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QUASI M-COMPACT SPACES

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0. Preliminaries

All spaces are assumed to be Tychonoff. If $f\colon X \to Y$ is a function, then $\bar{f}\colon \beta(X) \to \beta(Y)$ denotes the Stone-Čech extension of f. The Greek letters α, γ, λ and δ will denote infinite cardinal numbers. If α and γ are cardinal numbers, then $[\alpha]^{\gamma} = \{A \subseteq \alpha\colon |A| = \gamma\}$. As usual $\beta(\alpha)$ is identified with the set of ultrafilters on α and the remainder $\alpha^* = \beta(\alpha) \setminus \alpha$ is the set of free ultrafilters on α. If $A \subseteq \alpha$, then $\mathrm{cl}_{\beta(\alpha)}(A) = \{p \in \beta(\alpha)\colon A \in p\}$, which will be denoted by $\hat{A}$ and $A^* = \hat{A} \setminus A$. We say that $f\colon \gamma \to \beta(\alpha)$ is a *strong embedding* for $\gamma \leqslant \alpha$ if there is a partition $\{A_\xi\colon \xi < \gamma\}$ of α such that $f(\xi) \in \hat{A}_\xi$ for each $\xi < \gamma$. The norm of $p \in \alpha^*$ is defined by $\|p\| = \min\{|A|\colon A \in p\}$. The set of uniform ultrafilters on α is $U(\alpha) = \{p \in \alpha^*\colon \|p\| = \alpha\}$. The *Rudin-Keisler* (pre-)order on α^* is defined by $p \leqslant_{RK} q$ if there exists $\sigma\colon \alpha \to \alpha$ such that $\bar{\sigma}(q) = p$, for $p, q \in \alpha^*$. The Rudin-Keisler order induces an equivalence relation on α^* by defining $p \approx_{RK} q$ if $p \leqslant_{RK} q$ and $q \leqslant_{RK} p$, for $p, q \in \alpha^*$. It is not hard to prove that $p \approx_{RK} q$ iff there is a bijection $\sigma\colon \alpha \to \alpha$ such that $\bar{\sigma}(p) = q$. The equivalence class of $p \in \alpha^*$ is called the *type* of p and it is denoted by $T_{RK}(p) = \{q \in \alpha^*\colon q \approx_{RK} p\}$. For $p, q \in \alpha^*$, $p <_{RK} q$ means that $p \leqslant_{RK} q$ and p is not $\approx_{RK}$-equivalent to q. For $\emptyset \neq M \subseteq \alpha^*$, we let $P_{RK}(M) = \{q \in \alpha^*; \exists p \in M(q \leqslant_{RK} p)\}$. Notice that if $p \leqslant_{RK} q$ for $p, q \in \alpha^*$, then $\|p\| \leqslant \|q\|$; hence, if $q \approx_{RK} p \in U(\alpha)$, then $q \in U(\alpha)$. For $p, q \in \alpha^*$, their *tensor product* is defined by

$$p \otimes q = \{A \subseteq \alpha \times \alpha\colon \{\xi < \alpha\colon \{\zeta < \alpha\colon (\xi, \zeta) \in A\} \in q\} \in p\}.$$

Notice that $p \otimes q$ is an ultrafilter on $\alpha \times \alpha$ and can be considered as an ultrafilter on α via a fixed bijection between α and $\alpha \times \alpha$. Observe that $\|p \otimes q\| = \|p\|\|q\|$ for $p, q \in \alpha^*$. It is pointed out in [Ka] that $p <_{RK} p \otimes q$ and $q <_{RK} p \otimes q$ for $p, q \in \alpha^*$. The following result is essential in the application of $\otimes$ (for a proof see [CN, 16.5]): the case $\alpha = \omega$ was also proved by Booth [Bo].

Theorem 0.1. (Blass [Bl$_1$]). *Let $p, q \in \alpha^*$ with $p \in U(\gamma)$. If $e\colon \gamma \to T_{RK}(q)$ is a strong embedding, then $p \otimes q \approx_{RK} \bar{e}(p)$.*

It is not difficult to see that $\otimes$ is not an associative operation. However, Booth [Bo] noticed that $\otimes$ induces a semigroup structure on the set of types of ω^* by defining $T_{RK}(p) \otimes T_{RK}(q) = T_{RK}(p \otimes q)$ for $p, q \in \omega^*$. Hence, if $p \in \omega^*$, then p^n stands for any point in $T_{RK}(p)^n$ for $1 \leqslant n < \omega$. Booth [Bo] also defined the power $T_{RK}(p)^\nu$ for each $p \in \omega^*$ and each $\nu < \omega_1$ as follows: for every $\omega \leqslant \nu < \omega_1$ fix and increasing sequence $(\nu(n))_{n<\omega}$ of ordinals in ω_1 so that

(1) $\omega(n) = n$ for $n < \omega$;
(2) if ν is a limit ordinal, then $\nu(n) \nearrow \nu$;
(3) if $\nu = \mu + m$, where μ is a limit ordinal and $1 \leqslant m < \omega$, then $\nu(n) = \mu(n) + m$ for each $n < \omega$.

Let $p \in \omega^*$ and $\omega \leqslant \nu < \omega_1$, and assume $T_{RK}(p)^\mu$ has been defined for all $\mu < \nu$. If ν is a limit ordinal, then we define $T_{RK}(p)^\nu = T_{RK}(\bar{f}(p))$, where $f\colon \omega \to \omega^*$ is an embedding such that $f(n) \in T_{RK}(p)^{\nu(n)}$ for each $n < \omega$. If $\nu = \mu + 1$, then $T_{RK}(p)^\nu = T_{RK}(p)^\mu \otimes T_{RK}(p)$. As above, p^ν stands for an arbitrary point in $T_{RK}(p)^\nu$ for each $p \in \omega^*$ and each $\nu < \omega_1$.

We omit the proof of the following straightforward lemma.

Lemma 0.2. *If $f\colon \omega \to \beta(\omega)$ has infinite image, then there is a one-to-one function $\sigma\colon \omega \to \omega$ such that $f \circ \sigma\colon \omega \to \beta(\omega)$ is an embedding (i.e., $f \circ \sigma$ is one-to-one and $(f \circ \sigma)[\omega]$ is a discrete subset of $\beta(\omega)$).*

Z. Frolík [F$_2$] noticed that if $e\colon \omega \to \omega^*$ is an embedding, then $p <_{RK} \bar{e}(p)$ for every $p \in \omega^*$; that is, no type is produced by itself. Frolík's result and Theorem 9.2 (b) of [CN] imply:

Lemma 0.3. *If $f\colon \omega \to \beta(\omega)$ is an embedding and $p \in \omega^*$, then $p \leqslant_{RK} \bar{f}(p)$.*

Bernstein [B] introduced the class of p-compact spaces for $p \in \omega^*$. Later, Saks [Sa], Woods [W] and Kannan and Soundararajan [KS] considered the notion of p-compactness for various $p \in \alpha^*$ at the same time:

Definition 0.4. Let $\emptyset \neq M \subseteq \alpha^*$. A space X is *M-compact* if for every $f\colon \alpha \to X, \bar{f}(p) \in X$ for all $p \in M$.

If $p \in U(\alpha)$, then we simply write p-compact instead of $\{p\}$-compact. The basic properties of M-compactness are stated in the next theorem (for a proof see [V$_2$]): Bernstein [B] proved the same result for $p \in \omega^*$.

Theorem 0.5. *Let $\emptyset \neq M \subseteq \alpha^*$. Then*

(1) *every compact space is M-compact;*

(2) *M-compactness is closed under arbitrary products;*

(3) *M-compactness is closed-hereditary;*

(4) *if $f\colon X \to Y$ is a continuous surjection and X is M-compact, then Y is M-compact.*

It follows from clause (1)–(3) of Theorem 0.5 that, for every space X, the space $\beta_M(X) = \bigcap\{Y\colon X \subseteq Y \subseteq \beta(X),\ Y \text{ is } M\text{-compact}\}$ is the (M-compact)-reflection of X which satisfies the following properties:

(1) X is a dense subspace of $\beta_M(X)$;

(2) $\beta_M(X)$ is M-compact;

(3) if $f\colon X \to Z$ is continuous and Z is M-compact, then $\bar{f}[\beta_M(X)] \subseteq Z$;

(4) up to a homeomorphism fixing X pointwise the space $\beta_M(X)$ is the only space with properties (1), (2) and (3).

For $p \in U(\alpha)$, we write $\beta_p(X)$ instead of $\beta_{\{p\}}(X)$. For a space X and for $\emptyset \neq M \subseteq \alpha^*$, there is an alternative definition of $\beta_M(\alpha)$ which will be very useful: By transfinite induction, we define

$$X_0 = X \quad \text{and} \quad X_\eta = \{\bar{f}(p) \mid f\colon \alpha \to \bigcup_{\xi<\eta} X_\xi, p \in M\} \quad \text{for} \quad \eta < \alpha^*.$$

It is not hard to see that $\beta_M(X) = \bigcup_{\eta<\alpha^+} X_\eta$. Hence, we have that $M \subseteq \beta_M(\alpha)$ and $|\beta_M(\alpha)| \leqslant 2^\alpha \cdot |M|^\alpha$, for each $\emptyset \neq M \subseteq \alpha^*$.

In [G-F$_4$], Comfort introduced the next (pre-)order on ω^*.

Definition 0.6. For $p, q \in \alpha^*$, we define $p \leqslant_c q$ if every q-compact space is p-compact.

For $p, q \in \alpha^*$, we say that $p \approx_c q$ if $p \leqslant_c q$ and $q \leqslant_c p$. We have that $\approx_c$ is an equivalence relation on ω^* and the $\approx_c$-equivalence class of $p \in \alpha^*$ is called the *C-type* of p and is denoted by $T_c(p)$. The connection between $\leqslant_{RK}$ and $\leqslant_c$ were established in $[G - F_4]$. The next outstanding properties are the following:

Lemma 0.7.

(1) $\leqslant_{RK}\ \subseteq\ \leqslant_c$ *and* $\leqslant_{RK} \neq \leqslant_c$;

(2) *if $p \in \omega^*$, then $T_c(p)$ can be filled out with exactly 2^ω types;*

(3) *if $p \in \omega^*$, then $p^\nu \in T_c(p)$ for every $\nu < \omega_1$;*

(4) *if $p, q \in \alpha^*$, then $p \leqslant_c q \Leftrightarrow \beta_p(\alpha) \subseteq \beta_q(\alpha) \Leftrightarrow p \in \beta_q(\alpha)$;*

(5) *if $p \leqslant_c q$ for $p, q \in \alpha^*$, then $\|p\| \leqslant \|q\|$;*

(6) *if $p \in \beta_M(\alpha) \setminus \alpha$ for $\emptyset \neq M \subseteq \alpha^*$, then $T_{RK}(p) \subseteq P_{RK}(p) \subseteq \beta_M(\alpha)$ and $T_c(p) \subseteq P_c(p) \subseteq \beta_M(\alpha)$.*

Observe that a space X is M-compact, for $\emptyset \neq M \subseteq \alpha^*$, iff $\bar{f}(p) \in X$ for every $f\colon \gamma \to X$ with $\gamma \leqslant \alpha$, and for every $p \in U(\gamma) \cap T_c(q)$ with $q \in M$.

It is convenient to have the definition of initially α-compactness at our disposal ([St] and [G-F$_1$] offer a good survey on initially α-compact spaces), as follows:

Definition 0.8. (Smirnov [Sm]). A space X is *initially α-compact* if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leqslant \alpha$ has a finite subcover.

The authors of [GFW] introduced the notion of α-boundedness and Comfort [0] and Vaughan [V$_2$] generalized this concept as follows.

Definition 0.9. A space X is *$< \alpha$-bounded* if $\mathrm{cl}_X(A)$ is a compact for each $A \subseteq X$ with $|A| < \alpha$.

Notice that α-boudedness coincides with $< \alpha^+$-boudedness for each cardinal α. It is shown in [G-F$_1$] that a space X is $< \alpha$-bounded iff X is p-compact for every $p \in \alpha^* \setminus U(\alpha)$. Hence, every space X has a unique ($< \alpha$-bounded)-reflection which will be denoted by $B_\alpha(X)$.

1. Quasi M-compact spaces

We start this section with the following definition due to Kombarov [K$_2$] and Savchenko [S] which generalizes, in a very natural fashion, M-compactness.

Definition 1.1. Let $\emptyset \neq M \subseteq \alpha^*$. A space X is *quasi M-compact* if for every $f\colon \alpha \to X$ there is $p \in M$ such that $\bar{f}(p) \in X$.

In this paper, for $\emptyset \neq M \subseteq \alpha^*$, we shall use the name of quasi M-compact instead of the name of M-compact given by Kombarov in [K$_2$], since the name of M-compact is reserved for the concept stated in Definition 0.4. Kombarov [K$_1$] also introduced the notion of weakly M-compactness, for $\emptyset \neq M \subseteq \omega^*$, but it is shown in [G-F$_2$] that weakly M-compacntess coincides with quasi $\mathrm{cl}_{\omega^*}(M)$-compactness.

If $p \in U(\alpha)$, then quasi $\{p\}$-compactness coincides with p-compactness. If $M, N \subseteq \alpha^*$ and $\emptyset \neq M \subseteq N$, then every quasi M-compact space is quasi N-compact. In particular, quasi ω^*-compactness is precisely the concept of countably compactness (see [K$_2$]). For $\emptyset \neq M \subseteq \alpha^*$, we have that M-compactness implies quasi M-compactness. But these two concepts are different; indeed, we saw in Theorem 0.5 (2) that M-compactness is productive for every $\emptyset \neq M \subseteq U(\alpha)$, and it is well-known that countably compactness is not finitely productive (see [GJ, 9.15]). We shall show, in Example 2.8, that quasi $T_{RK}(p)$-compactness fails to be preserved under finite products for each $p \in \omega^*$ (see [K$_2$]). In the next theorem we give a necessary condition to gain the quasi M-compactness of the product of two quasi M-compact spaces. We need a lemma.

Lemma 1.2. (Ginsburg-Saks [GS]). *Let* $p \in U(\alpha)$, $X = \prod_{i \in I} X_i$ *and* $f : \alpha \to X$. *Then* $\bar{f}(p) = x = (x_i)_{i \in I}$ *iff* $\bar{\pi}_i \circ \bar{f}(p) = x_i$ *for every* $i \in I$, *where* $\pi_i : X \to X_i$ *is the projection map for each* $i \in I$.

Theorem 1.3. *Let* M *and* N *be nonempty subsets of* α^* *such that for every* $p \in M$ *there is* $q \in N$ *such that* $p \leqslant_c q$. *Then if* X *is* N*-compact and* Y *is quasi* M*-compact, then* $X \times Y$ *is quasi* M*-compact.*

Proof. Let $f : \alpha \to X \times Y$ and consider $f_0 = \pi_0 \circ f : \alpha \to X$ and $f_1 = \pi_1 \circ f : \alpha \to Y$, where $\pi_0 : X \times Y \to X$ and $\pi_1 : X \times Y \to Y$ are the projection maps. By assumption, there exists $p \in M$ such that $\bar{f}_1(p) \in Y$. Choose $q \in N$ such that $p \leqslant_c q$. Since X is q-compact, then X is p-compact and hence $\bar{f}_0(p) \in X$. From 1.2 it then follows that $\bar{f}(p) \in X \times Y$. □

In what follows, for a cardinal α, $\mathfrak{C}_\alpha$ will denote the class of all spaces X with the property that for every initially α-compact space Y, $X \times Y$ is initially α-compact. For $\alpha = \omega$, the class $\mathfrak{C}_\omega$ was introduced and studied by Frolík [F_1]. The author [F_1] also characterized the spaces of $\mathfrak{C}_\omega$ (see [V_2, Theorem 3.11]). Our next aim is to give a characterization of the spaces in $\mathfrak{C}_\alpha$ in terms of p-limits for an arbitrary cardinal α. The following concept is fundamental for our purposes. We need some notation and the next lemmas.

For a cardinal $\alpha, \mathscr{M} = \{M_i : i \in I\}$ will denote an arbitrary set of nonempty subsets of α^*, and if $\omega \leqslant \gamma \leqslant \alpha$, then $\mathscr{M}(\gamma, \alpha)$ will denote a set $\{M_\lambda : \gamma \leqslant \lambda \leqslant \alpha\}$ of nonempty subsets of α^* such that $M_\lambda \subseteq U(\lambda)$ for every $\gamma \leqslant \lambda \leqslant \alpha$.

A slight generalization of quasi M-compactness is the following:

Definition 1.4. Let $\mathscr{M} = \{M_i : i \in I\}$. Then a space X is said to be *quasi* $\mathscr{M}$*-compact* if X is quasi M_i-compact for each $i \in I$.

If $\emptyset \neq M \subseteq U(\alpha)$, then quasi $\{M\}$-compactness agrees with quasi M-compactness, and if $\mathscr{M} = \{\{p_i\} : p_i \in U(\gamma_i), \omega \leqslant \gamma_i \leqslant \alpha, i \in I\}$, then X is quasi $\mathscr{M}$-compact iff X is p_i-compact for all $i \in I$. If $p <_c q$ for $p, q \in \omega^*$, then $\beta_p(\omega)$ is p-compact, but it is not quasi $\{\{p\}, \{q\}\}$-compact since $\beta_p(\omega)$ cannot be q-compact.

It should be mentioned that for a space X the following conditions are equivalent (for a proof see [St]):

(1) X is initially α-compact;

(2) for every $\omega \leqslant \gamma \leqslant \alpha$ and for every $f : \gamma \to X$ there is $p \in U(\gamma)$ such that $\bar{f}(p) \in X$;

(3) for every $\omega \leqslant \gamma \leqslant \alpha$ and for every one-to-one function $f\colon \gamma \to X$ there is $p \in U(\gamma)$ such that $\bar{f}(p) \in X$.
(4) for every regular cardinal γ with $\omega \leqslant \gamma \leqslant \alpha$ and for every one-to-one function $f\colon \gamma \to X$ there is $p \in U(\gamma)$ such that $\bar{f}(p) \in X$.

In our context we have:

Lemma 1.5. *X is initially α-compact iff there is a set $\mathscr{M}(\omega, \alpha)$ of nonempty subsets of α^* such that X is quasi $\mathscr{M}(\omega, \alpha)$-compact.*

The proof of the following result is immediate.

Lemma 1.6. *If $\emptyset \neq K \subseteq \alpha^*$ and $X = \alpha \cup K \subseteq \beta(\alpha)$ is initially α-compact, then $K \cap U(\gamma) \neq \emptyset$ for every cardinal γ such that $\omega \leqslant \gamma \leqslant \alpha$.*

For cardinal α, we let

$$\mathscr{A}(\alpha) = \{K\colon K \subseteq \alpha^* \quad \text{and} \quad \alpha \cup K \quad \text{is initially} \quad \alpha\text{-compact}\},$$

and for $\omega \leqslant \gamma \leqslant \alpha, \mathscr{A}(\gamma, \alpha) = \{U(\gamma) \cap K\colon K \in \mathscr{A}(\alpha)\}$. Notice, by 1.6, that if $\omega \leqslant \gamma \leqslant \alpha$ and $K \in \mathscr{A}(\alpha)$, then $U(\gamma) \cap K \neq \emptyset$. We set $\mathscr{M}_\alpha = \bigcup\{\mathscr{A}(\gamma, \alpha)\colon \omega \leqslant \gamma \leqslant \alpha\}$. Henceforth, if $K \in \mathscr{A}(\alpha)$, then $I(K)$ will denote the subspace $\alpha \cup K$ of $\beta(\alpha)$. Now, we shall show that the sets $\mathscr{M}_\alpha$ characterizes the spaces of $\mathfrak{C}_\alpha$, First an easy lemma.

Lemma 1.7. *If $f\colon Y \to Z$ is a continuous function, Y is compact and $X \subseteq Z$ is initially α-compact, then $f^{-1}(X)$ is initially α-compact.*

Theorem 1.8. *For a space X the following conditions are equivalent.*
(1) X is quasi $\mathscr{M}_\alpha$-compact;
(2) $X \in \mathfrak{C}_\alpha$;
(3) $X \times I(K)$ is initially α-compact for each $K \in \mathscr{A}(\alpha)$.

Proof. (1)$\Rightarrow$(2). Let Y be initially α-compact and let $f\colon \gamma \to X \times Y$, where $\omega \leqslant \gamma \leqslant \alpha$. Set $f_0 = \pi_0 \circ f\colon \gamma \to X$ and $f_1 = \pi_1 \circ f\colon \gamma \to Y$, where $\pi_0\colon X \times Y \to X$ and $\pi_1 : X \times Y \to Y$ stand for the projection maps. Since Y is initially α-compact, by 1.7, $\bar{f}_1^{-1}(Y) \subseteq \beta(\gamma)$ is initially α-compact. We then have that $\alpha \cup \bar{f}_1^{-1}(Y) \cup (\widehat{\alpha \setminus \gamma})$ is initially α-compact and $U(\gamma) \cap [\alpha \cup \bar{f}_1^{-1}(Y) \cup (\widehat{\alpha \setminus \gamma})] = U(\gamma) \cap \bar{f}_1^{-1}(Y)$. Since X is quasi $[U(\gamma) \cap \bar{f}_1^{-1}(Y)]$-compact there is $p \in U(\gamma) \cap \bar{f}_1^{-1}(Y)$ such that $\bar{f}_0(p) \in X$. Hence, by 1.2, $\bar{f}(p) \in X \times Y$, since $\bar{f}_1(p) \in Y$.

(2) $\Rightarrow$ (3). This is evident.

(3) $\Rightarrow$ (1). Let $\omega \leqslant \gamma \leqslant \alpha, U(\gamma) \cap K \in \mathscr{A}(\gamma, \alpha)$ and $f\colon \gamma \to X$. Let $g\colon \gamma \to X \times I(K)$ be defined by $g(\xi) = (f(\xi), \xi)$ for each $\xi < \alpha$. Since $X \times I(K)$ is initially

α-compact there is $p \in U(\gamma)$ such that $\bar{g}(p) = (x, q) \in X \times I(K)$. It then follows that $\bar{f}(p) = x \in X$ and $p = q \in U(\gamma) \cap K$. Therefore, X is quasi $\mathscr{A}(\gamma, \alpha)$-compact. $\square$

Next, we shall prove that a cardinal number α is singular if and only if $B_\alpha(\alpha) \in \mathfrak{C}_\alpha$. First some preliminary results.

Lemma 1.9. (Stephenson-Vaughan [SV]). *Let α be a singular cardinal and let X be a space. If X is initially κ-compact for each $\omega \leqslant \kappa < \alpha$, then X is initially α-compact.*

The next lemma is a special case of the Theorem 2.2 of [SV] (see [Sa, 5.2]). For the sake of completeness we include a proof.

Lemma 1.10. *If X is $< \alpha$-bounded and Y is initially α-compact, then $X \times Y$ is initially κ-compact for every $\omega \leqslant \kappa < \alpha$.*

Proof. Let κ be a cardinal such that $\omega \leqslant \kappa < \alpha$ and let $f\colon \kappa \to X \times Y$. Set $f_0 = \pi_0 \circ f\colon \kappa \to X$ and $f_1 = \pi_1 \circ f\colon \kappa \to Y$. Since Y is initially κ-compact, there is $p \in U(\kappa)$ such that $\bar{f}_1(p) \in Y$. The $< \alpha$-boundedness of X implies that $\bar{f}_0(p) \in \mathrm{cl}_x(f[\kappa]) \subseteq X$. Thus, by 1.2, we have that $\bar{f}(p) \in X \times Y$. This shows that $X \times Y$ is initially κ-compact. $\square$

In [GT], several topological conditions are shown to be equivalent to the singularity of a cardinal number. In particular, we have:

Lemma 1.11. *For a cardinal α, the following an equivalent.*
(1) *α is singular;*
(2) *B_α is initially α-compact.*

The next theorem is an immediate consequence of 1.9, 1.10 and 1.11.

Theorem 1.12. *For a cardinal α, the following statements are equivalent.*
(1) *α is singular;*
(2) *every $< \alpha$-bounded space lies in $\mathfrak{C}_\alpha$;*
(3) *$B_\alpha(\alpha) \in \mathfrak{C}_\alpha$;*
(4) *$B_\alpha(X) \in \mathfrak{C}_\alpha$ for every space X.*

We know that if α is a regular cardinal, then $B_\alpha(\alpha) = \beta(\alpha) \setminus U(\alpha)$ and it cannot be initially α-compact. It is pointed out implicitly in [SS, proof of 4.11] that if α is a strong limit singular cardinal, then initial α-compactness agrees with $< \alpha$-boundedness (see [G-F$_1$, 2.4]). This result implies the following.

Theorem 1.13. *If α is a strong limit singular cardinal, then every initially α-compact space is a member of $\mathfrak{C}_\alpha$.*

We turn now to study when $\beta_M(\alpha) \in \mathfrak{C}_\alpha$ for $\emptyset \neq M \subseteq \alpha^*$. The proof of the next easy lemma is omitted.

Lemma 1.14. *For $\emptyset \neq M \subseteq \alpha^*$ and $q \in U(\gamma)$ for $\omega \leqslant \gamma \leqslant \alpha$, the following conditions are equivalent.*

(1) *$\beta_M(\alpha)$ is q-compact;*

(2) *$q \in \beta_M(\alpha)$;*

(3) *$T_c(q) \cap \beta_M(\alpha) \neq \emptyset$.*

Applying 1.8 and 1.14, we have:

Theorem 1.15. *For $\emptyset \neq M \subseteq \alpha^*$, the following are equivalent.*

(1) *$\beta_M(\alpha) \in \mathfrak{C}_\alpha$;*

(2) *$\beta_M(\alpha) \times I(K)$ is initially α-compact for each $K \in \mathscr{A}(\alpha)$;*

(3) *for every $K \in \mathscr{A}(\alpha)$ and for every $\omega \leqslant \gamma \leqslant \alpha$ there is $q \in U(\gamma) \cap K$ such that $\beta_M(\alpha)$ is q-compact;*

(4) *for every $K \in \mathscr{A}(\alpha)$ and for every $\omega \leqslant \gamma \leqslant \alpha$, $\beta_M(\alpha) \cap K \cap U(\gamma) \neq \emptyset$.*

Proof. (1) $\Rightarrow$ (2) is evident and (3) $\Leftrightarrow$ (4) follows from 1.14. (2) $\Rightarrow$ (4). Let $K \in \mathscr{A}(\alpha)$. Since $\beta_M(\alpha) \times I(K)$ is initially α-compact and $\beta_M(\alpha) \cap I(K)$ is homeomorphic to the diagonal of $\beta_M(\alpha) \times I(K)$, we have that $\beta_M(\alpha) \cap I(K)$ is initially α-compact. By lemma 1.6., we have that $\beta_M(\alpha) \cap I(K) \cap U(\gamma) \neq \emptyset$ for each $\omega \leqslant \gamma \leqslant \alpha$.

(3) $\Rightarrow$ (1). According to 1.8, it is enough to prove that $\beta_M(\alpha)$ is quasi $\mathscr{M}_\alpha$-compact. Let $K \in \mathscr{A}(\alpha)$ and $\omega \leqslant \gamma \leqslant \alpha$. By assumption, there is $q \in \beta_M(\alpha) \cap K \cap U(\gamma)$ such that $\beta_M(\alpha)$ is q-compact. Thus $\beta_M(\alpha)$ is quasi $\mathscr{M}_\alpha$-compact. □

The equivalences between clauses (1) and (2) of the next lemma is a direct consequence of 1.6 and the equivalence among the others may be established by an easy argument: the case $M = \{p\}$ for $p \in U(\alpha)$ is stated in [G-F$_3$]. We recall the reader that $p \in U(\alpha)$ is called *decomposable* if for every $\omega \leqslant \gamma \leqslant \alpha$ there is $q \in U(\gamma)$ such that $q \leqslant_{RK} p$: for further information about decomposable ultrafilters the reader is referred to [BS].

Lemma 1.16. *For $\emptyset \neq M \subseteq U(\alpha)$, the following are equivalent,*

(1) *$\beta_M(\alpha)$ is initially α-compact;*

(2) *$\beta_M(\alpha) \cap U(\gamma) \neq \emptyset$ for every $\omega \leqslant \gamma \leqslant \alpha$;*

(3) *there are $p \in M$ and $q \in T_c(p)$ such that q is decomposable;*

(4) *there is* $p \in U(\alpha)$ *decomposable such that* $\beta_M(\alpha)$ *is p-compact.*

It is shown in [G-F$_1$] that all powers of space X are initially α-compact iff there is $p \in U(\alpha)$ decomposable such that X is p-compact, and we proved in [G-F$_3$] that if α is a strong limit and $p \in U(\alpha)$ is indecomposable, then $\beta_p(\alpha)$ is a p-compact space which is not initially α-compact: we also remarked in [G-F$_3$] that in the Core model K a space X is p-compact, for $p \in U(\alpha)$, iff all powers of X are initially α-compact.

Lemma 1.17. *Let* $p \in U(\alpha), \omega \leqslant \gamma \leqslant \alpha$ *and* $\sigma\colon \gamma \to U(\alpha)$ *a strong embedding. If* $p \leqslant_{RK} \sigma(\xi)$ *for each* $\xi < \gamma$*, then* $p \leqslant_{RK} \bar{\sigma}(q)$ *for each* $q \in U(\gamma)$*.*

Proof. Let $\{A_\xi : \xi < \gamma\}$ be a partition of α such that $\sigma(\xi) \in A_\xi^*$ for each $\xi < \gamma$. Then there is $f_\xi\colon A_\xi \to \alpha$ such that $\bar{f}_\xi(\sigma(\xi)) = p$ for each $\xi < \gamma$. If $f = \bigcup_{\xi<\gamma} f_\xi$, then $\bar{f}(\bar{\sigma}(q)) = p$ for each $q \in U(\gamma)$. □

Lemma 1.18. *For every* $p \in \omega^*$ *there is* $K \in \mathscr{A}(\omega)$ *such that* $p \leqslant_{RK} q$ *for every* $q \in K$*.*

Proof. Let $p \in \omega^*$. Define $K_0 = T_{RK}(p)$ and, by transfinite induction, let $K_\nu\{\bar{e}(p) \mid e\colon \omega \to \bigcup_{\mu<\nu} K_\mu$ is an embedding$\}$ for $\nu < \omega_1$. Then, we put $K = \bigcup_{\nu<\omega_1} K\nu$. First, we shall verify that $K \in \mathscr{A}(\omega)$. Suppose that $\omega \cup K$ is not countably compact. Then, there exists an embedding $e\colon \omega \to \omega \cup K$ such that $\bar{e}[\omega^*] \cap (\omega \cup K) = \emptyset$. Set $A = \{n < \omega\colon e(n) \in \omega\}$ and $B = \{n < \omega\colon e(n) \in K\}$. If $A \in p$, then $\bar{e}(p) \in T_{RK}(p) = K_0 \subseteq K$ which is impossible. Thus $B \in p$. Without loss of generality, we may assume that $B = \omega$. For each $n < \omega$ choose $\nu_n < \omega_1$ such that $e(n) \in K_{\nu_n}$ and set $\nu = \sup\{\nu_n\colon n < \mu\}$. By definition, we have that $\bar{e}(p) \in K_\nu \subseteq K$ which is a contradiction. Therefore, $K \in \mathscr{A}(\omega)$. We shall prove that K satisfies the required condition. Assume that $p \leqslant_{RK} r$ for every $r \in K_\mu \cap U(\alpha)$ and for every $\mu < \nu < \omega_1$. Let $q \in K_\nu \cap U(\alpha)$. Then, there is an embedding $\sigma\colon \omega \to \bigcup_{\mu<\nu} K_\mu$ such that $\bar{\sigma}(p) = q$. By induction hypothesis, we have that $p \leqslant_{RK} \sigma(n)$ for every $n < \omega$. Applying Lemma 1.17, we obtain that $p \leqslant_{RK} \bar{\sigma}(p) = q$. □

Theorem 1.19. *If* $\beta_M(\omega) \in \mathfrak{C}_\omega$ *for* $\emptyset \neq M \subseteq \omega^*$*, then* $\beta_M(\omega) = \beta(\omega)$*.*

Proof. Assume that $\beta_M(\omega) \in \mathfrak{C}_\omega$ for $\emptyset \neq M \subseteq \omega^*$. Fix $p \in \omega^*$. By 1.18, there is $K \in \mathscr{A}(\omega)$ such that $p \leqslant_{RK} r$ for all $r \in K$. It follows from 1.8 that $\beta_M(\omega)$ is quasi K-compact and hence there is $q \in \beta_M(\omega) \cap K$. By Lemma 0.7 (6), $p \in P_{RK}(q) \subseteq \beta_M(\alpha)$. Thus, $\beta_M(\omega) = \beta(\omega)$. □

For a singular cardinal α, we have the following:

(1) $B_\alpha(\alpha) \in \mathfrak{C}_\alpha$, by 1.12;

(2) $B_\alpha(\alpha) = \beta_N(\alpha)$, where $N = \alpha^* \setminus U(\alpha)$ (see [G-F$_1$, Theorem 1.3]); and
(3) $B_\alpha(\alpha) \neq \beta(\alpha)$ (see [GT, Corollary 2.4 (b)]).
This leads us to ask:

Question 1.20. Let $\alpha > \omega$ be a regular cardinal. Is there $\emptyset \neq M \subseteq \alpha^*$ such that $\beta_M(\alpha) \in \mathfrak{C}_\alpha$ and $\beta_M(\alpha) \neq \beta(\alpha)$?

The following example shows that there exists $\emptyset \neq M \subseteq \omega^*$ such that $|\beta_M(\omega)| = 2^{2^\omega}$ and $\beta_M(\omega) \neq \beta(\omega)$.

Lemma 1.21. *Let $\emptyset \neq M \subseteq \omega^*$. If p is a weak P-point of ω^* and $p \in \beta_M(\omega)$, then $p \leqslant_{RK} q$ for some $q \in M$.*

Proof. We indicate in the preliminary section that $\beta_M(\omega) = \bigcup_{\nu<\omega_1} X_\nu$, where $x_0 = \omega$ and $X_\nu = \{\bar{f}(q) \mid f\colon \omega \to \bigcup_{\mu<\nu} X_\mu, q \in M\}$ for $0 < \nu < \omega_1$. By a slight modification of the X_ν's, we obtain that $\beta_M(\omega) = \bigcup_{\nu<\omega_1} Z_\nu$, where $Z_0 = \omega, Z_1 = \bigcup_{q\in M} P_{RK}(q)$ and $Z_\nu = \{\bar{f}(q) \mid f\colon \omega \to \bigcup_{\mu<\nu} Z_\mu,\ f[\omega] \subseteq \omega^*,\ q \in M$ and $\bar{f}(q) \notin f[\omega]\}$ for $1 < \nu < \omega_1$. Assume that $p \in \beta_M(\omega)$ and p is a weak P-point. Let ν be the least ordinal $\nu < \omega_1$ such that $p \in Z_\nu$. Since p is a weak P-point of ω^*, we have that $\nu = 1$ and so $p \in \bigcup_{q\in M} P_{RK}(q)$. □

Example 1.22. K. Kunen [Ku] proved that there is a set W of weak P-points of ω^* such that $|W| = 2^{2^\omega}$ and the elements of W are pairwise RK-incomparable. Choose $M \subseteq W$ so that $|M| = |W \setminus M| = 2^{2^\omega}$ and enumerate M as $\{p\xi\colon \xi < 2^{2^\omega}\}$. Then, $2^{2^\omega} = |M| = |\beta_M(\omega)| = |\beta(\omega)|$ and, by Lemma 1.21, $W \setminus M \subseteq \beta(\omega) \setminus \beta_M(\omega)$.

The author introduced in [G-F$_5$] the notion of (γ, M)-compactness for a cardinal $1 \leqslant \gamma$ and $\emptyset \neq M \subseteq \omega^*$, and proved that X^γ is countably compact iff X is (γ, M)-compact for some $\emptyset \neq M \subseteq \omega^*$. The situation for cardinal numbers higher that ω is completely similar as it is stated in the following two results.

Definition 1.23. Let $\emptyset \neq M \subseteq U(\alpha)$ and γ a cardinal. A space X is said to be *(γ, M)-compact* if for every γ-sequence $(f_\zeta)_{\zeta<\gamma}$ of functions in ${}^\alpha X$, there is $p \in M$ such that $\bar{f}_\zeta(p) \in X$ for each $\zeta < \gamma$.

As a direct sequence from 1.2 we have:

Theorem 1.24. *Let X be a space and γ a cardinal. Then X^γ is initially α-compact iff for each cardinal δ with $\omega \leqslant \delta \leqslant \alpha$, there is $\emptyset \neq M_\delta \subseteq U(\delta)$ such that X is (γ, M_δ)-compact.*

Theorem 1.25. *Let $\emptyset \neq M \subseteq U(\alpha)$, let $1 \leqslant \gamma$ be a cardinal number and let X be a (γ, M)-space.*

(1) *If $|M| \leqslant \gamma$, then there is $p \in M$ such that X is p-compact.*

(2) *If there exist $p \in U(\alpha)$ and a surjection $\sigma\colon \alpha \to \alpha$ such that $M \subseteq \sigma^{-1}(p)$, then X is p-compact.*

2. Almost M-compact spaces

It follows from the definition that quasi M-compactness implies quasi $P_{RK}(M)$-compactness for $\emptyset \neq M \subseteq \alpha^*$. The next theorem establishes the similarity between these two concepts.

Theorem 2.1. *Let $\emptyset \neq M \subseteq U(\alpha)$. A space X is quasi $P_{RK}(M)$-compact if and only if for every $f\colon \alpha \to X$ there is $p \in M$ and $\sigma\colon \alpha \to \alpha$ such that $\bar{\sigma}(p) \in \alpha^*$ and $\bar{f}(\bar{\sigma}(p)) \in X$.*

For $\emptyset \neq M \subseteq U(\alpha)$, we simply say *almost M-compact* space instead of quasi $P_{RK}(M)$-space (2.1 justifies the name almost M-compact). Thus, an almost p-compact space is a quasi $P_{RK}(p)$-space for $p \in U(\alpha)$. We shall give in 2.3 an example of an almost p-compact space which is not p-compact.

Theorem 2.2. *If X_ξ is initially α-compact and $|X_\xi| \leqslant 2^\alpha$ for $\xi < 2^\alpha$, then there is $p \in U(\alpha)$ such that X_ξ is almost p-compact for each $\xi < 2^\alpha$.*

P r o o f. We have that $|\bigcup_{\xi<2^\alpha} X_\xi| \leqslant 2^\alpha$. Since X_ξ is initially α-compact, for every $f\colon \alpha \to X_\xi$, there is $p_f \in U(\alpha)$ such that $\bar{f}(p_f) \in X_\xi$, for each $\xi < 2^\alpha$. We have that $|\{p_f \mid f\colon \alpha \to X_\xi, \xi < 2^\alpha\}| \leqslant 2^\alpha$. Hence, by Theorem 10.9 of [CN], there is $p \in U(\alpha)$ such that $p_f \leqslant_{RK} p$ for every $f\colon \alpha \to X_\xi$ and for every $\xi < 2^\alpha$. We claim that X_ξ is almost p-compact for every $\xi < 2^\alpha$. Indeed, fix $\xi < 2^\alpha$ and $f\colon \alpha \to X_\xi$. Since $p_f \leqslant_{RK} p$, there is $\sigma\colon \alpha \to \alpha$ such that $\bar{\sigma}(p) = p_f$ and so $\bar{f}(\bar{\sigma}(p)) \in X_\xi$. □

For every $p \in \omega^*$, we have that p-compactness $\Rightarrow$ quasi $T_{RK}(p)$-compactness $\Rightarrow$ almost p-compactness $\Rightarrow$ quasi $T_c(p)$-compactness $\Rightarrow$ countable compactness. The following three examples and theorem show that they are different each other except for the case when $p \in \omega^*$ is RK-minimal (in this case we have that quasi $T_{RK}(p)$-compactness coincides with almost p-compactness).

Example 2.3. For each $p \in \omega^*$, there is an almost p-compact space Γ_p which is not p-compact. Fix $p \in \omega^*$. The space Γ_p will be constructed inside $\beta(\omega)$ by induction and by a well-known standard method. Put $\Gamma_0 = \omega$ and assume that Γ_μ

has been defined for each $\mu < \nu < \omega_1$. Then, define $\Gamma_\nu = \{\bar{f}(q) \mid f\colon \omega \to \bigcup_{\mu<\nu} \Gamma_\mu$ is an embedding, $\bar{f}(q) \neq p$, $q \in T_{RK}(p)\}$. We set $\Gamma_p = \bigcup_{\nu<\omega_1} \Gamma_\nu$. Since $p \notin \Gamma_p$, the space Γ_p is not p-compact. Now, in order to prove that Γ_p is almost p-compact we let $f\colon \omega \to \Gamma_p$ be such that $f[\omega]$ is infinite. By 0.2, there is a one-to-one function $\sigma\colon \omega \to \omega$ such that $f \circ \sigma\colon \omega \to \Gamma_p$ is an embedding. So we may find $q \in T_{RK}(p)$ such that $\bar{f}(\bar{\sigma}(q)) \neq p$. Since $(f \circ \sigma)[\omega] \subseteq \bigcup_{\mu<\nu} \Gamma_\mu$ for some $\nu < \omega_1$, we have that $\bar{f}(\bar{s}(q)) \in \Gamma_\nu \subseteq \Gamma_P$ and $\bar{\sigma}(q) \in T_{RK}(p)$. This shows that Γ_p is quasi $T_{RK}(p)$-compact and so Γ_p is almost p-compact.

Example 2.4. If $p \in U(\omega)$, then $\Delta_p = \beta(\omega) \setminus T_c(p)$ is a countably compact space which is not quasi $T_c(p)$-compact. Indeed, since $|P_{RK}(p)| \leqslant 2^\omega$, the space Δ_p is countably compact (it is well-known that for every subspace X of $\beta(\omega)$ such that $|X| \leqslant 2^\omega$, $\beta(\omega) \setminus X$ is countably compact). If $f\colon \omega \to \omega$ is an arbitrary bijection, then $\bar{f}[T_c(p)] \subseteq T_c(p) \subseteq \beta(\omega) \setminus \Delta_p$.

Example 2.5. For $p \in \omega^*$, the space $\Omega_p = \beta(\omega) \setminus P_{RK}(p)$ is quasi $T_c(p)$-compact and is not almost p-compact. It is evident that Ω_p cannot be almost p-compact. Now, let $f\colon \omega \to \Omega_p$ be with infinite image. By 0.2, there is a one-to one function $\sigma\colon \omega \to \omega$ such that $f \circ \colon \omega \to \Omega_p$ is an embedding. Then, by 0.3, either $p <_{RK} p^2 <_{RK} \bar{f}(\bar{\sigma}(p^2))$ or $\bar{f}(\bar{\sigma}(p^2)) \approx_{RK} p^2$. In both cases we have that $\bar{f}(\bar{\omega}(p^2)) \in \Omega_p$ and $\bar{\sigma}(p^2) \in T_{RK}(p^2) \subseteq T_c(p)$.

Theorem 2.6. *For $p \in \omega^*$, the following are equivalent,*

(1) *p is RK-minimal;*

(2) *quasi $T_{RK}(p)$-compactness and almost p-compactness are the same.*

P r o o f. Only (2) $\Rightarrow$ (1) requires proof. Suppose that p is not RK-minimal. Then there is $q \in \omega^*$ such that $q <_{RK} p$. Consider the space $X = \beta(\omega) \setminus T_{RK}(p)$. This space cannot be quasi $T_{RK}(p)$-compact. We shall verify that the space is almost p-compact. Indeed, let $f\colon \omega \to X$ be with infinite image. By 0.2, we may choose a one-to-one function $\sigma\colon \omega \to \omega$ such that $f \circ \sigma\colon \omega \to X$ is an embedding. If there is an infinite $A \subseteq \omega$ such that $f \circ \sigma[A] \subseteq \omega$, then $\bar{f} \circ \bar{\sigma}(\hat{A} \cap T_{RK}(q)] \subseteq T_{RK}(q) \subseteq X$. If $(f \circ \sigma[\omega]) \cap \omega$ is finite, by Lemma 0.3, then we have that $p <_{RK} \bar{f}(\bar{\sigma}(p)) \in X$ and $\bar{\sigma}(p) \in T_{RK}(p)$. $\square$

The following example shows that almost p-compactness is not preserved by finite products, for $p \in \omega^*$. Nevertheless, we proved in Theorem 1.3 above that the product of a p-compact space and an almost p-compact space is almost p-compact as well for every $p \in U(\alpha)$. We need a lemma.

Lemma 2.7. (Blass [Bl_2]). *Let $f, g\colon \omega \to \omega^*$ be embeddings and let $p \in \omega^*$. If $\{n < \omega\colon f(n) <_{RK} g(n)\} \in p$, then $\bar{f}(p) <_{RK} \bar{g}(p)$.*

Example 2.8. For every $p \in \omega^*$, there is quasi $T_{RK}(p)$-compact space C_p such that $C_p \times C_p$ is not countably compact. Fix $p \in \omega^*$, we proceed as in the example 9.15 of [GJ]. Let $\omega = A \cup B$, where $|A| = |B| = \omega$ and $A \cap B = \emptyset$. Let $\delta\colon A \to B$ be a bijection and define $\tau\colon \beta(\sigma) \to \beta(\sigma)$ by $\tau\big|_{\beta(A)} = \bar{\delta}$ and $\tau\big|_{\beta(B)} = \bar{\delta}^{-1}$. Observe that $q \approx_{RK} \tau(q)$ for $q \in \omega^*$. It suffices to construct a quasi $T_{RK}(p)$-compact space C_p so that $C_p \times C_p \cap \{(q, \tau(q))\colon q \in \omega^*\} = \emptyset$ (see [GJ, 9.15]). We proceed by induction. Let $C_0 = \omega$ and assume that $C_\nu = \{p(\nu, \xi)\colon \xi < 2^\omega\} \subseteq \omega^*$ has been defined for each $\nu < \theta < \omega_1$ so that:

(1) $p(\nu, \xi) = \bar{f}^\nu_\xi(q(\nu, \xi))$ for some $q(\nu, \xi) \in T_{RK}(p)$, for $f^\nu_\xi \in \mathscr{F}_\nu$, $\nu < \theta$ and $\xi < 2^\omega$, where if $\nu = \mu + 1$, then $\mathscr{F}_\nu = \{f \mid f\colon \omega \to X_\mu$ is an embedding$\}$ and if ν is a limit ordinal, then $\mathscr{F}_\nu = \{f \mid f\colon \omega \to \bigcup_{\mu<\nu} X_\mu$ is an embedding with $f(n) \in X_{\mu_n}$ for each $n < \omega$ and $\mu_n \nearrow \nu\}$, and $\{f^\nu_\xi\colon \xi < 2^\omega\}$ is an enumeration of $\mathscr{F}_\nu$;
(2) $p(\nu, \xi) \neq \tau(p(\nu, \xi))$ for $\nu < \theta$ and $\xi < \zeta < 2^\omega$;
(3) $p(\mu, \zeta) <_{RK} p(\nu, \xi)$ for $\mu < \nu < \theta$ and $\xi, \zeta < 2^\omega$.

We consider two cases.

Case I. $\theta = \mu + 1$. In this case, we let $\mathscr{F}_\theta = \{f^\theta_\xi \mid f^\theta_\xi\colon \omega \to X_\mu$ is an embedding, $\xi < 2^\omega\}$ and put $p(\theta, 0) = \bar{f}^\theta_0(p)$. Suppose that, for each $\xi < \zeta < 2^\omega$, $p(\theta, \xi)$ has been defined so that $p(\theta, \xi) = \bar{f}^\theta_\xi(q(\theta, \xi))$ for some $q(\theta, \xi) \in T_{RK}(p)$ and (1)–(3) hold for $\nu < \theta$, for $\xi < 2^\omega$ and for $\{p(\theta, \xi)\colon \xi < \zeta < 2^\omega\}$. Since $|\{\tau(p(\theta, \xi))\colon \xi < \zeta\}| < 2^\omega$ and $|T_{RK}(p)| = 2^\omega$, there is $q(\theta, \zeta) \in T_{RK}(p)$ such that $p(\theta, \zeta) = \bar{f}^\theta_\zeta(q(\theta, \zeta)) \neq \tau(p(\theta, \xi))$ for every $\xi < \zeta$. In order to verify (3) we fix $\nu \leqslant \mu$ and ξ, $\zeta < 2^\omega$. By definition, we have that $f^\nu_\xi(n) <_{RK} f^\theta_\zeta(n)$ for each $n < \omega$. From 2.7 it then follows that

$$p(\nu, \xi) = \bar{f}^\nu_\xi(q(\nu, \xi)) <_{RK} \bar{f}^\theta_\zeta(q(\theta, \zeta)) = p(\theta, \zeta).$$

Case II. θ is a limit ordinal. We put $\mathscr{F}_\theta = \{f^\theta_\xi \mid f^\theta_\xi : \omega \to \bigcup_{\nu<\theta} X_\nu$ is an embedding with $f^\theta_\xi(n) \in X_{\nu_n}$ for each $n < \omega, \nu_n \nearrow \nu, \xi < 2^\omega\}$. Define $p(\theta, 0) = \bar{f}^\theta_0(p)$ and assume that for each $\xi < \zeta < 2^\omega, p(\theta, \xi)$ has been defined so that $p(\theta, \xi) = \bar{f}^\theta_\xi(p)$ and (1)-(3) hold for $\nu < \theta$, for $\xi < 2^\omega$ and for $\{p(\theta, \xi)\colon \xi < \zeta < 2^\omega\}$. Since $|\{\tau(p(\theta, \xi))\colon \xi < \zeta\}| < 2^\omega$ and $|T_{RK}(p)| = 2^\omega$, there is $q(\theta, \zeta) \in T_{RK}(p)$ such that $p(\theta, \zeta) = \bar{f}^\theta_\zeta(q(\theta, \zeta)) \neq \tau(p(\theta, \xi))$ for every $\xi < \zeta$. Only (3) requires proof. Let $\nu < \theta$ and $\xi, \zeta < 2^\omega$. It suffices to prove that $p(\nu + 1, \xi) <_{RK} p(\theta, \zeta)$. In fact, since $\nu < \theta$ and $\nu_n \nearrow \theta$ there is $m < \omega$ such that $\nu < \nu_n < \theta$ for each $m \leqslant n < \omega$. By assumption, we have that $f^{\nu+1}_\xi(n) < f^\theta_\zeta(n)$ for each $m \leqslant n < \omega$. By Lemma 2.7, we obtain that

$$p(\nu + 1, \xi) = \bar{f}^{\nu+1}_\xi(q(\nu + 1, \xi)) <_{RK} \bar{f}^\theta_\zeta(q(\theta, \zeta)) = p(\theta, \zeta).$$

Our example is the space $C_p = \bigcup_{\theta<\omega_1} X_\theta$ with the topology inherited from $\beta(\omega)$. First, we show that C_p is quasi $T_{RK}(p)$-compact. Let $f\colon \omega \to C_p$ be with infinite image. If we may find a one-to-one function $\sigma\colon \omega \to \omega$ and $\theta < \omega_1$ so that $f \circ \sigma\colon \omega \to X_\theta$ is an embedding, then there is $\xi < 2^\omega$ for which $f \circ \sigma = f_\xi^\theta$ and so $\bar{f}(\bar{\sigma}(q(\theta,\xi))) = \bar{f}_\xi^\theta(q(\theta,\xi)) = p(\theta,\xi) \in X_{\theta+1} \subseteq C_p$ and $q(\theta,\xi) \in T_{RK}(p)$. In the preceding case does not hold, then we may find a one-to-one function $\sigma\colon \omega \to \omega$ and a sequence of ordinals $(\nu_n)_{n<\omega}$ in ω_1 so that $\nu_n \nearrow \nu$ and $f \circ \sigma\colon \omega \to C_p$ is an embedding, and $f(n) \in X_{\nu_n}$ for each $n < \omega$. Now, we proceed as above. This proves that C_p is quasi $T_{RK}(p)$-compact. Let $q \in X_\theta \subseteq C_p$ for some $\theta < \omega_1$. Since $\tau(q) \approx_{RK} q$, we have that $\tau(q) \notin X_\nu$ for each $\nu < \omega_1$ with $\nu \neq \theta$, because of (3). It follows from (2) that $\tau(q) \notin X_\theta$ as well. Thus,

$$C_p \times C_p \cap \{(q,\tau(q))\colon q \in \omega^*\} = \emptyset.$$

The following example is needed to show that RK-order can be expressed in terms of almost p-compact properties.

Example 2.9. For $p \in \omega^*$, we define $\Xi_p = \omega \cup \{q \in \omega^*\colon \exists\nu < \omega_1(p \leqslant_{RK} q \leqslant_{RK} p^\nu)\}$. We claim that Ξ_p is quasi $T_{RK}(p)$-compact. In fact, let $f\colon \omega \to \Xi_p$ be with infinite image. We consider two cases.

Case I. There is a one-to-one function $\sigma\colon \omega \to \omega$ such that $f \circ \sigma[\omega] \subseteq \omega$ and $f \circ \sigma\colon \omega \to \Xi_p$ is an embedding. Then, by [CN, 9.2 (b)], we have that $p \approx_{RK} \bar{\sigma}(p) \approx_{RK} \bar{f}(\bar{\sigma}(p)) \in \Xi_p$.

Case II. There is a one-to-one function $\sigma\colon \omega \to \omega$ such that $f \circ \sigma\colon \omega \to \Xi_p$ is an embedding and $f \circ \sigma[\omega] \subseteq \omega^*$. For each $n < \omega$, choose $\nu_n < \omega_1$ such that $f(\sigma(n)) \leqslant_{RK} p^{\nu_n}$ and let $\nu = \lim \nu_n$. Hence, $p \leqslant_{RK} f(\sigma(n)) \leqslant_{RK} p^\nu$ for each $n < \omega$. Applying Lemma 2.28 (see also [B1, clause (6) p. 34]) and Lemma 2.29 of [G-F$_5$], we obtain $p \otimes p \leqslant_{RK} \bar{f}(\bar{\sigma}(p)) \leqslant_{RK} p \otimes p^\nu \leqslant_{RK} p^\mu$ for a limit ordinal $\nu < \mu < \omega_1$; hence, $\bar{f}(\bar{\sigma}(p)) \in \Xi_p$ and $\bar{\sigma}(p) \in T_{RK}(p)$.

Theorem 2.10. *For $p, q \in \omega^*$, the following are equivalent,*

(1) $p \leqslant_{RK} q$;

(2) *every almost p-compact space is almost q-compact.*

Proof. (1) $\Rightarrow$ (2). Suppose that X is almost p-compact. Let $\tau\colon \omega \to \omega$ be such that $\bar{\tau}(q) = p$ and $f\colon \omega \to X$. By 2.1, there is $\sigma\colon \omega \to \omega$ such that $\bar{\sigma}(p) \in \omega^*$ and $\bar{f}(\bar{\sigma}(p)) \in X$. Then $\bar{\sigma}(\bar{\tau}(q)) \in \omega^*$ and $\bar{f}(\bar{\sigma}(\bar{\tau}(p))) \in X$. It then follows from 2.1 that X is almost q-compact.

(2) $\Rightarrow$ (1). By 2.9, we have that Ξ_p is almost p-compact. By assumption, Ξ_p is almost q-compact. Hence, by 2.1, there is $\sigma\colon \omega \to \omega$ such that $\bar{\sigma}(q) \in \sigma^*$ and $\bar{\sigma}(p) \in \Xi_p$. So $p \leqslant_{RK} \bar{\sigma}(q) \leqslant_{RK} q$. $\square$

It is shown in [G-F$_4$, 3.4] that $T_c(p)$ is countably compact for each $p \in \omega^*$. We improve this result as follows. First, we recall the definition of the Rudin-Frolík order on ω^*:

We say that $p <_{RF} q$ if there is an embedding $e\colon \omega \to \omega^*$ such that $\bar{e}(p) = q$, for $p, q \in \omega^*$. It is known that $<_{RF}$ and $\leqslant_{RK}$ are in fact distinct and $<_{RF} \subseteq \leqslant_{RK}$ (see [CN, Chapter 16]).

Theorem 3.1. *For every $p \in \omega^*$, we have that $T_c(p)$ is almost p-compact.*

Proof. It suffices to prove that $T_c(p)$ is quasi $T_{RK}(p)$-compact. Indeed, let $f\colon \omega \to T_c(p)$. Without loss of generality, we may assume that $f[\omega]$ is infinite. It is clear that we may find an infinite subset A of ω such that $f\big|_A\colon A \to T_c(p)$ is an embedding. Now, choose a bijection $\sigma\colon \omega \to A$ and set $e = f \circ \sigma$. Then, we have that $p <_{RF} q = \bar{e}(p)$ and hence $p \leqslant_c q$. Since $\beta_p(\omega)$ is p-compact, we obtain that $q = \bar{e}(p) \in \beta_p(\omega)$. It then follows from Lemma 0.7.
(4) that $q \leqslant_c p$. Therefore, $\bar{f}(\bar{\sigma}(p)) = q \in T_c(p)$ and $\bar{\sigma}(p) \in T_{RK}(p)$. □

It is also proved in [G-F$_4$, 3.8] that if $p \in \omega^*$ is a P-point, then $T_c(p)$ is p-compact. But we could not answer the following question which is taken from [G-F$_4$, 3.9. (1)].

Question 3.2. Is $T_c(p)$ a p-compact space for every $p \in \omega^*$? The topological behavior of the Comfort types for a cardinal number α bigger than ω is only known when $p \in U(\alpha)$ is RK-minimal in α^*: If such an uniform ultrafilter p exists on α, then α is measurable (see [CN, Lemma 9.5]). In fact, the author proved in [G-F$_3$, 3.16] that $T_c(p) = \bigcup_{1 \leqslant n < \omega} T_{RK}(p^n)$ provided that $p \in U(\alpha)$ is RK-minimal in α^* and $\alpha > \omega$; hence, by Lemma 3.13, of [G-F$_3$], we have that every element of $T_c(p)$ is α-complete and so $T_c(p)$ cannot be countably compact, since no point of $T_c(p)$ is the accumulation point of a countable subset of $T_c(p)$. This leads us to ask:

Question 3.3. If $p \in U(\alpha)$ is not RK-minimal in α^*, must $T_c(p)$ be countably compact? The next question is posed in [G-F$_4$, 3.9. (3)].

Question 3.4. For $p, q \in \omega^*$, is $T_c(p) \times T_c(q)$ a countably compact space? In connection with this question we have the next Theorem. We need the following lemmas.

Lemma 3.5. *Let $p, q \in \omega^*$. If $T_c(p) \times T_c(q)$ is countably compact, then there are $s \in T_c(p)$, $t \in T_c(q)$ and $r \in \omega^*$ such that $r <_{RF} s$ and $r <_{RF} t$.*

Proof. Suppose that $T_c(p) \times T_c(q)$ is countably compact. Let $\{A_n\colon n < \omega\}$ be a partition of ω in infinite subsets. For each $n < \omega$, choose $p_n \in A_n^* \cap T_c(p)$ and

$q_n \in A_n^* \cap T_c(q)$ and define $f\colon \omega \to T_c(p) \times T_c(q)$ by $f(n) = (p_n, q_n)$ for each $n < \omega$. By assumption, there is $r \in \omega^*$ such that $\bar{f}(r) = (s,t) \in T_c(p) \times T_c(q)$. Then, we have that $\pi_0(\bar{f}(r)) = s$ and $\pi_1(\bar{f}(r)) = t$, where $\pi_i\colon \beta(\omega) \times \beta(\omega) \to \beta(\omega)$ is the projection map for $i = 0,1$. Since $\pi_0 \circ f\colon \omega \to T_c(p)$ and $\pi_1 \circ f\colon \omega \to T_c(q)$ are embeddings, $r <_{RF} s$ and $r <_{RF} t$. □

Lemma 3.6. (G-F$_4$. Theorem 2.17). *Let $p \in \omega^*$. Then p is RK-minimal iff p is C-minimal (i.e., $\leqslant_c$-minimal) and P-point.*

Lemma 3.7. (G-F$_4$). Theorem 2.10). *Let $p,q \in \omega^*$. If $p \leqslant_c q$ and p is a weak p-point, then $p \leqslant_{RK} q$.*

Theorem 3.8. *If $p,q \in \omega^*$ are RK-minimal and RK-incomparable, then $T_c(p) \times T_c(q)$ is not countably compact.*

Proof. Assume that $T_c(p) \times T_c(q)$ is countably compact. By Lemma 3.4, there are $s \in T_c(p), t \in T_c(q)$ and $r \in \omega^*$ such that, in particular, $r <_{RK} s$ and $r <_{RK} t$. According to Lemma 3.6, we have that p and q are C-minimal. Hence, $p \approx_c r \approx_c q$, but this is impossible by Lemma 3.7. □

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