3 = \mathrm{d}y + \bar{x}\,\mathrm{d}w. \tag{7.34}$$

Finally, consider the new variables

$$X = \bar{x} + A(w), \quad Y = y + B(w), \tag{7.35}$$

and try to determine the functions $A(w)$, $B(w)$ so that

$$\omega^2 = dX - Y\,\mathrm{d}w, \quad \omega^3 = dY + X\,\mathrm{d}w. \tag{7.36}$$

We obtain a system of ordinary differential equations

$$A'(w) = B(w) - \varphi(w), \quad B'(w) = -A(w), \tag{7.37}$$

which is easy to solve. Then we get the expression (7.29) in the new variables.

Suppose now that (M, g) is planar, i.e., $h = H = 0$ and $rq = 0$. Because $q \neq 0$, we get $r = 0$ and $C = C(w, x)$. We see that all functions A, C, f depend on w, x only, and $\omega^3 = \mathrm{d}y$; thus we obtain a product metric.

The rest of Theorem 7.7 has been proved in [KTV 1]. □

Remark 7.8. It is easy to show that the spaces described in Theorem 7.7 are exactly *the parabolic spaces in the sense of Szabó* (see the previous Remark 6.12).

Remark 7.9. Let us recall (see [KTV 1,2,3]) that a Riemannian manifold (M, g) is said to be *curvature homogeneous* if, for any two points $p, q \in M$, there is a linear isometry $\varphi\colon T_pM \to T_qM$ such that $\varphi^*(R_q) = R_p$ holds for the corresponding curvature tensors. In [KTV 2] the authors have proved that, in dimension n, all irreducible and locally nonhomogeneous curvature homogeneous spaces (M, g) which have the same curvature tensor as a fixed symmetric space must be foliated SSS. (The proof is based on the paper [Sz 1]). The symmetric model space is then either of the form $S^2(\lambda^2) \times \mathbb{R}^{n-2}$, or of the form $H^2(-\lambda^2) \times \mathbb{R}^{n-2}$ (cf. Section 1).

In dimension 3 we can now describe all such spaces explicitly. In fact, if (M, g) is curvature homogenous (and semi-symmetric), then the scalar curvature $Sc(g)$ is constant on the whole space, and if (M, g) is not locally a direct product, then it must belong to the family described in Theorem 7.7. Then the formula (4.3) implies that the function $f(w, x)$ must be either of the form $f = a(w)e^{\lambda x} + b(w)e^{-\lambda x}$, or of the form $f = a(w)\cos \lambda x + b(w)\sin \lambda x$ (see [KTV 1] for the details).

We are now left with the planar case.

Theorem 7.10. *All locally irreducible planar metrics are locally determined by an orthonormal coframe*

$$\omega^1 = f(w, x)y\,\mathrm{d}w, \quad \omega^2 = y\,\mathrm{d}x, \quad \omega^3 = \mathrm{d}y. \tag{7.38}$$

The local isometry classes are parametrized by the function $f(w, x)$ modulo 2 arbitrary functions of 1 variable.

Proof. According to Corollary 6.9 we have $f = \lambda(w, x)A$, $\quad C = \mu(w, x)A$ and $H = 0$. We can write hence

$$t = \lambda p, \quad u = \lambda q, \tag{7.39}$$

$$r = \mu p, \quad s = \mu q. \tag{7.40}$$

If we substitute (7.39) and $h = 0$ into the equation (7.7), we get

$$pq'_x - qp'_x = 0. \tag{7.41}$$

The equation (7.3) can be written again in the form (7.25). Substituting here from (7.4) and using (7.40), (7.41), we get

$$pq'_w - qp'_w = 0. \tag{7.42}$$

Hence, if $p \neq 0$, (7.41) and (7.42) imply $q/p =$ constant. We can express the orthonormal coframe (2.1) in the form

$$\begin{cases} \omega^1 = \lambda p(y + c)\,\mathrm{d}w, \\ \omega^2 = p(y + c)\,\mathrm{d}x + \mu p(y + c)\,\mathrm{d}w, \\ \omega^3 = \mathrm{d}y. \end{cases} \tag{7.43}$$

Substituting the new variable $\bar{y} = y + c$, we eliminate the constant c. Let us introduce the new variable $\bar{x} = \bar{x}(w, x)$ as a potential function of the Pfaffian equation $\mathrm{d}x + \mu\,\mathrm{d}w = 0$. Then we can write (7.43) in the form

$$\omega^1 = \bar{t}\bar{y}\,\mathrm{d}w, \quad \omega^2 = \bar{p}\bar{y}\,\mathrm{d}\bar{x}, \quad \omega^3 = \mathrm{d}\bar{y}. \tag{7.44}$$

Now, solving a 1st order linear PDE, we can find a function $\varphi(w, \bar{x})$ such that $(\cos\varphi)t\,\mathrm{d}w + (\sin\varphi)\bar{p}\,\mathrm{d}\bar{x}$ is (locally) a total differential, say dX. Using a new orthonormal coframe $(\bar{\omega}^1, \bar{\omega}^2, \omega^3)$ where $\bar{\omega}^1 = \sin\varphi\cdot\omega^1 - \cos\varphi\cdot\omega^2, \bar{\omega}^2 = \cos\varphi\cdot\omega^1 + \sin\varphi\cdot\omega^2$, and denoting by W a potental function of $\bar{\omega}^1 = 0$, we obtain (7.38) up to a notation.

Suppose now $p = 0$. Then by the analogous calculation we obtain the coframe

$$\omega^1 = f(w, x)\,\mathrm{d}w, \quad \omega^2 = \mathrm{d}x, \quad \omega^3 = \mathrm{d}y, \tag{7.45}$$

which gives a direct product metric. It is obvious that to classify the isometry classes of the metrics (7.45) is (locally) the same as to classify the surfaces in E^3 up to an isometry. This problem was solved (in the analytic case) by E. Cartan: All surfaces in E^3 which are (locally) isometric to a fixed generic surface $M_2 \subset E^3$ depend on 2 arbitrary functions of 1 variable (see [Ca2], Part II, Problem V).

Now we see that the problem to characterize the local isometry classes of the metrics (7.38) is equivalent to the same problem for the metrics (7.45). This concludes the proof of Theorem 7.10. □

Remark 7.11. The direct product metrics (7.45) are semi-symmetric for a trivial reason. The metrics (7.38) are obviously warped products. Introducing a new variable by $y = \mathrm{e}^v$ we see that they are conformaly equivalent to the product metrics.

Finally, we have the following result about the completeness:

Theorem 7.12. *A hyperbolic SSS is never complete. A parabolic SSS can be complete only if it belongs to the family* (7.29), *and a planar space can be complete only if it is a direct product.*

(This is an a full accordance with the more general results by Z. Szabó, [Sz 2], if one takes into cosideration Remark 6.12).

P r o o f. If (M, g) is complete, then all principal geodesics must be defined for $y \in (-\infty, +\infty)$. This means that we have the formula $Sc(g) = 2\sigma(w, x)/(Ky^2 + Ly + M)$ in an infinite 3-dimensional strip. If $Ky^2 + Ly + M = 0$ has a real root for some $(w, x) = (w_0, x_0)$, then the corresponding principal geodesic meets a singularity. □

The elliptic case is much more difficult to deal with because the coefficients f, A, C in (2.1) cannot be linearized and they are always *algebraic* functions of y. We are not able to solve the classification problem explicitly, but we can still prove that the local isometric classes in the generic case depend essentially on 3 arbitrary functions of 2 variables. Also, we give examples of explicit families depending on 1 arbitrary function of 2 variables.

We see first that the functions a_0, φ_0 and h are always nonzero in the elliptic case (cf. (6.1) and Proposition 6.10). From (4.13) and (4.18) we see that $a_1 a_3 > 0$, $\varphi_1\varphi_3 > 0$, and from Proposition 6.7 we see that $KM > 0$. Thus A^2, $f^2 + C^2$ and fA are proper quadratic polynomials w.r.t. to y (with the imaginary roots).

Now, we have

Proposition 8.1. *Every elliptic metric g can be expressed locally, using the convenient coordinates and the convenient coframe, in the form* (2.1), *where either*

$$\text{(Case I)} \qquad L = 0, \quad a_2 \neq 0, \quad b_2 = 0,$$

or

$$\text{(Case II)} \qquad L = 0, \quad a_2 = 0, \quad b_1 = b_3 = 0.$$

Proof. First, if $L \neq 0$, we substitute the new variable $\bar{y} = y + L/2K$ in (2.1). Then we can introduce new functions $\bar{w}$ and $\overline{H}$ such that

$$\mathrm{d}\bar{y} + \overline{H}\,\mathrm{d}\bar{w} = \mathrm{d}y + H\,\mathrm{d}w. \tag{8.1}$$

In fact, is suffices to fix $\bar{w}$ as a potential function of the equation

$$(H - (L/2K)'_w)\,\mathrm{d}w - (L/2K)'_x\,\mathrm{d}x = 0. \tag{8.2}$$

Now, it is easy to find an orthonormal coframe $(\bar{\omega}^1, \bar{\omega}^2, \omega^3)$ such that, using a new local coordinate system $(\bar{w}, \bar{x}, \bar{y})$,

$$\bar{\omega}^1 = \bar{f}\,\mathrm{d}\bar{w}, \quad \bar{\omega}^2 = \bar{A}\,\mathrm{d}\bar{x} + \bar{C}\,\mathrm{d}\bar{w}, \quad \omega^3 = \mathrm{d}\bar{y} + \overline{H}\,\mathrm{d}\bar{w}. \tag{8.3}$$

Due to (5.15), $(Ky^2 + Ly + M)/K$ is a Riemannian invariant, and hence

$$\frac{\bar{K}\bar{y}^2 + \bar{L}\bar{y} + \overline{M}}{\bar{K}} = \frac{Ky^2 + Ly + M}{K} = \frac{K\bar{y}^2 + (M - L^2/4K)}{K}, \tag{8.4}$$

which implies $\bar{L} = 0$. □

Now, we restore the original notation as in (2.1) and continue as follows:

a) Suppose first $a_2 \neq 0$. Let us fix a potential function $\bar{x} = \bar{x}(w, x)$ of the equation $a_2\,\mathrm{d}x + b_2\,\mathrm{d}w = 0$; then there is another function $P(w, x)$ such that

$$P(w, x)\,\mathrm{d}\bar{x} = a_2\,\mathrm{d}x + b_2\,\mathrm{d}w. \tag{8.5}$$

We obtain $\omega^2 = \bar{A}\,\mathrm{d}\bar{x} + \bar{C}\,\mathrm{d}w$, where $\bar{A} = (P/a_2)A$, $\bar{C} = C - (b_2/a_2)A$, and hence

$$\bar{A}\omega^2 = (\bar{a}_1 y^2 + \bar{a}_2 y + \bar{a}_3)\,\mathrm{d}\bar{x} + (\bar{b}_1 y^2 + \bar{b}_3)\,\mathrm{d}w, \tag{8.6}$$

as required.

b) Suppose now $a_2 = 0$ in a neighborhood, then $(4.23)_2$ for $L = a_2 = 0$ means $b_1 = \lambda a_1$, $b_3 = \lambda a_3$ and hence

$$A\omega^2 = (a_1 y^2 + a_3)(\,\mathrm{d}x + \lambda\,\mathrm{d}w) + b_2 y\,\mathrm{d}w. \tag{8.7}$$

Introducing a new variable $\bar{x}$ as a potential function of the equation $\mathrm{d}x + \lambda\,\mathrm{d}w = 0$, we conclude the proof.

We shall now study the "fine structure" of the differential equations

$$(A1)\ (A\alpha)'_y + \beta'_x = 0, \quad (A2)\ R'_y - \beta'_w = 0\ (\beta \neq 0),$$

i.e., we shall rewrite (A1) and (A2) as a system of PDE for the functions of 2 variables only.

First, using $(3.3)_1$ we express αA in the form

$$\alpha A = \frac{1}{2}\psi[(A^2)'_w - 2(AC)'_x + (AC)A^{-2}(A^2)'_x - H(A^2)'_y]. \tag{8.8}$$

Then we substitute for $\psi = 1/fA, A^2, AC$ from (4.2), (4.12) and (4.19). Hence we obtain αA as a rational function w.r. to y with the coefficients depending on a_i, b_i, K, L, M, H and their first partial derivatives w.r. to w, x. Thus the coefficients are functions of w, x only.

Further, using (3.4) and$(3.3)_2$, we express R in the form

$$R = \frac{1}{2}\psi[(f^2 + C^2)'_x + H(H'_x + (AC)'_y) - (AC)A^{-2}(A^2)'_w], \tag{8.9}$$

and we substitute here from (4.2), (4.12), (4.19) and (4.17), putting also $H'_x = h$. We obtain another rational function w.r. to y; the coefficients depend, in addition, on the functions $(\varphi_i)'_x$, $i = 1, 2, 3$. Finally, we put

$$\beta = a_0(a_1 y^2 + a_2 y + a_3)^{-1} \tag{8.10}$$

as follows from (4.11), (4.12).

Substituting (8.8)–(8.10) into (A1) and (A2), and using the common denominator $(Ky^2 + Ly + M)^2(a_1y^2 + a_2y + a_3)^2$ in each case, we see that the corresponding numerators are polynomials of degree 7 w.r. to y. Now, these polynomials have to be zero and hence all the coefficients must vanish.

For the equation (A1), each coefficient equation involves a linear combination of $(a_i)'_w$, $(a_i)'_x$, $(b_i)'_x$ $(i = 1,2,3)$, $(a_0)'_x$ and H. For the equation (A2), each coefficient equation involves a linear combination of $(a_i)'_w$, $(\varphi_i)'_x$ $(i = 1,2,3)$, $(a_0)'_w$ and H. Taking suitable linear combinations of the coefficient equations we see that the number of these equations is reduced to five, in each case.

We can make some additional simplifications using the equations (4.23) and (4.33), (4.34). After a very long but routine calculation we obtain (in the generic case $a_1a_3 \neq 0$, which includes the elliptic case) the final form of the equation (A1) as the PDE system

$$\sum_{i=1}^{3} P^i_\alpha V_i + \sum_{i=1}^{3} Q^i_\alpha a'_{ix} + R_\alpha a'_{0x} = 0 \quad (\alpha = 1,\dots,5), \tag{8.11}$$

where

$$V_1 = a'_{1w} - 2b'_{1x}, \quad V_2 = a'_{2w} - 2b'_{2x} - 2a_1H, \; V_3 = a'_{3w} - 2b'_{3x} - a_2H, \tag{8.12}$$

$$(P^i_\alpha) = \begin{bmatrix} a_1L & -a_1K & 0 \\ a_2L + 2a_1M & -a_2K & -2a_1K \\ 2a_2M + a_3L & a_1M - a_3K & -(2a_2K + a_1L) \\ 2a_3M & a_2M & -(2a_3K + a_2L) \\ 0 & a_3M & -a_3L \end{bmatrix}, \tag{8.13}$$

$$(Q^i_\alpha, R_\alpha) = \begin{bmatrix} b_1L - hk & -b_1K & 0 & 2K^2 \\ (b_2-h)L + 2b_1M & -(b_2+h)K & -2b_1K & 4KL \\ (2b_2-h)M + b_3L & -b_3K - hL + b_1M & -[(2b_2+h)K + b_1L] & 4KM + 2L^2 \\ 2b_3M & (b_2-h)M & -[(2b_3K + (b_2+h)L] & 4LM \\ 0 & b_3M & -(b_3L + hM) & 2M^2 \end{bmatrix}, \tag{8.14}$$

Analogously, the equation (A2) gives the PDE system

$$\sum_{i=1}^{3} P^i_\alpha W_i - \sum_{i=1}^{3} Q^i_\alpha a'_{iw} - R_\alpha a'_{0w} = 0 \quad (\alpha = 1,\dots,5) \tag{8.15}$$

where

$$W_1 = \varphi'_{1x}, \quad W_2 = \varphi'_{2x} + 2b_1 H, \quad W_3 = \varphi'_{3x} + (b_2 + h)H. \tag{8.16}$$

Now, if $a_0 \neq 0$ (which is our case), (4.13) implies

$$a'_{0x} = \frac{1}{2a_0}\Big(a_3 a'_{1x} - \frac{1}{2}a_2 a'_{2x} + a_1 a'_{3x}\Big), \tag{8.17}$$

and a similar expression for a'_{0w}. If we substitute into (8.11) and (8.15), respectively, each of the new equations will be a linear combination of six terms only, with modified coefficients. The following Proposition can be checked by a direct (but rather long) computation:

Proposition 8.2. *The rank of the matrix*

$$\Big[P^1_\alpha,\ P^2_\alpha,\ P^3_\alpha,\ Q^1_\alpha + \frac{a_3}{2a_0}R_\alpha,\ Q^2_\alpha - \frac{a_2}{4a_0}R_\alpha,\ Q^3_\alpha + \frac{a_1}{2a_0}R_\alpha\Big]$$

is not greater than 2.

Hence we have

Corolary 8.3. *If $a_0 \neq 0$ and the partial derivatives of a_0 are eliminated in* (8.11) *and* (8.15), *then each of the new PDE system contains at most two linearly independent equations.*

Thus, in the most general case, the equations (A1), (A2) are reduced assentially to 4 PDE in two variables. We shall see later that we can make an additional reduction to only two equations (one of the form (8.11) and one of the form (8.15)). This will be in the full accord with the nonelliptic case, in which we also had two PDE, namely (7.3) and (7.7).

THE ELLIPTIC CASE I (THE GENERIC CASE)

Suppose $a_2 \neq 0$, $L = b_2 = 0$, which is the generic case of Proposition 8.1. We have first

Proposition 8.4. *The following algebraic formulas must hold due to algebraic equations of Theorem* 4.8:

$$\varphi_1 = \mathcal{V} a_1, \quad \varphi_2 = -\mathcal{V} a_2, \quad \varphi_3 = \mathcal{V} a_3, \tag{8.18}$$

where

$$\mathscr{V} = (h^2 + (\varphi_5)^2)/4a_1a_3, \quad \mathscr{V} = -\varphi_0/a_0. \tag{8.19}$$

Further

$$K = \frac{2a_0h + a_2\varphi_5}{4a_3}, \quad L = 0, \quad M = \frac{2a_0h - a_2\varphi_5}{4a_1}, \tag{8.20}$$

$$b_1 = \frac{-a_2h + 2a_0\varphi_5}{4a_3}, \quad b_2 = 0, \quad b_3 = \frac{a_2h + 2a_0\varphi_5}{4a_1}. \tag{8.21}$$

On the other hand, if a_1, a_2, a_3, h, φ_5 are arbitrary functions and if the other basic functions are defined by (8.18)-(8.21), then all algebraic equations of Theorem 4.8 hold.

Proof. We shall show only the necessity of (8.18)–(8.21); the sufficiency will be proved by the direct check.

The second and the fourth equation (4.31) imply $\varphi_1a_2+\varphi_2a_1 = 0, \varphi_2a_3+\varphi_3a_2 = 0$, i.e. (8.18). The first and the last equations (4.22) and (4.23) form a system of linear equations from which b_1, b_3, K and M can be expressed by means of a_1, a_2, a_3, a_0, h and φ_5. Hence we obtain (8.20) and (8.21). Substituting for b_1 and K into the first equation (4.31), we obtain formula $(8.19)_1$. Finally, from (8.18), (4.13) and (4.18) we get $(\varphi_0)^2 = \mathscr{V}^2(a_0)^2$. Here $\mathscr{V} > 0$ due to $(8.19)_1$ and $4a_0\varphi_0 = \Delta - (\varphi_5)^2 < 0$ because $\Delta < 0$. Hence we get formula $(8.19)_2$. □

Now, consider the PDE systems (8.11) and (8.15) in which the derivatives a'_{0x}, a'_{0w} are eliminated by means of (8.17) and its analogue. According to proposition 8.2, it suffices to use *any* two lines of the new coefficient matrix. The first two lines lead to the (equivalent) coefficient matrix

$$\begin{pmatrix} 0 & 2a_0 & 0 & 2M & \varphi_5 & -2K \\ 2M & -\varphi_5 & -2K & 0 & 2\varphi_0 & 0 \end{pmatrix}. \tag{8.22}$$

Hence we get the corresponding PDE system in the form

$$\begin{cases} 2a_0V_2 + 2Ma'_{1x} + \varphi_5a'_{2x} - 2Ka'_{3x} = 0, \\ 2a_0W_2 - 2Ma'_{1w} - \varphi_5a'_{2w} + 2Ka'_{3w} = 0, \\ 2MV_1 - \varphi_5V_2 - 2KV_3 + 2\varphi_0a'_{2x} = 0, \\ 2MW_1 - \varphi_5W_2 - 2KW_3 - 2\varphi_0a'_{2w} = 0, \end{cases} \tag{8.23}$$

where V_i, W_i are defined by (8.12) and (8.16).

Substituting for φ_i and b_i from (8.18) and (8.21) we see, after a lengthy but routine calculation, that the last two equations of (8.23) are consequences of the first two ones.

Due to (8.20) we see that

$$2Ma'_{1x} + \varphi_5 a'_{2x} - 2Ka'_{3x} = a_0 h[\ln(a_1/a_3)]'_x + \frac{1}{2}\varphi_5 a_2[\ln({a_2}^2/a_1a_3)]'_x. \tag{8.24}$$

and a similar formula holds for $2Ma'_{1w} + \varphi_5 a'_{2w} - 2Ka'_{3w}$. Introducing the new functions $U(w,x)$, $V(w,x)$, $V < \ln 4$, by the formulas

$$a_1/a_3 = \mathrm{e}^U, \quad {a_2}^2/a_1a_3 = \mathrm{e}^V, \tag{8.25}$$

we can rewrite the first two equations of (8.23) in the form

$$\begin{cases} 2a_0(a'_{2w} - 2a_1H) + a_0hU'_x + \frac{1}{2}\varphi_5 a_2 V'_x = 0, \\ 2a_0(\varphi'_{2x} + 2b_1H) - a_0hU'_w - \frac{1}{2}\varphi_5 a_2 V'_w = 0. \end{cases} \tag{8.26}$$

From (8.25) we get

$$a_0 = a_2\Big(\mathrm{e}^{-V} - \frac{1}{4}\Big)^{1/2} \tag{8.27}$$

and hence the equations (8.26) take on the form

$$a'_{2w} - 2a_1H + hU'_x = -\frac{1}{4}\varphi_5 V'_x\Big(\mathrm{e}^{-V} - \frac{1}{4}\Big)^{-1/2}, \tag{8.28}$$

$$\varphi'_{2x} + 2b_1H - hU'_w = \frac{1}{4}\varphi_5 V'_w\Big(\mathrm{e}^{-V} - \frac{1}{4}\Big)^{-1/2}. \tag{8.29}$$

Moreover, we can substitute

$$a_1 = a_2\mathrm{e}^{(1/2)(U-V)} \tag{8.30}$$

into (8.28), and

$$\varphi_2 = -\frac{h^2 + \varphi_5^2}{4a_2}\mathrm{e}^V, \quad 2b_1H = \mathrm{e}^{\frac{1}{2}(U+V)}\Big[\Big(\mathrm{e}^{-V} - \frac{1}{4}\Big)^{\frac{1}{2}}\varphi_5 - \frac{1}{2}h\Big]H \tag{8.31}$$

into (8.29).

Let now H, U, V be arbitrary analytic functions. Expressing φ_5 from (8.28) in the form $\varphi_5 = f_1a'_{2w} + f_2a_2 + f_3$, where f_i are known functions, and substituting this in (8.29) (which has been transformed by (8.31)), we obtain a PDE of the form

$$a''_{2xw} = F(a'_{2w}, a'_{2x}, a_2, w, x), \tag{8.32}$$

where F is a fixed analytic function of 5 variables. Then general solution of (8.32) depends on two arbitrary (analytic) functions of 1 variable. Thus *the elliptic family* I *depends on* 3 *arbitrary functions of* 2 *variables*, namely on H, U and V.

The equation (8.32) *cannot* be solved explicitly, in general. Yet, we shall give an explicit example in this place. Let us choose $H = H(w, x)$ as arbitrary function, and U, V as functions of w satisfying the equation $V'(w) = \mathrm{e}^{\frac{1}{2}(U+V)}(\mathrm{e}^{-V} - \frac{1}{4})$. Then (8.28), (8.29) are reduced to the form

$$a'_{2w} - g_1 a_2 = 0, \quad \varphi'_{2x} + g_2 = 0, \tag{8.33}$$

where g_1, g_2 are known functions of w, x. Then we obtain, using also (8.18),

$$a_2 = \exp\Big(\int g_1 \,\mathrm{d}w\Big), \quad \mathscr{V} = \Big(\int g_2 \,\mathrm{d}x\Big)\Big/ a_2. \tag{8.34}$$

Hence φ_5 can be determined using (8.19), and all basic functions are determined.

Remark. If $\varphi_5 = 0$ holds in (8.28), (8.29), we can choose H, U as arbitrary analytic functions and then express $\mathrm{e}^{(1/2)V}$ from (8.28) as a function of a'_{2w}, a_2, w, x. Substituting in (8.29), we obtain again a PDE for a_2 in the form (8.32).

Now, we shall prove the main theorem of this Section:

Theorem 8.5. *The local isometry classes of generic elliptic metrics are parametrized by the arbitrary functions H, U, V of 2 variables modulo two arbitrary functions of one variable.*

P r o o f. It suffices to prove that each local isometry class depend only on 2 arbitrary functions of 1 variable. We get first the formula

$$\begin{aligned}(\omega^1)^2 + (\omega^2)^2 = \mathscr{V}(a_1 y^2 - a_2 y + a_3)\,\mathrm{d}w^2 + (a_1 y^2 + a_2 y + a_3)\,\mathrm{d}x^2 \\ + 2(b_1 y^2 + b_3)\,\mathrm{d}x\,\mathrm{d}w.\end{aligned} \tag{8.35}$$

Let $(\overline{M}, \bar{g})$ be another elliptic space which is isometric to (M, g). Then the corresponding isometry is given by formulas of the form (5.18). Now, because $1/K\psi$ is a Riemannian invariant (see (5.15)), we get

$$\frac{\overline{K}\bar{y}^2 + \overline{M}}{\overline{K}} = \frac{Ky^2 + M}{K}. \tag{8.36}$$

Substituting from (5.18) in (8.36) we get $\Phi(w, x) = 0$ and hence $\bar{y} = \varepsilon y$. Then we obtain $\bar{\omega}^3 = \varepsilon\omega^3$ and $\overline{H}\,\mathrm{d}\bar{w} = H\,\mathrm{d}w$. We get finally

$$\bar{y} = \varepsilon y, \quad \bar{w} = \varphi(w), \quad \overline{H}\varphi'(w) = \varepsilon H. \tag{8.37}$$

Next, analogously to (8.35), we obtain

$$(8.38)\qquad \begin{cases} (\bar\omega^1)^2 + (\bar\omega^2)^2 = \overline{V}(\bar a_1\bar y^2 - \bar a_2\bar y + \bar a_3)\,\mathrm{d}\bar w^2 + (\bar a_1\bar y^2 + \bar a_2\bar y + \bar a_3)\,\mathrm{d}\bar x^2 \\ \qquad + 2(\bar b_1\bar y^2 + \bar b_3)\,\mathrm{d}\bar x\,\mathrm{d}\bar w \\ \quad = \overline{V}(\varphi'(w))^2(\bar a_1\bar y^2 - \bar a_2\bar y + \bar a_3)\,\mathrm{d}w^2 \\ \qquad + (\bar a_1\bar y^2 + \bar a_2\bar y + \bar a_3)((\bar x'_x)^2\,\mathrm{d}x^2 \\ \qquad + 2\bar x'_x\bar x'_w\,\mathrm{d}x\,\mathrm{d}w + (\bar x'_w)^2\,\mathrm{d}w^2) \\ \qquad + 2(\bar b_1\bar y^2 + \bar b_3)\varphi'(w)(\bar x'_x\,\mathrm{d}x\,\mathrm{d}w + \bar x'_w\,\mathrm{d}w^2). \end{cases}$$

Now, we can compare the coefficients of the expressions (8.35) and (8.38). Comparing the coefficients of $y\,\mathrm{d}x\,\mathrm{d}w$, we obtain $\bar a_2\bar x'_x\bar x'_w = 0$. Because we assume $\bar a_2 \neq 0$, and we have $\bar x'_x = \frac{D(\bar w,\bar x)}{D(w,x)}(\varphi'(w))^{-1} \neq 0$, this implies $\bar x'_w = 0$. Hence

$$(8.39)\qquad \bar x = g(x),\ \mathrm{d}\bar x = g'(x)\,\mathrm{d}x.$$

Comparing the coefficients of $\mathrm{d}x^2$ we obtain

$$(8.40)\qquad A^2 = \overline{A}^2(g'(x))^2, \quad \text{i.e.,} \quad A = \varepsilon'\overline{A}g'(x),$$

and comparing the coefficients of $\mathrm{d}x\,\mathrm{d}w$ we obtain

$$(8.41)\qquad AC = \overline{AC}\varphi'(w)g'(x), \quad \text{i.e.,} \quad C = \varepsilon'\overline{C}\varphi'(w).$$

Because $\overline{H}\varphi'(w) = \varepsilon H$, we get $\varepsilon H'_x = \overline{H}_x\varphi'(w) = \overline{H}'_{\bar x}g'(x)\varphi'(w)$, i.e.,

$$(8.42)\qquad \varepsilon h = \bar h g'(x)\varphi'(w).$$

Finally according to (5.14) we have

$$(8.43)\qquad \overline{A}\bar f/\bar h = \bar\varepsilon Af/h, \quad \text{i.e.,} \quad Af = \varepsilon\bar\varepsilon\overline{A}\bar f g'(x)\varphi'(w).$$

Using (8.40) we infer hence

$$(8.44)\qquad f = \varepsilon_0\bar f\varphi'(w), \quad \varepsilon_0 = \varepsilon\varepsilon'\bar\varepsilon.$$

We conclude that the functions A, C, f, H can be obtained from the functions $\overline{A}$, $\overline{C}$, $\bar f$, $\overline{H}$ (expressed in the same variables w, x, y!) by a transformation involving only two arbitrary functions of 1 variable and some signs. Hence Theorem 8.5 follows. □

We shall study also this nongeneric case because it gives some more examples. Here we have, in an open domain,

$$L = a_2 = 0, \quad b_1 = b_3 = 0. \tag{8.45}$$

First, we need information about the algebraic structure:

Proposition 8.6. *In the elliptic case* II, *the following algebraic formulas must hold:*

$$\varphi_2 = \varphi_5 = 0, \tag{8.46}$$

$$\varphi_0 = -\varepsilon\sqrt{\varphi_1\varphi_3}, \quad a_0 = \varepsilon\sqrt{a_1 a_3}, \tag{8.47}$$

$$K = \bar{\varepsilon}\sqrt{\varphi_1 a_1}, \quad M = \bar{\varepsilon}\sqrt{\varphi_3 a_3}, \tag{8.48}$$

$$b_2 = \varepsilon\bar{\varepsilon}(\sqrt{\varphi_3 a_1} - \sqrt{\varphi_1 a_3}, \quad h = \varepsilon\bar{\varepsilon}(\sqrt{\varphi_3 a_1} + \sqrt{\varphi_1 a_3}) \tag{8.49}$$

where ε, $\bar{\varepsilon} = \pm 1$ *are some signs.*

On the other hand, if a_1, a_3, φ_1, φ_3 *are arbitrary positive functions, and other basic functions are defined by* (8.45)–(8.49), *then all algebraic equations of Theorem* 4.8 *hold.*

Proof. Using (8.45) and having still in mind that $a_1 a_3 = {a_0}^2 > 0$, we get $\varphi_5 = 0$ from (4.36) and $\varphi_2 = 0$ from $(4.31)_2$. Also we have $\varphi_1\varphi_3 = {\varphi_0}^2 > 0$ and $\varphi_0 a_0 < 0$; hence (8.47) follows. Further, (8.48) is a consequence of $(4.31)_1$, $(4.31)_5$ and of the inequality $KM > 0$. Finally, (8.49) follows from $(4.28)_2$ and (4.33).

The second part of the proof is a direct check. □

Now we pass over to the PDE systems (8.11) and (8.15). Considering the first two equations of each system and substituting here all formulas (8.45)–(8.49), we see that the given four equations are reduced to the following two equations:

$$\begin{cases} \sqrt{a_3\varphi_3}\,a'_{1w} - \sqrt{a_1\varphi_1}\,a'_{3w} = 0, \\ \sqrt{a_3\varphi_3}\,\varphi'_{1x} - \sqrt{a_1\varphi_1}\,\varphi'_{3x} - 2a_1\sqrt{\varphi_1\varphi_3}H = 0. \end{cases} \tag{8.50}$$

Let us put

$$p = \sqrt{a_1}, \quad q = \sqrt{a_3}, \quad r = \sqrt{\varphi_1}, \quad s = \sqrt{\varphi_3}. \tag{8.51}$$

Then we can rewrite (8.50) in the form

$$sp'_w - rq'_w = 0, \quad qr'_x - ps'_x - p^2 H = 0. \tag{8.52}$$

Moreover, we have the algebraic relations

$$s = \frac{h + b_2}{2p}, \quad r = \frac{h - b_2}{2q} \tag{8.53}$$

as consequence of (8.49). Then (8.52) takes on the form

$$h(qp'_w - pq'_w) + b_2(qp'_w + pq'_w) = 0, \quad h \neq 0, \tag{8.54}$$

$$-2p^2q^2b'_{2x} + h(p^3q'_x - q^3p'_x) + b_2(p^3q'_x + q^3p'_x) = 2p^4q^2H. \tag{8.55}$$

If p, q are now fixed arbitrary functions of two variables such that $(p/q)'_w \neq 0$, $(pq)'_w \neq 0$, we can express b_2 through h from (8.54) and then substitute into (8.55). We obtain a PDE of the form

$$BH''_{xx} + DH'_x = EH, \quad B \neq 0, \tag{8.56}$$

where B, D, E are fixed functions.

Hence our metrics depend on two arbitrary (analytic) functions of 2 variables and two additional arbitrary functions of 1 variable. By the same method that was used in the generic case, we can show that the local isometry classes in the elliptic case II are parametrized by two arbitrary functions of 2 variables modulo two arbitrary functions of 1 variable.

In the particular cases, we can get new explicit families of solutions.

a) If $p/q = f(x)$ and $b_2 = 0$, then (8.54) is identically satisfied and (8.55) takes on the form $DH'_x = EH$, which can be solved by an explicit formula.

b) If $p = p(x)$, $q = q(x)$, and $b_2 = b_2(w, x)$ are arbitrary but fixed, we get (8.55) in the form $BH'_x = CH + D$, which can be solved by an explicit formula, again.

We shall conclude with an additional example which does not involve integration. Let us consider the "singular" case of Proposition 8.4, namely the case $a_2 = \varphi_2 = 0$. Then the first two PDE of (8.23) can be written in the form

$$4a_1H - h[\ln(a_1/a_3)]'_x = 0, \quad 4b_1H - h[\ln(a_1/a_3)]'_w = 0. \tag{8.57}$$

Let $H = H(w, x)$ and $U = \ln(a_1/a_3)$ be fixed arbitrary functions. Then we obtain

$$a_1 = hU'_x/4H, \quad b_1 = hU'_w/4H \ (h = H'_x) \tag{8.58}$$

and hence

$$A^2 = \frac{hU'_x}{4H}(y^2 + e^{-U}), \quad AC = \frac{hU'_w}{4H}(y^2 + e^{-U}), \quad Af = \frac{1}{2}he^{\frac{1}{2}U}(y^2 + e^{-U}). \tag{8.59}$$

By the similar computation as in the proof of Theorem 8.5 we can see that the local isometry classes are parametrized by the arbitrary function $U(w, x)$ only. Thus we can put $H = x$, $h = 1$ in (8.59).

Remark 8.7. According to [Sz 2], Theorem 4.5, a foliated SSS of *arbitrary* dimension can be *complete* only if it is (generically) a 3-dimensional elliptic space (i.e., 3-dimensional hyperbolic space in the Szabó's terminology), or a k-dimensional parabolic space in the sense of Z. Szabó. The last spaces can be characterized as foliated SSS with constant scalar curvature along each $(k-2)$-dimensional Euclidean leaf (see [BKV]). These spaces have been constructed explicitly and studied already in [KTV 2], including simple criteria for the completeness. As concerns the complete 3-dimensional elliptic spaces, a geometrical construction was described in [Sz 3], which *may* produce elliptic semi-symmetric hypersurfaces in $\mathbb{R}^4$. Yet, to the author's knowledge, the only known *explicit* example so far was that by H. Takagi, [T]. A new family of examples was found recently by E. Boeckx (to appear in Tsukuba Math. J.).

9. Semi-symmetric spaces with the prescribed scalar curvature

From the formulas (2.2), (4.2) and Remark 2.2 we see that the scalar curvature of a 3-dimensional foliated SSS is locally of the form

$$Sc(g) = \frac{2\sigma(w,x)}{Ky^2 + Ly + M}, \tag{9.1}$$

where K, L, M are functions of two variables w, x only. We shall now state the converse.

Theorem 9.1. *Let $k_i = k_i(w,x)$, $i = 1,2,3$, be analytic functions in a domain $W \subset \mathbb{R}^2(w,x)$ such that $\sum(k_i)^2 > 0$. Let $(w_0, x_0) \in W$ be a fixed point and $y_0 \in \mathbb{R}$ a fixed number such that*

$$k_1(w_0,x_0){y_0}^2 + k_2(w_0,x_0)y_0 + k_3(w_0,x_0) \neq 0. \tag{9.2}$$

Then there is a neighborhood $U \subset W \times \mathbb{R} \subset \mathbb{R}^3(w,x,y)$ of the point (w_0,x_0,y_0) and a semi-symmetric metric g defined on U whose scalar curvature is given by the formula

$$Sc(g) = \frac{1}{k_1y^2 + k_2y + k_3}. \tag{9.3}$$

In particular, if the quadratic polynomial $k_1y^2 + k_2y + k_3$ has two imaginary roots for $(w,x) = (w_0,x_0)$, then there exists a neighborhood $\widetilde{W} \subset W$ of (w_0,x_0) and an elliptic metric g defined in the infinite strip $\widetilde{W} \times R \subset \mathbb{R}^3$ such that its scalar curvature *is given by* (9.3). *The local isometry classes of all such that metrics are parametrized by one arbitrary function* H *of* 2 *variables modulo* 2 *arbitrary functions of* 1 *variable.*

Remark. The last assertion is not always true if the prescribed scalar curvature is not "of elliptic type". For example, if $Sc(g)$ is prescribed as a nonzero constant, then the corresponding (local) isometry classes of semi-symmetric metrics depend on two arbitrary functions of 1 variable modulo some constants (see Remark 7.9 and [KTV 1]).

Proof. We shall only prove the second part of our theorem (the elliptic case). In the nonelliptic situations (and, in particular, if $k_1 k_3 = 0$), the proof can be done case by case, using the same main argument.

Thus we can assume, e.g., $k_1 > 0$, $k_3 > 0$ at (w_0, x_0). Using a substitution $\bar{y} = y + k_2/2k_1$, we can reduce our problem to the case $k_2 = 0$ in some neighborhood $W' \subset W$ of (w_0, x_0) (which corresponds to the assumption $L = 0$). We shall look for the solutions in the elliptic family I; thus we shall assume that the formulas (8.18)–(8.21) hold. Now, we have to satisfy the following conditions:

a) The first two partial differential equations of the system (8.23).

b) The proportion formula $M/K = k_3/k_1$.

c) The partial differential equation

$$[(A\alpha)'_w + R'_x]_{y=0} + K = -K/2k_1. \tag{9.4}$$

Here the equation (9.4) follows from (4.3) and (9.1), (9.3), because $\sigma = K/2k_1$.

We shall write down our system of three PDE a) and c) in a more explicit form:

$$2a_0(a'_{2w} - 2a_1 H) + 2Ma'_{1x} + \varphi_5 a'_{2x} - 2Ka'_{3x} = 0, \tag{9.5}$$

$$2a_0(\varphi'_{2x} + 2b_1 H) - 2Ma'_{1w} - \varphi_5 a'_{2w} + 2Ka'_{3w} = 0, \tag{9.6}$$

$$\begin{aligned} &\frac{\partial}{\partial w}\Big\{\frac{1}{2M}[a'_{3w} - 2b'_{3x} + (b_3/a_3)a'_{3x} - a_2 H]\Big\} \\ &+\frac{\partial}{\partial x}\Big\{\frac{1}{2M}[\varphi'_{3x} + hH - (b_3/a_3)a'_{3w}]\Big\} + K = -K/2k_1. \end{aligned} \tag{9.7}$$

We shall now assume that the neighborhood W' is of the form

$$W' = (w_0 - \delta, w_0 + \delta) \times (x_0 - \varepsilon, x_0 + \varepsilon) \subset W.$$

Thus, in W' we have $k_1 > 0, k_3 > 0, k_2 = 0$.

We fix an arbitrary analytic function $H = H(w, x)$ in W' and put $M = (k_3/k_1)K$. Then we see easily from (8.18)–(8.21) and (4.13) that all basic functions involved in (9.5)–(9.7) can be expressed as *fixed* functions of a_1, a_2, a_3 and w, x. On the other hand, the functions a_i remain independent. Then (9.5) and (9.6) can be written in the form

$$a'_{2w} = g_1(a'_{ix}, a_i, w, x), \tag{9.8}$$

$$-Ma'_{1w} + Ka'_{3w} = g_2(a'_{ix}, a_i, w, x), \tag{9.9}$$

$$M = M(a_i, w, x), \quad K = K(a_i, w, x),$$

and (9.7) takes on the form

$$a''_{3ww} = g_3(a''_{iwx}, a''_{ixx}, a'_{iw}, a'_{ix}, a_i, w, x), \tag{9.10}$$

where g_1, g_2, g_3 are fixed analytic functions.

Differentiating (9.8) and (9.9) with respect to w, we obtain finally a system of 2nd order PDE of the form

$$a''_{jww} = f_j(a''_{iwx}, a''_{ixx}, a'_{iw}, a'_{ix}, a_i, w, x) \quad (j = 1, 2, 3) \tag{9.11}$$

where f_j are fixed analytic functions.

Now, it is always possible to choose analytic functions $\varphi_i(x)$, $\psi_i(x)$ $(i = 1, 2, 3)$ on $(x_0 - \varepsilon, x_0 + \varepsilon)$ such that the functions $f_j(\psi'_i(x), \varphi''_i(x), \psi_i(x), \varphi'_i(x), \varphi_i(x), w_0, x)$ are defined on the whole of $(x_0 - \varepsilon, x_0 + \varepsilon)$, and

$$\varphi_1 > 0, \quad \varphi_3 > 0, \quad \varphi_1\varphi_3 - \frac{1}{4}(\varphi_2)^2 > 0; \tag{9.12}$$

$$\psi_2(x) = g_1(\varphi'_i(x), \varphi_i(x), w_0, x), \tag{9.13}$$

$$-M_0\psi_1(x) + K_0\psi_3(x) = g_2(\varphi'_i(x), \varphi_i(x), w_0, x), \tag{9.14}$$

where M_0, K_0 denote the functions M, K from (9.9) in which a_i is replaced by $\varphi_i(x)$ for $i = 1, 2, 3$ and w is replaced by w_0.

According to the Cauchy-Kowalewski Theorem, there exists a unique solution (a_1, a_2, a_3) of (9.11) in a neighborhood $W'' \subset W'$ of the set $\{w_0\} \times (x_0 - \varepsilon, x_0 + \varepsilon)$ such that

$$a_i(w_0, x) = \varphi_i(x), a'_{iw}(w_0, x) = \psi_i(x), i = 1, 2, 3. \tag{9.15}$$

Due to (9.13), (9.14), the equations (9.8), (9.9) are satisfied for $w = w_0$, and because the derivatives of these equations with respect to w are also satisfied (as a part of (9.11)), the equations (9.8), (9.9) are satisfied identically. Moreover, because (9.12) holds, then $a_0 \neq 0$ and a_1, $a_3 > 0$ holds on the set $\{w_0\} \times (x_0 - \varepsilon, x_0 + \varepsilon)$, and hence on a neighborhood $\widetilde{W} \subset W''$ of this set. Then $a_1 y^2 + a_2 y + a_3 > 0$ holds on the strip $\widetilde{W} \times R$.

Thus a semi-symmetric metric g of the elliptic type is well-defined on $\widetilde{W} \times R$ by the standard formulas (2.1), where A is given by (4.12), C by (4.19), f is equal to $K(y^2 + k_3/k_1)/A$ and H is the function which was fixed in advance.

The rest of the proof now follows from Theorem 8.5. □

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Author's address: MFF UK, Sokolovská 83, 186 00 Praha, Czech Republic.