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A NOTE ON MEASURABLE SETS

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We present here a strengthenning of the previous results of the author concerning the measurability of some sets with respect to an outer measure, some examples and a remark about the Hausdorff measure in abstract spaces.

Theorem 1. *Let H be a hereditary σ-ring*), γ be an outer measure on H. Let $\mathscr{R}$ be a symmetric relation on the system of all subsets of X such that $E\mathscr{R}F$, $E_1 \subset E$, $F_1 \subset F$ implies $E_1 \mathscr{R} F_1$. Let $\gamma(E \cup F) = \gamma(E) + \gamma(F)$ whenever $E\mathscr{R}F$. Let $\{V_n\}$ be a non-ascendent sequence of the sets of H, $(V_n - V_{n+1})\mathscr{R} V_{n+2}$ $(n = 1, 2, \ldots)$. Let C be a set $C \subset V_n$, $C\mathscr{R}(X - V_n)$ $(n = 1, 2, \ldots)$, $\gamma(\bigcap_{n=1}^{\infty} V_n - C) = 0$.*

Then C is γ-measurable.

Proof. Since we only modify a well-known proof (c. f. [2], Theorem 2, [3] § 11, exercise 8, [5], Theorem 1), we present only basic ideas. Let $E \in H$ be an arbitrary set. Then

$$\gamma(E - V_n) \leqq \gamma(E - C) \leqq \gamma(\bigcap_{i=1}^{\infty} V_i - C) + \gamma(E - V_{2k}) + \tag{1}$$

$$+ \sum_{i=k}^{\infty} \gamma(E \cap (V_{2i} - V_{2i+1})) + \sum_{i=k}^{\infty} \gamma(E \cap (V_{2i+1} - V_{2i+2}))$$

for all k and n. If both series in (1) converge, we get

$$\gamma(E - C) = \lim \gamma(E - V_n). \tag{2}$$

In the reverse case we use to prove (2) the following relations:

$$\gamma(E - V_{2k}) \geqq \sum_{i=2}^{k} \gamma(E \cap (V_{2i-1} - V_{2i-2})),$$

$$\gamma(E - V_{2k}) \geqq \sum_{i=1}^{k} \gamma(E \cap (V_{2i} - V_{2i-1})).$$

*) We use the terminology according to [3].

The last two inequalities follow from the relation $(V_n - V_{n+1})\mathscr{R} V_{n+2}$. From the relation $C\mathscr{R}(X - V_n)$ and from (2) it follows

$$\gamma(E) \geqq \lim \gamma((E \cap C) \cup (E - V_n)) = \gamma(E \cap C) + \lim \gamma(E - V_n) =$$
$$= \gamma(E \cap C) + \gamma(E - C)$$

for all $E \in H$.

Theorem 2. *Let H be a hereditary σ-ring of subsets of X, γ an outer measure on H, Y some γ-measurable set of H. Let $\mathscr{R}$ and γ fulfil the assumptions of Theorem 1. Let $U \subset Y$, $\{D_n\}$ be a non-descendent sequence of the sets $D_n \subset U$, $D_n \mathscr{R}(D_{n+2} - D_{n+1})$, $D_n \mathscr{R}(Y - U)$ $(n = 1, 2, \ldots)$, $\gamma(U - \bigcup_{n=1}^{\infty} D_n) = 0$.*

Then U is γ-measurable.

Proof. Put $C = Y - U$, $V_n = Y - D_n$. Then C is γ-measurable according to Theorem 1. Since Y is γ-measurable and $U = Y - C$, U is γ-measurable too.

Corollary ([2], Theorem 2). *Let (X, T) be a topological space, γ be an outer measure such that $\gamma(E \cup F) = \gamma(E) + \gamma(F)$, whenever $\bar{E} \cap \bar{F} = \emptyset$. Let A be an open set and $\{A_n\}$ be such a sequence of sets in X that $\bar{A}_n \subset A$, $\overline{A_{n+2} - A_{n+1}} \cap \cap \bar{A}_n = \emptyset$ and $\gamma(A - \bigcup_{n=1}^{\infty} A_n) = 0$. Then A is γ-measurable.*

Another corollary of our theorems (especially of Theorem 1) is the previous result of the author in [5], Theorem 1 (c. f. also [6], Theorem 1). There are a few known as well as a few unknown examples in [5], [6]. Here we present further ones.

Example 1. Let X be a linear space. We shall say that two sets A, B are separated if for every $x \in A$, $y \in B$ the segment $[x, y]$ contains a subsegment $[u, v] \subset [x, y]$ $(u \neq v)$ such that $[u, v] \subset A' \cup B'$. Let $\gamma(A \cup B) = \gamma(A) + \gamma(B)$ for any A, B which can be separated. Then the radial kernel (see [4]) of any convex set is γ-measurable.

Proof. Let A be the radial kernel of a convex set B, $a \in A$. Then $A - a$ is the radial kernel of $U = B - a$. Let p be the Minkowski functional for U. Then $\{x : p(x) < 1\} = A - a$. Hence if we define a functional f by the formula $f(x) = p(x - a)$, then $A = \{x : f(x) < 1\}$. We can put $D_n = \{x : f(x) \leqq 1 - 1/n\}$ and $E\mathscr{R}F$, whenever E, F can be separated.

Example 2. Let A be a convex body (see [4]) in a linear topological space, $\gamma(E \cup F) = \gamma(E) + \gamma(F)$, whenever $\bar{E} \cap \bar{F} = 0$. Then $\bar{A}$ is γ-measurable.

Proof. Here $\bar{A} = \{x : f(x) \leqq 1\}$ for a continuous function f (see [4], Theorem 13.2).

Example 3 ([2], Theorem 3). If f is a continuous function on a topological space X, $\gamma(E \cup F) = \gamma(E) + \gamma(F)$, whenever $\bar{E} \cap \bar{F} = \emptyset$, then $\{x : f(x) = 0\}$ is γ-measurable.

Proof. $E\mathscr{R}F$ iff $\bar{E} \cap \bar{F} = \emptyset$, $V_n = \{x : f(x) < 1/n\}$.

Example 4 ([7], Theorem 2.4). In the example 3 the condition $\bar{E} \cap \bar{F} = \emptyset$ can be replaced by a stronger one (hence a theorem with a weaker assumption holds): there are open disjoint U, V including $\bar{E}$, resp. $\bar{F}$.

Proof. $E\mathscr{R}F$ iff there are open disjoint U, V including $\bar{E}$ resp. $\bar{F}$.

Example 5. Let X be a topological space, $\gamma(E \cup F) = \gamma(E) + \gamma(F)$, whenever $\bar{E} \cap F = \emptyset$ or $E \cap \bar{F} = \emptyset$. Then every closed G_δ set is γ-measurable.

Theorem 3. *Let γ and $\mathscr{R}$ satisfy the assumptions of Theorem 1. Let $D \subset X$. Let to any $\delta > 0$ and any $T \subset X$, with $\gamma(T) < \infty$ there be V_n, $C_n \subset X$ such that $V_n \supset V_{n+1}$, $C_{n+1} \subset C_n \subset V_n$, $(V_n - V_{n+1})\mathscr{R}V_{n+2}$, $C_n \subset D$ $(n = 1, 2, \ldots)$, $\gamma(D \cap T - C_1 \cap T) < \delta/2$, $\gamma(C_n \cap T - C_{n+1} \cap T) < \delta/2^{n+1}$, $\gamma(\bigcap_{n=1}^{\infty} V_n \cap T - \bigcap_{n=1}^{\infty} C_n \cap T) = 0$.*

Then D is γ-measurable.

Proof. Put $\pi(E) = \gamma(E \cap T)$, $C = \bigcap_{n=1}^{\infty} C_n$. C is π-measurable according to Theorem 1. Further

$$\pi(D - C) \leqq \pi(D - C_1) + \sum_{i=1}^{\infty} \pi(C_i - C_{i+1}) \leqq \delta.$$

Hence to any i there is a π-measurable set $M_i \subset D$ with $\pi(D - M_i) < 1/i$. $M = \bigcup_{i=1}^{\infty} M_i$ is then π-measurable, $M \subset D$, $\pi(D - M) = 0$, therefore D is π-measurable. Now if $\gamma(K) = \infty$, then $\infty = \gamma(K) \leqq \gamma(K \cap D) + \gamma(K - D)$. If $\gamma(K) < \infty$, then put $K = T$. Then $\gamma(K) = \gamma(K \cap T) = \pi(K) = \pi(K \cap D) + \pi(K - D) = \gamma(K \cap D) + \gamma(K - D)$.

Theorem 3 is a generalization of Theorem 1. For obtaining Theorem 1 it suffices to put $C_n = D$ $(n = 1, 2, \ldots)$. Then $\gamma(\bigcap_{n=1}^{\infty} V_n \cap T - D \cap T) = 0$, $\gamma(D - D) = 0 < \delta/2^{n+1}$. The formulation of Theorem 3 is quite complicated but we can get directly from it the Bledsoe-Morse theorem on the measurability of closed G_δ (resp. open F_σ) sets in so-called γ-normal spaces (see [1], Theorem 2.17 also [5], Theorem 7).

Finally we should like to say a few words about the Hausdorff measure. The well-known classical definition was generalized in [1] for arbitrary topological spaces. We intended to generalize it further on abstract spaces (using

Theorem 1). We actually obtained some results concerning the measurability and the comparison of various definitions in a uniform space. But after reading an excellent and thoroughly exhaustive paper [8] we found that all our results had been contained explicitely or implicitely in [8].

Therefore we present here only a direct proof of a measurability theorem with respect to a Hausdorff measure in an abstract space.

Definition 1. *Let T be a set of indices, X be an abstract space, K_t $(t \in T)$ be a system of subsets of X, $\emptyset \in K_t$, τ be a non-negative function on $K = \bigcup_{t \in T} K_t$, $\tau(\emptyset) = 0$. Then for any $A \subset X$ put*

$$\nu_t(A) = \inf \{\sum_{i=1}^{\infty} \tau(A_i) : \bigcup_{i=1}^{\infty} A_i \supset A,\ A_i \in K_t\},$$

$$\nu(A) = \sup \{\nu_t(A) : t \in T\}.$$

Definition 2. *For $A \subset X$ and a family B of subsets of X put $B[A] = \cup \{E : E \in B, E \cap A \neq \emptyset\}$ and say that $B[A]$ is the B-star of A.*

Theorem 4. *ν is an outer measure (for any K_t, $t \in T$). If T is directed, $t < s$ implies $K_t \supset K_s$ and A, B are subsets of X with disjoint stars (i. e. there are s, t such that $K_t[A] \cap K_s[B] = \emptyset$), then $\nu(A \cup B) = \nu(A) + \nu(B)$.*

Proof. The first assertion is evident. Let $K_t[A] \cap K_s[B] = \emptyset$, $u > s, t$. Then $K_u[A] \subset K_t[A]$, $K_u[B] \subset K_s[B]$, $K_u[A] \cap K_u[B] = \emptyset$. Let $\nu(A \cup B) < \infty$, $\delta > 0$. Then there are $E_i \in K_u$ $(i = 1, 2, \ldots)$ such that

$$\nu_u(A \cup B) + \delta > \sum_{i=1}^{\infty} \tau(E_i) \geqq \sum_{E_i \cap A \neq 0} \tau(E_i) + \sum_{E_i \cap B \neq 0} \tau(E_i) \geqq \nu_u(A) + \nu_u(B).$$

The rest of the proof is now evident.

Theorem 5. *Let K be a system of subsets of X, $\emptyset \in K$, τ, d be two non-negative functions on K, $\tau(\emptyset) = 0$. For any $A \subset X$, $r > 0$ put*

$$h_r(A) = \inf \{\sum_{i=1}^{\infty} \tau(A_i) : \bigcup_{i=1}^{\infty} A_i \supset A,\ A_i \in K,\ d(A_i) < r\}$$

and

$$h(A) = \sup \{h_r(A) : r > 0\}.$$

Let d satisfy the following condition:

(C) *If $E_1, E_2 \in K$, $d(E_1) < r_1$, $d(E_2) < r_2$, $E_1 \cap E_2 \neq \emptyset$, then there is $E \in K$ such that $E \supset E_1 \cup E_2$ and $d(E) < r_1 + r_2$.*

Now if A is an intersection of r-stars (i. e. of sets $A_r = \cup \{E \in K : E \cap$

$\cap A \neq \emptyset,\ d(E) < r\}$), *or if more generally* $h(\cap A_r - A) = 0$, *then* A *is h-measurable.*

Proof. h is a special case of γ; $T = (0, \infty)$, $K_r = \{E \in K : d(E) < r\}$. Put $U_n = \cup \{E \in K : E \cap A \neq \emptyset, d(E) < 1/n\}$. We shall prove that U_{n+1}, U'_n can be separated. Put $R = 1/2\ (1/n - 1/(n+1))$ and construct R-stars of U_{n+1}, resp. U'_n. Denote them by U, resp. V. We assert that $U \cap V = \emptyset$.

Prove it indirectly. Let $x \in U \cap V$. Then there are $E_1, E_2 \in K$ such that $d(E_i) < R$, $E_1 \cap U_{n+1} \neq \emptyset$, $E_2 \cap U'_n \neq \emptyset$, $x \in E_1 \cap E_2$. According to the assumption there is $E_3 \in K$, $E_3 \supset E_1 \cup E_2$, such that $d(E_3) < 2R = 1/n - 1/(n+1) = \alpha$. Since $E_3 \supset E_2$, there is $y \in U'_n$, $y \in E_3$. Since $E_3 \cap U_{n+1} \neq \emptyset$, there is $E_4 \in K$, $E_4 \cap A \neq \emptyset$, $E_4 \cap E_3 \neq \emptyset$, $d(E_4) < 1/(n+1)$. Then according to the assumption there is $B \in K$, $B \supset E_4 \cup E_3$, $d(B) < 1/(n+1) + \alpha = 1/n$. Since $B \cap A \supset E_4 \cap A \neq \emptyset$, $d(B) < 1/n$, we have $B \subset U_n$. Since $B \supset E_3$, we have $y \in B$. Hence $y \in U_n$ which is a contradiction.

Now it is clear that $U_n - U_{n+1}$, U_{n+2} can be separated. The possibility to separate A, $X - U_n$ can be proved similarly. Hence we can use Theorem 1.

Remark. If K satisfies the condition (C), then K satisfies also the condition (5II) from [8], hence A is ν-measurable $= h$-measurable according to corollary 7.4 of [8]. Hence Theorem 5 is obtained implicitely in [8].

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