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SESQUIREGULAR MEASURES IN PRODUCT SPACES AND CONVOLUTION OF SUCH MEASURES

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In the theory of measure in locally compact spaces some attention has been devoted to a "nondirect" product of measures [3, 7, 8, 13, 14, 15]. In this paper some slight generalizations will be given in presence of sesquiregularity [4]. Also some applications to the direct poduct of measures and to the convolution of measures will be given.

1. Throughout, S and T denote locally compact Hausdorff spaces. We follow the terminology of [1, 2, 3, 4]. In particular, the class of Baire [Borel; weakly Borel] sets in S is the σ-ring $\mathscr{B}_0(S)$ [$\mathscr{B}(S)$, $\mathscr{B}_w(S)$] generated by the compact G_δ [compact; closed] sets in S. A weakly Borel measure τ on S [that is a measure defined on $\mathscr{B}_w(S)$ and finite for the compact sets] will be called sesquiregular if it is outer regular, $\tau(A) = \inf\{\tau(U) : U \supset A,\ U \text{ is open}\}$, $A \in \mathscr{B}_w(S)$, and if $\tau(U) = \sup\{\tau(C) : C \subset U,\ C \text{ is compact}\}$ for all open sets U. The definition of sesquiregularity coincides with the definition of regularity in [10, pp. 122 and 230; 11. p. 127] (cf. also [17]).

The following theorem gives a generalization of a theorem proved in [3], p. 139. The result is useful in the theory of spectral and vector-valued measures [6]. We confine ourselves to the finite measures. Then $\tau(A) = \sup\{\tau(C),\ C \subset A, C \text{ is compact}\}$ for all sets A in $\mathscr{B}_w(S)$ [4, Th. 3]. The following proof is a modification of the proof in [3 p. 139].

Theorem 1. *Suppose that λ is a nonnegative finite measure on the σ-algebra $\mathscr{B}_w(S) \times \mathscr{B}_w(T)$ such that* (i) *for each closed set C in S, the correspondence*

$$F \to \lambda(C \times F), \quad (F \in \mathscr{B}_w(T)),$$

is a sesquiregular weakly Borel measure on T, and (ii) *for each closed set D in T, the correspondence*

$$E \to \lambda(E \times D), \quad (E \in \mathscr{B}_w(S)),$$

is a sesquiregular weakly Borel measure on S. Then λ may be extended to one and only one sesquiregular weakly Borel measure μ_w on $S \times T$.

Proof. The uniqueness of μ_w follows from the fact that the domain of definition of λ includes the Baire sets of $S \times T$ and that every Baire [regular Borel] measure has the unique regular Borel [sesquiregular weakly Borel] extension [1, Th. 1, Sec. 62; 4, Th. 2 and Cor.].

The restriction λ_+ of λ to $\mathscr{B}(S) \times \mathscr{B}(T)$ satisfies the conditions of the theorem in [3, Th. 3] and hence there is the unique regular Borel extension λ_1 of λ_+ (coinciding with λ on $\mathscr{B}(S) \times \mathscr{B}(T)$) [$\lambda_1$ is the extension of the Baire restriction λ_0 of λ_+ [3, Th. 3]]. Let μ_w be the unique sesquiregular weakly Borel extension of λ_1 [also of λ_- and λ_0] [4, Th. 2 and Cor.]. We shall show that

$$(+) \qquad \mu_w(H) = \lambda(H),$$

for all sets H in $\mathscr{B}_w(S) \times \mathscr{B}_w(T)$.

Let $E \times F$ be a closed rectangle in $\mathscr{B}_w(S) \times \mathscr{B}_u(T)$. By the assumption (i) and by the proof of a theorem in [4, Th. 3] there is a σ-compact set D in T such that

$$\lambda(E \times F) = \lambda((E \times F) \cap D)$$

and by the assumption (ii) and [4, Th. 3] there is a σ-compact set C in S such that

$$\lambda(E \times F) = \lambda((E \times F) \cap D) = \lambda((E \cap C) \times (F \cap D) .$$

Since $E \cap C$ and $F \cap D$ are σ-compact, then $(E \cap C) \times (F \cap D)$ is in $\mathscr{B}(S) \times \times \mathscr{B}(T)$ and we have

$$\lambda(E \times F) = \lambda((E \cap C) \times (F \cap D)) = \lambda_+((E \cap C) \times (F \cap D)) =$$
$$= \mu_w((E \cap C) \times (F \cap D)) \leqq \mu_w(E \times F).$$

On the other hand, since μ_w is a finite sesquiregular weakly Borel measure on $S \times T$, then according to [4, Th. 3] there is a σ-compact set K in $S \times T$ such that

$$\mu_w(E \times F) = \mu_w((E \times F) \cap K).$$

If P_S and P_T are the projection mappings of $S \times T$ onto S and T, respectively, then we have that $P_S K$ and $P_T K$ and also $E \cap P_S K$, $F \cap P_T K$ are σ-compact, $K \subset P_S K \times P_T K$, and

$$\mu_w(E \times F) = \mu_w((E \times F) \cap K) \leqq \mu_w((E \times F) \cap (P_S K \times P_T K)) =$$
$$= \mu_w((E \cap P_S) \times (F \cap P_T K)) = \lambda_+((E \cap P_S K) \times (F \cap P_T K)) =$$
$$= \lambda((E \cap P_S K) \times (F \cap P_T K)) \leqq \lambda(E \times F).$$

Thus $\mu_w(E \times F) = \lambda(E \times F)$ for all closed rectangles.

Let now $\mathscr{R}_w(S)$ be the ring generated by the closed sets in S. Every set in $\mathscr{R}_w(S)$ is a finite disjoint union of "proper differences" $C - C^*$, where C and C^* are closed sets such that $C \supset C^*$ [1, Th. 1, Sec. 58]. Similarly for $\mathscr{R}_w(T)$.

Let $\mathscr{R}_w(S \times T)$ be the ring generated by the class of all rectangles $E \times F$ with sides in $\mathscr{R}_w(S)$ and $\mathscr{R}_w(T)$, respectively. Each set in $\mathscr{R}_w(S \times T)$ can be written as a finite dijoint union of sets of the form

$$(C - C^*) \times (D - D^*),$$

where both of the indicated differences are proper [1, Th. 1, Sec. 34]. Such a set can be written in the form

$$(1) \qquad (C \times D - C \times D^*) - (C^* \times D - C^* \times D^*),$$

where each of the indicated differences is proper.

We have verified $(+)$ for rectangles $H = E \times F$ with closed sides; it follows from (1) that $(+)$ holds for all sets in $\mathscr{R}_w(S \times T)$ [9, p. 37] and therefore for all sets in the σ-ring generated by $\mathscr{R}_w(S \times T)$ [9, p. 54], in other words for all sets in $\mathscr{B}_w(S) \times \mathscr{B}_w(T)$ [1, p. 118).

The following theorem represents a generalization of a result proved in [13].

Theorem 2. *Suppose that λ is a nonnegative finite set function defined on the system of the sets of the form $E \times F$, $E \in \mathscr{B}_w(S)$, $F \in \mathscr{B}_w(T)$ such that*

(i) *for each E in $\mathscr{B}_w(S)$, the correspondence ${}_E\lambda : F \to \lambda(E \times F)$ is a sesquiregular weakly Borel measure on T,*

(ii) *for each F in $\mathscr{B}_w(T)$, the correspondence $\lambda_F : E \to \lambda(E \times F)$ is a sesquiregular weakly Borel measure on S.*

Then λ is σ-additive on the system of the sets $E \times F \in \mathscr{B}_w(S) \times \mathscr{B}_w(T)$ and on $\mathscr{B}_w(S \times T)$ there is one and only one sesquiregular weakly Borel measure μ_w coinciding with λ for $E \times F$ in $\mathscr{B}_w(S) \times \mathscr{B}_w(T)$.

Proof. Denote by λ_1 the unique additive extension of λ to the ring $\mathscr{R}$ generated by the sets of the form $E \times F$, E in $\mathscr{B}_w(S)$, F in $\mathscr{B}_w(T)$. We shall prove, in a standard manner, that λ_1 is σ-additive on $\mathscr{R}$. To prove the σ-additivity of λ_1 take an arbitrary decreasing sequence G_n, $G_n \in \mathscr{R}$, $n = 1, 2, \ldots$ of the form

$$G_n = \bigcup_{i\ 1}^{k_n} E_i^n \times F_i^n$$

with $0 < \varepsilon < \lambda(G_n)$, $n = 1, 2, \ldots$

From the inner regularity of λ and λ_F it follows [4, Th. 3 or 10, Th. 2.40] that for every n there exist compact sets C_i^n, D_i^n, $C_i^n \subset E_i^n$, $D_i^n \subset F_i^n$, $i = 1, \ldots, k_n$, such that

$$\lambda_1(G_n - Y_n) < \frac{\varepsilon}{2^n}, \ n = 1, 2, \ldots,$$

where

$$Y_n = \bigcup_{i=1}^{k_n} C_i^n \times D_i^n.$$

Denote

$$X_n = \bigcap_{i=1}^{n} Y_i.$$

Then $m \leqq n$ implies $X_n \subset X_m$ and

$$\lambda_1(G_n - X_n) = \lambda_1 (\bigcup_{i=1}^{n} (G_n - Y_i)) \leqq \sum_{i=1}^{k_n} \lambda_1(G_i - Y_i) < \varepsilon.$$

It follows that $\lambda_1(X_n) > 0$, $n = 1, 2, \ldots$, that is the sets X_n are nonempty and $X_{n+1} \subset X_n$. Since X_n are compact we have $\bigcap_{n=1}^{\infty} G_n \supset \bigcap_{n=1}^{\infty} X_n \neq \emptyset$. From this and from the finite additivity of λ_1 the σ-additivity of λ_1 follows.

The measure λ_1 has the unique extension to the measure ν on the σ-algebra $\mathscr{B}_w(S) \times \mathscr{B}_w(T)$ generated by $\mathscr{R}$. The measure ν fulfils the assumptions of Theorem 1 and thus we may complete the proof using Theorem 1.

Both Theorem 1 and Theorem 2 involve a measure defined on the σ-algebra $\mathscr{B}_w(S) \times \mathscr{B}_w(T)$ that is not a weakly Borel measure if $\mathscr{B}_u(S) \times \mathscr{B}_w(T) \neq \mathscr{B}_w(S \times T)$. Nevertheless this measure is regular in the following sense [cf. 10, Th. 21.18].

Theorem 3. *Let λ be a nonnegative finite set function defined on the system of the sets $E \times F \in \mathscr{B}_w(S) \times \mathscr{B}_w(T)$ such that the assumptions* (i) *and* (ii) *of Theorem* 2 *are satisfied. Let τ be the extension of τ to the measure on $\mathscr{B}_u(S) \times \mathscr{B}_w(T)$ existing according to Theorem 2. Then τ is regular on $\mathscr{B}_w(S) \times \mathscr{B}_w(T)$ in the sense that*

(a) $\tau(E) = \inf\{\tau(U) : E \subset U,\ U \in \mathscr{B}_w(S) \times \mathscr{B}_w(T),\ U \text{ is open}\}$,

(b) $\tau(E) = \sup\{\tau(F) : E \supset F,\ F \in \mathscr{B}_w(S) \times \mathscr{B}_w(T),\ F \textit{ is compact}\}$.

Proof. [cf. 10, Th. 21.18]. Let $\mathscr{R}$ be the family of all sets $E \in \mathscr{B}_u(S) \times \mathscr{B}_u(T)$ for which the assertions (a) and (b) hold. We will prove that $\mathscr{R} = \mathscr{B}_u(S) \times \mathscr{B}_w(T)$. It easy to prove that $\mathscr{R}$ is closed under the formation of countable unions.

We shall prove that $\mathscr{R}$ contains each rectangle $E \times F \in \mathscr{B}_w(S) \times \mathscr{B}_u(T)$. By the assumptions (i) and (ii) we have

$$\tau(E \times F) = \lambda(E \times F) = \sup\{\lambda(C \times F) : C \subset E,\ C \text{ is compact}\}$$

$$\sup\{\lambda(C \times D) : C \subset E,\ D \subset F,\ C \text{ and } D \text{ are compact}\} \leq$$

$$\leqq \sup \{\tau(K) : K \in \mathscr{B}_w(S) \times \mathscr{B}_w(T),\ K \subset E \times F,\ K \text{ is compact}\}.$$

Let $K \subset E \times F$ be an arbitrary compact set in $\mathscr{B}_w(S) \times \mathscr{B}_w(T)$. We have

$$\tau(K \ \leqq \tau(E \times F),$$

$$\sup \{\tau(K) : K \subset E \times F,\ K \in \mathscr{B}_u(S) \times \mathscr{B}_w(T),\ K \text{ is compact}\} \leqq \tau(E \times F).$$

Further we have

$$\tau(E \times F) = \inf \{\lambda(E \times V) : V \subset F,\ V \text{ is open}\} =$$

$$= \inf \{\lambda(U \times V) : U \subset E,\ V \subset F,\ U \text{ and } V \text{ are open}\} \geqq$$

$$\geqq \inf \{\lambda(O) : O \supset E \times F,\ O \in \mathscr{B}_w(S) \times \mathscr{B}_w(T),\ O \text{ is open}\}.$$

On the other hand, let O be an arbitrary open set in $\mathscr{B}_w(S) \times \mathscr{B}_w(T)$, $O \supset$ $\supset E \times F$. We have

$$\tau(E \times F) \leqq \tau(O),\ \ \tau(E \times F) \leqq \inf \{\tau(O) : O \supset E \times F,\ O \in \mathscr{B}_w(S) \times \mathscr{B}_w(T),\ O$$

is open$\}$.
Thus we have

$$\tau(E \times F) \quad \inf \{\tau(O) : O \in \mathscr{B}_w(S) \times \mathscr{B}_w(T),\ O \supset E \quad F,\ O \text{ is open}\},$$

$$\tau(E \times F) \quad \sup \{\tau(K) : K \in \mathscr{B}_w(S) \times \mathscr{B}_w(T),\ K \subset E \times F,\ K \text{ is compact}\}.$$

We shall prove that $\mathscr{R}$ is closed under complementation. Let $E \in \mathscr{R}$ and let $\varepsilon > 0$ be given. There is a compact set F and an open set U, both in $\mathscr{B}_u(S) \times$ $\times \mathscr{B}_w(T)$ such that $F \subset E \subset U$ and $\lambda(U \cap F^c) < \varepsilon$. Since $S \times T \in \mathscr{R}$, there exists a compact set K in $\mathscr{B}_w(S) \times \mathscr{B}_w(T)$ such that $K \subset S \times T$ and $\tau(K^c) < \varepsilon$. Now F^c is open and $K \cap U^c$ is compact. Further it is clear that $K \cap U^c \subset$ $\subset E^c \subset F^c$ and that

$$\tau(F^c) - \tau(K \cap U^c) - \tau(F^c \cap (K \cap U^c)^c) \leqq \tau(F^c \cap K^c) + \tau(F^c \cap U) < \varepsilon +$$
$$+ \varepsilon \quad 2\varepsilon.$$

This proves that E^c is in $\mathscr{R}$ if E i in $\mathscr{R}$. Thus we have $\mathscr{R} = \mathscr{B}_w(S) \times \mathscr{B}_w(T)$ and the proof is completed.

2. We shall give some applications of the preceding theorems. It is known that $\mathscr{B}_w(S) \times \mathscr{B}_w(T) \subset \mathscr{B}_w(S \times T)$, where the inclusion can be proper [3, p. 136]. Now, if we have two sesquiregular weakly Borel measures μ and ν on S and T, respectively, then their product $\lambda \quad \mu \times \nu$, as defined in [1 or 9] (that is regular in the sense as in Theorem 3 [10, Th. 21.18]) cannot be a weakly Borel measure on $S \times T$ when $\mathscr{B}_w(S) \quad \mathscr{B}_w(T) \neq \mathscr{B}_w(S \times T)$. In order to obtain a weakly Borel measure we may use Theorem 1 or Theorem 2. Namely, we may take the unique sesquiregular weakly Borel extension of $\lambda = \mu \quad \nu$. It is easy to verify that the conditions of Theorem 1 (or those of Theorem 2) are satisfied using the fact that

$$\lambda(E \times F) = \mu(E)\nu(F),$$

for all rectangles with closed (weakly Borel) sides. Thus we have the following.

Theorem 4. *Let μ be a finite nonnegative sesquiregular weakly Borel measure on S and ν be a finite nonnegative sesquiregular weakly Borel measure on T. Then there is one and only one sesquiregular weakly Borel measure λ_w on $S \times T$ that extends $\lambda = \mu \times \nu$. More explicitely, we will write $\lambda_w = \mu \otimes \nu$.*

3. A complex measure μ defined on the σ-ring $\mathscr{B}_0(S)$ [the σ-ring $\mathscr{B}(S)$; the σ-algebra $\mathscr{B}_w(S)$] is said to be a complex regular Baire [regular Borel: sesquiregular weakly Borel] measure on S if its total variation, $|\mu|$ [necessarily bounded, [10, p. 360]] is a Baire [regular Borel; sesquiregular weakly Borel] measure on S. We may now give a generalization of Theorem 1 (and also of Theorem 2) for complex measures.

Theorem 5. *Suppose that λ is a complex measure on the σ-algebra $\mathscr{B}_w(S) \times \times \mathscr{B}_w(T)$ such that* (i) *for each closed set C in S, the correspondence*

$$_C\lambda : F \to \lambda(C \times F), \quad (F \in \mathscr{B}_w(T)),$$

is a complex sesquiregular weakly Borel measure on T and (ii) *for each closed set D in F, the correspondence*

$$\lambda_D : E \to \lambda(E \times D), \quad (E \in \mathscr{B}_w(S)),$$

is a complex sesquiregular weakly Borel measure on S.

Then λ may be extended to one and only one complex sesquiregular weakly Borel measure μ on $S \times T$.

Proof. Let $\sum_{j=1}^{4} a_j\lambda_j$ be the Jordan decomposition of λ [10, p. 311]. The measures λ_j, $j = 1, 2, 3, 4$ all satisfy the conditions of Theorem 1 using the fact that $\nu = \sum_{j=1}^{4} a_j\nu_j$ is a complex sesquiregular weakly Borel measure if and only if all ν_j, $j = 1, 2, 3, 4$ [or $|\nu|$] are nonnegative sesquiregular weakly Borel measures [10, p. 360]. Take now a sesquiregular weakly Borel extension μ_j of λ_j, $j = 1, 2, 3, 4$. Then $\sum_{j=1}^{4} a_j\mu_j$ gives a required measure μ.

It is also possible to give a generalization of Theorem 2 for a complex case. In this case one supposes that λ is bounded on the algebra generated by the sets of the form $E \times F$, $E \in \mathscr{B}_w(S)$, $F \in \mathscr{B}_w(T)$ [8, p. 242].

From Theorem 5 we obtain the following.

Theorem 6. *Let μ be a complex sesquiregular weakly Borel measure on S and ν be a complex sesquiregular weakly Borel measure on T. Then there is one and only one complex sesquiregular weakly Borel measure λ_w on $S \times T$ which extends $\lambda = \mu \times \nu$. We will write, more explicitely, $\lambda_w = \mu \otimes \nu$.*

From the Riesz representation theorem [10, p. 346] it follows that the measure $\mu \otimes \nu$ from Theorem 6 is the unique complex sesquiregular weakly Borel measure on $S \times T$ such that for every continuous function f on $S \times T$ vanishing at infinity [i. e. $f \in C_0(S \times T)$] we have

$$\int_{S \times T} f \mathrm{d}\mu \otimes \nu = \int_T \{ \int_S f(s, t) \mathrm{d}\mu(s) \} \mathrm{d}\nu(t) = \int_S \{ \int_T f(s, t) \} \mathrm{d}\nu(t) \mathrm{d}\mu(s).$$

In particular, the measure $\mu \otimes \nu$ coincides with the product measure constructed in [11, p. 182]. Therefore from [11, Th. 14.24] we have the following.

Theorem 7. *Let μ, ν and $\mu \otimes \nu$ be as in Theorem 6. Then we have $|\mu \otimes \nu| = |\mu| \otimes |\nu|$.*

4. We shall give some connections with Fubini's theorem. We recall that the real-valued function on S is called the weakly Borel function if it is measurable with respect to the σ-algebra of weakly Borel sets [1, p. 181]. If μ, ν and $\mu \otimes \nu$ are as in Theorem 6 and f is a bounded weakly Borel function on $S \times T$, then f is $|\mu \otimes \nu|$-integrable and it follows from Fubini's theorem in [11, Th. 14.25] that $f(s, t)$ qua function of s is $|\mu|$-integrable for $|\nu|$-almost all $t \in T$ and the function $t \to \int_S f(s, t) \mathrm{d}\mu(s)$ is $|\nu|$-integrable and we have

$$\int_{S \times T} f \mathrm{d}\mu \otimes \nu = \int_S \{ \int_T f(s, t) \mathrm{d}\nu(t) \} \mathrm{d}\mu(s) = \int_T \{ \int_S f(s, t) \mathrm{d}\mu(s) \} \mathrm{d}\nu(t).$$

In particular, if G is a weakly Borel set in $S \times T$, we have

$$\mu \otimes \nu(G) = \int_{S \times T} \chi_G \mathrm{d}\mu \otimes \nu = \int_S \{ \int_T \chi_G(s, t) \mathrm{d}\nu(t) \} \mathrm{d}\mu(s) = \int_T \{ \int_S \chi_G(s, t) \mathrm{d}\mu(s) \} \mathrm{d}\nu(t).$$

Sometimes it may be interesting to know that if f is a bounded weakly Borel function on $S \times T$, then $s \to f(s, t)$ is a bounded weakly Borel function on S and $t \to \int_S f(s, t) \mathrm{d}\mu(s)$ is a bounded weakly Borel function on T.

If P_S and P_T are the projection mappings of $S \times T$ onto S and T, respectively, then

$$G^t = \{ s \in S : (s, t) \in G \} = P_S[G \cap (S \times \{t\})],$$

$$G_s = \{ t \in T : (s, t) \in G \} = P_T[G \cap (\{s\} \times T)].$$

If $G \in \mathscr{B}_w(S) \times \mathscr{B}_w(T)$, then $G^t \in \mathscr{B}_w(S)$ for all $t \in T$, and $G_s \in \mathscr{B}_w(T)$ for all $s \in S$ [9, 34A]. The same result holds for any Borel set [12] and we shall show that for any weakly Borel set G in $S \times T$, too.

Lemma 1. *If $G \in \mathscr{B}_w(S \times T)$, then $G^t \in \mathscr{B}_w(S)$ for all $t \in T$, and $G_s \in \mathscr{B}_w(T)$ for all $s \in S$.*

Proof. Let $\mathscr{R}$ be the class of all $G \subset S \times T$ such that $G^t \in \mathscr{B}_w(S)$ for all $t \in T$, and $G_s \in \mathscr{B}_w(T)$ for all $s \in S$. Since sections preserve countable unions and set-theoretic differences, $\mathscr{R}$ is a σ-algebra. We shall show that $\mathscr{R}$ contains the closed sets in $S \times T$.

Let G be a closed set in $S \times T$, and let $t \in T$. Then $S \times \{t\}$ is closed, the restriction $P_S \mid_{S \times \{t\}}$ of P_S on $S \times \{t\}$ is a homeomorphism from $S \times \{t\}$ onto S, and

$$G^t = P_S[G \cap (S \times \{t\})],$$

is closed because $G^t = P_S \mid_{S \times \{t\}} G$. Hence $G^t \in \mathscr{B}_w(S)$ for all $t \in T$. Similarly, $G_s \in \mathscr{B}_w(T)$ for all $s \in S$. Hence $G \in \mathscr{R}$ and Lemma 1 then follows.

Lemma 2. *Let f be a weakly Borel function on $S \times T$. Then $f_s : t \to f(s, t)$ is a weakly Borel function on T and $f^t : s \to f(s, t)$ is a weakly Borel function on S, for all $s \in S$ and $t \in T$.*

Proof. If G is any set of real numbers, then $(f_s)^{-1}(G) = (f^{-1}(G))_s$ and $(f^t)^{-1}(G) = (f^{-1}(G))^t$. The lemma now follows directly from Lemma 1.

Let now G be any set in $\mathscr{B}_w(S \times T)$. Then the characteristic function χ_G is a bounded nonnegative weakly Borel function on $S \times T$. For complex sesquiregular weakly Borel measures μ and ν we may write

$$\int_T \chi_G(s, t) \mathrm{d}\nu(t) = \int_T \chi_{G_s}(t) \mathrm{d}\nu(t) = \nu(G_s),$$

$$\int_S \chi_G(s, t) \mathrm{d}\mu(s) = \int_S \chi_{G^t}(s) \mathrm{d}\mu(s) = \mu(G^t).$$

We wish to prove that the function $f_G : s \to \nu(G_s)$ is weakly Borel on S, and $h_G : t \to \mu(G^t)$ is weakly Borel on T.

Theorem 8. *Let μ and ν be complex sesquiregular weakly Borel measures on S and T, respectively. Then $f_G : s \to \nu(G_s)$ is a weakly Borel function on S and $h_G : t \to \mu(G^t)$ is a weakly Borel function on T for every $G \in \mathscr{B}_w(S \times T)$.*

Proof. We may suppose that μ and ν are nonnegative measures. It will suffice to prove that f_G is a weakly Borel function for all open sets in $S \times T$.

Suppose G is a nonvoid open set in $S \times T$. Let F be the set of all functions $f \in C_{00}^+(S \times T)$ [i. e. nonnegative continuous functions on $S \times T$ with the compact support] such that $f \leqq \chi_G$. Since G is an open set, Urysohn's theorem [10, Th. 6.80] implies that $\sup\{f : f \in F\} = \chi_G$. For every $f \in F$, for each fixed $s \in S$, the function $t \to f(s, t)$ is in $C_{00}^+(T)$, the function $s \to \int_T f(s, t) \mathrm{d}\nu(t)$ is in $C_{00}^+(S)$. Further for each fixed $s_0 \in S$ we have

$$\chi_G(s_0, t) = \sup\{f(s_0, t) : f \in F\}, \text{ for all } t \in T.$$

Every function $t \to f(s_0, t)$ is in $C^+_{00}(T)$. It is obvious that the set of functions

$$\{s \to \int_T f(s, t) \mathrm{d}\nu(t) : f \in F\},$$

is directed upward. Applying [10, Th. 9.11 and Th. 12.35] we have

$$\nu(G_{s_0}) = \int_T \chi_G(s_0, t) \mathrm{d}\nu(t) = \sup \{ \int_T f(s_0, t) \mathrm{d}\nu(t) : f \in F\}.$$

This being true for all $s_0 \in S$ we have that the function

$$s \to \int_T \chi_G(s, t) \mathrm{d}\nu(t) = \nu(G_s),$$

is lower semicontinuous [10, Th. 7.22] and hence is a weakly Borel function [10, Cor. 11.5].

According to Theorem 8 we may form the iterated integrals

$$\int_S \{ \int_T \chi_G(s, t) \mathrm{d}\nu(t)\} \mathrm{d}\mu(s), \quad \int_T \{ \int_S \chi_G(s, t) \mathrm{d}\mu(s)\} \mathrm{d}\nu(t).$$

Define a set function $\mu . \nu$ by the relation

$$\mu . \nu(G) = \int_S \{ \int_T \chi_G(s, t) \mathrm{d}\nu(t)\} d\mu(s), \quad G \in \mathscr{B}_w(S \times T).$$

It is obvious that $\mu . \nu$ is a weakly Borel measure on $S \times T$. We shall prove that $\mu . \nu = \mu \otimes \nu$ on $\mathscr{B}_w(S \times T)$.

Theorem 9. *Let μ and ν be complex sesquiregular weakly Borel measures on S and T, respectively. Then for all $G \in \mathscr{B}_w(S \times T)$ we have $\mu . \nu(G) = \mu \otimes \nu(G)$.*

Proof. Take μ and ν nonnegative. It will suffice to prove that $\mu . \nu(G) = \mu \otimes \nu(G)$ for all open sets in $S \times T$. Let F be the set of functions from the proof of Theorem 8. It is a corollary of the Stone-Weierstrass theorem that every $f \in F$ is $\mathscr{B}_w(S) \times \mathscr{B}_w(T)$-measurable and f is also $\mu \times \nu$-integrable on $S \quad T$, and we may use Fubini's theorem. The measure $\mu \otimes \nu$ coincides with $\mu \times \nu$ on the σ-algebra $\mathscr{B}_w(S) \times \mathscr{B}_w(T)$. Applying, similarly as in proving Theorem 8, [10, Th. 9.11] we have

$$\mu \otimes \nu(G) = \int_{S \times T} \chi_G \mathrm{d}\mu \otimes \nu = \sup \{ \int_{S \times T} f \mathrm{d}\mu \otimes \nu : f \in F\} =$$

$$= \sup \{ \int_{S \ T} f \mathrm{d}\mu \times \nu : f \in F\} = \sup \{ \int_T \{ \int_S f(s, t) \mathrm{d}\mu(s)\} \mathrm{d}\nu(t) : f \in F\} =$$

$$= \int_T \{ \int_S \chi_G(s,t) \mathrm{d}\mu(s)\} \mathrm{d}\nu(t) = \mu . \nu(G).$$

Let $\mathscr{R}$ be the collection of all weakly Borel subsets H of $S \times T$ for which $\mu . \nu(H) = \mu \otimes \nu(H)$ holds. By the monotone convergence theorem $\mathscr{R}$ is a monotone class in the sense of [9, p. 26]. Since $\mathscr{R}$ contains all open sets of $S \times T$, then $\mathscr{R}$ obtains all weakly Borel sets of $S \times T$.

We have thus

$$\mu \otimes \nu(G) = \mu . \nu(G) = \int_S \{ \int_T \chi_G(s, t) \mathrm{d}\nu(t) \} \mathrm{d}\mu(s) = \int_T \{ \int_S \chi_G(s, t) \mathrm{d}\mu(s) \} \mathrm{d}\nu(t),$$

for all $G \in \mathscr{B}_w(S \times T)$. We have thus the following.

Theorem 10. *Let μ be a complex sesquiregular weakly Borel measure on S and ν be a complex sesquiregular weakly Borel measure on T. Let f be a bounded weakly Borel function on $S \times T$. Then*

(1) $$s \to \int_T f(s, t) \mathrm{d}\nu(t),$$

is a weakly Borel function on S and

(2) $$\int_{S \times T} f \mathrm{d}\mu \otimes \nu = \int_S \{ \int_T f(s, t) \mathrm{d}\nu(t) \} \mathrm{d}\mu(s).$$

Proof. The assertion of Theorem 10 is valid for characteristic functions of weakly Borel sets. Since each bounded weakly Borel function on $S \times T$ is a uniform limit of linear combinations of χ_E for E weakly Borel set, the assertion is valid for all bounded weakly Borel functions.

5. Let now G be a locally compact Hausdorff group; μ and ν complex sesquiregular weakly Borel measures on G. Their convolution $\mu * \nu$ is a complex sesquiregular weakly Borel measure on G which can be defined in two equivalent ways [16]. The first definition uses the Riesz representation theorem and $\mu * \nu$ is taken to be the unique complex sesquiregular weakly Borel measure on G such that

$$\int_G f(z) \mathrm{d}\mu * \nu(z) = \int_G \{ \int_G f(st) \mathrm{d}\mu(s) \} \mathrm{d}\nu(t) = \int_{G \times G} f(st) \mathrm{d}\mu \bigcirc \nu(s, t),$$

for all continuous functions f on G which vanish at infinity. In the second definition, for each weakly Borel subset D of G, $\mu * \nu\ (D)$ is defined to be $\mu \otimes \nu(E)$, where $E = \{(s, t) : st \in D\}$ and $\mu \otimes \nu$ is the unique complex sesquiregular weakly Borel measure on $G \times G$ satisfying

$$\int_{G \times G} g \mathrm{d}(\mu \otimes \nu) = \int_G \{ \int_G g(s, t) \mathrm{d}\mu(s) \} \mathrm{d}\nu(t),$$

for all continuous functions on $G \times G$ vanishing at infinity. The second definition makes it possible to give $\mu * \nu(D)$ explicitly.

Theorem 11. *Let G be a locally compact Hausdorff group; μ and ν be complex sesquiregular weakly Borel measures on G. Then, for each weakly Borel subset D of G,*

$$t \to (Dt^{-1}),\ s \to (s^{-1}D), \tag{1}$$

are weakly Borel functions on G and

$$\mu * \nu(D) = \int_G \mu(Dt^{-1})\mathrm{d}\nu(t) = \int_G \nu(s^{-1}D)\mathrm{d}\mu(s). \tag{2}$$

Remark. The formula (2) is stated, for D weakly Borel, in [16, p. 351] but no assertion about the measurability of (1) is made and no proof is given. The proof that (1) is $|\nu|$-integrable, resp. $|\mu|$-integrable and that of (2) is given in [11, p. 269]. Our contribution here is the proof of the weak Borel measurability of (1). For compact spaces this is done in [5].

Proof. Let E be the subset $\{(s, t) : st \in D\}$ of $G \times G$. E is weakly Borel since the mapping $(s, t) \to st$ of $G \times G$ into G is continuous. We have

$$\int_G \chi_E(s, t)\mathrm{d}\mu(s) = \mu(Dt^{-1})$$

and the weak Borel measurability of (1) follows from the first assertion of Theorem 10 and (2) is a consequence of the second assertion of Theorem 10 and the fact that

$$\mu * \nu(D) = \mu \otimes \nu(E) = \int_G \int_G \chi_E \mathrm{d}\mu \otimes \nu = \int_G \{\int_G \chi_E(s, t)\mathrm{d}\mu(s)\}\mathrm{d}\nu(t).$$

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