Anton Dekrét

On a Pair of Connections on a Principal Fibre Bundle

Matematický časopis, Vol. 24 (1974), No. 1, 59--68

Persistent URL: http://dml.cz/dmlcz/127060

Mat. čas. 24, 1974, No 1, 59—68

ON A PAIR OF CONNECTIONS ON A PRINCIPAL FIBRE BUNDLE

ANTON DEKRÉT

Kolář [3] introduced the difference tensor $\Delta(X)$ of an arbitrary semi-holonomic jet X. In this paper it is first shown that the mapping $X \to \Delta(X)$ can be extented on some subset of the non-holonomic jets. Futher, some properties of a pair of the connections on a principal fibre bundle are found. All our considerations are in the category C^∞. We use the standart terminology and notations of the theory of jets (see [2]) with the following notational conventions. We write $j^r_{x_0}(y) = j^r_{x_0}(x \to y)$ for a fixed y and j^k_r, $k < r$, denotes the natural projection of $\tilde{J}^r(M, N)$ into $\tilde{J}^k(M, N)$.

1. Let V, M, N be real manifolds. Let t^s or x^i or y^p be the local coordinates on V, or on M, or on N determined by local charts τ, or ξ, or ζ, respectively. Denote by $(t^s, x^i_{00}, x^i_{s_10}, x^i_{0s_2}, x^i_{s_1s_2})$ or $(x^i, y^p_{00}, y^p_{i_20}, y^p_{0i_2}, y^p_{i_1i_2})$, where s, s_1, $s_2 = 1, \ldots, \dim V = v$; i, i_1, $i_2 = 1, \ldots, \dim M = m$; $p = 1, \ldots, \dim N = n$, the natural coordinates on $\tilde{J}^2(V, M)$, or $\tilde{J}^2(M, N)$, respectively (see [5]). Let $X = (t^s, x^i_{00}, x^i_{s_10}, x^i_{0s_1}, x^i_{s_1s_2}) \in \tilde{J}^2(V, M)$, $Y = (x^i, y^p_{00}, y^p_{i_10}, y^p_{0i_2}, y^p_{i_1i_2}) \in \tilde{J}^2_{\beta X}(M, N)$. Then the composition $Z = YX \in \tilde{J}^2(V, N)$ has the coordinates $(t^s, z^p_{00}, z^p_{s_10}, z^p_{0s_2}, z^p_{s_1s_2})$, where

$$z^p_{s_10} = y^p_{i_10}x^{i_1}_{s_10}, \; z^p_{0s_2} = y^p_{0i_2}x^{i_2}_{0s_2}, \tag{1}$$

$$z^p_{s_1s_2} = y^p_{i_1i_2}x^{i_1}_{s_10}x^{i_2}_{0s_2} + y^p_{i_10}x^{i_1}_{s_1s_2}.$$

Lemma 1. *Let $X = (x^i, y^p_{00}, y^p_{i0}, y^p_{0i_2}, y^p_{i_1i_2}) \in \tilde{J}^2(M, N)$. Denote by $\Delta(X)$ the set of real numbers $y^p_{[i_1i_2]} = y^p_{i_1i_2} - y^p_{i_2i_1}$. Then $\Delta(X)$ is an element of $T_{\beta X}(N) \otimes \times \Lambda^2 T^*_{\alpha X}(M)$ if and only if*

$$y^{p_1}_{i_10}y^{p_2}_{0i_2} = y^{p_1}_{0i_1}y^{p_2}_{i_20}. \tag{2}$$

Proof. Let $a \in H^2_{\alpha X}(M)$, $b \in H^2_{\beta X}(N)$ be the holonomic 2-frames determined by local charts ξ, or ζ, respectively. Then the jet $b^{-1}Xa$ has the coordinates $(y^p_{i0}, y^p_{0i_2}, y^p_{i_1i_2})$. Let $A = (a^i_{i_1}, a^i_{i_1i_2}) \in L^2_m$, $B = (b^p_{p_1} = b^p_{p_10} = b^p_{0p_1}, b^p_{p_1p_2}) \in L^2_n$, $p_1, p_2 = 1, \ldots, n$. Let $Bb^{-1}XaA$ have the coordinates $(c^p_{i_10}, c^p_{0i_2}, c^p_{i_1i_2})$.

It is necessary to show that

$$(3) \qquad (c^p_{[i_1 i_2]} = b^p_{p_1} y^{p_1}_{[k_1 k_2]} a^{k_1}_{i_1} a^{k_2}_{i_2}) \Leftrightarrow (2).$$

Using (1), we obtain

$$c^p_{[i_1 i_2]} = b^p_{p_1 p_2} a^{k_1}_{i_1} a^{k_2}_{i_2} (y^{p_1}_{k_1 0} y^{p_2}_{0 k_2} - y^{p_1}_{0 k_1} y^{p_2}_{k_2 0}) + b^p_{p_1} y^{p_1}_{[k_1 k_2]} a^{k_1}_{i_1} a^{k_2}_{i_2},$$

where $k, k_1, k_2 = 1, \ldots, m$. That is why (3) is correct for any $A \in L^2_m$, $B \in L^2_n$ if and only if the jet X has the property (2).

Definition 1. *The non-holonomic jets having the property* (2) *will be said to be quasi-semi-holonomic. The tensor $\Delta(Y)$ determined by the quasi-semi-holonomic jet Y will be called the difference tensor of Y. If $\Delta(y) = 0$, we shall say that Y is quasi-holonomic.*

Remark. Let $Y \in \tilde{J}^2(M, N)$, $Y = j^1_{\alpha(Y)}\sigma$. Then the jets $j^1_2 Y$ and $l^1_2(Y) = j^1_{\alpha(Y)}(\beta\sigma)$ determine the homomorfisms

$$L(j^1_2 Y),\ L(l^1_2(Y)) \in Hom\ (T_{\alpha(Y)}(M), T_{\beta(Y)}(N)).$$

It is easy to see that Y has the property (2) if and only if $L(j^1_2 Y)[T_{\alpha Y}(M)] = 0$ or if there is such a real number λ that

$$L(l^1_2(Y)) = \lambda L(j^1_2 Y).$$

If $L(j^1_2 Y)[T_{\alpha Y}(M)] \neq 0$ and $L(l^1_2(Y)) = \lambda L(j^1_2 Y)$, the jet Y will be said to be *quasi-semi-holonomic with the coefficient* λ. In the case of $L(j^1_2 Y)[T_{\alpha Y}(M)] = 0$, Y will be called *quasi-semi-holonomic without a coefficient.* We introduce two examples. Let $X \in J^1(M, N)$, $X = j^1_{\alpha X}\sigma$, then $X^{(2)} = j^1_{\alpha X}(u \to j^1_u[\sigma(u)])$ is quasi-semi-holonomic without a coefficient. Further, denote by $J^1(M, N)_y$ the set of 1-jets of M into N with the target $y \in N$. Then $Y = j^1_{\alpha Y}\sigma$, where σ is a local cross-section of the fibre manifold $(J^1(M, N)_y, \alpha, M)$, is quasi-semi-holonomic with the coefficient 0.

Some properties of the difference tensor $\Delta(Y)$, formulated in [4] for the semi-holonomic case, can be easy generalized for the quasi-semi-holonomic case.

Lemma 2. *Let $X \in \tilde{J}^2(V, M)$, $Y \in \tilde{J}^2_{\beta X}(M, N)$ be quasi-semi-holonomic with the coefficients λ_1, λ_2 (one of them is without a coefficient). Then YX is quasi-semi-holonomic with the coefficient $\lambda_1 . \lambda_2$ (is without coefficient) and*

$$(4) \qquad \Delta(YX) = \lambda_1 \Delta(Y) L(j^1_2 X) + L(j^1_2 Y)\Delta(X).$$

Using (1), the proof is clear.

Now, let $X \in \tilde{J}^2_x(M, W)$, $Y \in \tilde{J}^2_x(M, N)$, $(X, Y) \in \tilde{J}^2_x(M, W \times N)$. If X, Y are quasi-semi-holonomic, (X, Y) need not be quasi-semi-holonomic. But if X, Y are quasi-semi-holonomic with the same coefficient λ (X, Y are withouf a coefficient), then (X, Y) is quasi-semi-holonomic with the coefficient λ (without a coefficient).

Lemma 3. *If* $X \in \tilde{J}^2_x(M, W)$, $Y \in \tilde{J}^2_x(M, N)$ *are quasi-semi-holonomic with the same coefficient or without a coefficient then*

$$\Delta(X, Y) = i_{1*}\Delta(X) + i_{2*}\Delta(Y), \tag{5}$$

where $i_1 : W \to W \times N$, $i_1(w) = (w, \beta Y)$,
$i_2 : N \to W \times N$, $i_2(y) = (\beta X, y)$.
The proof is obvious.

Lemma 4. *Let* G *be a Lie group. Let* $X, Y \in \tilde{J}^2_x(M, G)$, $\beta X = \beta Y = e$, *be quasi-semi-holonomic with the same coefficient or without a coefficient. Then*

$$\Delta(X \,.\, Y) = \Delta(X) + \Delta(Y),$$

where $X \,.\, Y$ *denotes the extension of the group operation on* G.

Proof. Let $f : G \times G \to G$ be the group operation on G. Using (4) and (5), we get

$$\Delta(X \,.\, Y) = f_*\Delta(X, Y) = f_*(i_{1*}\Delta(X) + i_{2*}\Delta(Y)) = \Delta(X) + \Delta(Y),$$

because $\beta X = \beta Y = e$ is the unit of G and thus $f(i_1(g)) = f(g, e) = g, f(i_2(g)) = f(e, g) = g$.

2. Let N be a parallelizable manifold and let

$$\omega_0^\alpha, \alpha, \beta, \gamma, \delta. \ldots = 1, \ldots, r = dim\ N$$

be a basis of $T^*(N)$. Consider the trivial fibre manifold $E = R^m \times N$ with the base R^m; the elements of R^m will be denoted by $(x^1, \ldots, x^m)$. Then $\omega^\alpha = pr_2^*\omega_0^\alpha$, $dt^i \quad pr_1^*dx^i$ is a basis of $T^*(E)$. Let X be quasi-semi-holonomic. We will need the coordinates of $\Delta(X)$ at the basis dx^i and the basis dual to ω^α. dt^i. Every element $Y \in J_0^1E$, $\beta Y = z$, can be identified with the subspace $Im\ L(Y) \subset \subset T_z(E)$ determined by

$$(\omega^\alpha)_z = A_i^\alpha(dt^i)_z, \text{ see [3].} \tag{6}$$

We get some real functions A_i^α on J^1E. Every $Y \in J_0^1E$ is uniquely determined by the point $\beta Y = z \in E$ and by the real numbers $A_i^\alpha(Y)$. Let $X \in \tilde{J}_0^2E$. $X = j_0^1\sigma$, $\beta X = z$. It is obvious that X is uniquely determined by the jets $l_2^1(X) = j_0^1(\beta\sigma)$, $j_2^1(X) = \sigma(o)$ and by the real numbers A_{ij}^α determined by

$$dA_i^\alpha(\sigma)_0 = A_{ij}^\alpha(dx^j)_0.$$

Denoting $A_i^\alpha(\sigma(o))$ by A_{io}^α.

$$(\omega^\alpha)_z = A_{io}^\alpha(dt^i)_z \text{ or} \tag{7}$$

$$(\omega^\alpha)_z = A^\alpha_{oi}(dt^i)_z,$$

are the equations of the subspace $Im\ L(j^1_2X)$, or $Im\ L(l^1_2(X))$, respectively. Let (x^i, z^α) be a local chart on E. Then the natural coordinates of X are $(z^\alpha, a^\alpha_{io}, a^\alpha_{oi}, a^\alpha_{ij})$ and thus

(8) $$(dz^\alpha)_z = a^\alpha_{io}(dt^i)_o \text{ or}$$

$$(dz^\alpha)_z = a^\alpha_{oi}(dt^i)_o,$$

determine $Im\ L(j^1_2X)$, or $Im\ L(l^1_2(X))$, respectively. The numbers a^α_{ij} are given by

$$da^\alpha_i(\sigma)_o = a^\alpha_{ij}(dx^j)_o,$$

where a^α_i are the coordinate functions of the chart $(x^i, z^\alpha, a^\alpha_i)$ on J^1E. Let $\omega^\alpha = B^\alpha_\beta\ dz^\beta$, $dB^\alpha_\beta = B^\alpha_{\beta\gamma}\ dz^\gamma$ and let $\tilde{B}^\alpha_\beta\ B^\beta_\gamma = \delta^\alpha_\gamma$. Using (6), (7), (8), we can compute

$$a_{ij} = -\ \tilde{B}^\alpha_\xi B^\xi_{\zeta\gamma}\ \tilde{B}^\zeta_\beta \tilde{B}^\gamma_\delta A^\delta_{0j} A^\beta_{i0} + \tilde{B}^\alpha_\beta A^\beta_{ij}.$$

If X is quasi-semi-holonomic, $A^\alpha_{ko} = 0$ or $A^\alpha_{ok} = \lambda\ .\ A^\alpha_{ko}$. Therefore, if X is quasi-semi-holonomic,

$$a_{[ij]} = -\ \tilde{B}^\alpha_\xi B^\xi_{[\zeta\gamma]} \tilde{B}^\zeta_\beta \tilde{B}^\gamma_\delta A^\beta_{io} A^\delta_{jo} \lambda + \tilde{B}^\alpha_\beta A^\beta_{[ij]}.$$

Let $d\omega^\xi = K^\xi_{\mu\nu}\omega^\mu \wedge \omega^\nu$, $K^\xi_{\mu\nu} = -\ K^\xi_{\nu\mu}$. Then for $\gamma < \zeta$,

$$B^\xi_{[\zeta\gamma]} dz^\gamma \wedge dz^\zeta = 2K^\xi_{\mu\nu} B^\mu_\gamma B^\nu_\zeta dz^\gamma \wedge dz^\zeta.$$

We have $B^\xi_{[\zeta\gamma]} = 2K^\xi_{\mu\nu} B^\mu_\gamma B^\nu_\zeta$. Now,

$$a^\alpha_{[ij]} = 2\tilde{B}^\alpha_\xi K^\xi_{\beta\delta} A^\beta_{i0} A^\delta_{j0} \lambda + \tilde{B}^\alpha_\beta A^\beta_{[ij]}.$$

Denote by E_α, E_i the basis of $T(E)$ dual to ω^α, dt^i. Then

(9) $$\Delta(X) = (2K^\alpha_{\beta\gamma} A^\beta_{io} A^\gamma_{jo} \lambda + A^\alpha_{[ij]})\ dx^i \wedge dx^j \otimes E_\alpha,\ i < j.$$

3. Let G be a Lie group and let $\mathfrak{G}$ be its Lie-algebra. Let e_α ($\alpha, \beta, \gamma, \ldots = 1, \ldots r = dim\ G$) be a basis of $\mathfrak{G}$ and let $[e_\alpha, e_\delta] = -\ c^\gamma_{\alpha\delta} e_\gamma$. Let (x^i) be a local chart on M defined on some neighbourhood of $x_0 \in M$. Let $Y \in J^1_{x_0}(M, G)$, $Y = j^1_{x_0}\varrho(x)$. Y can be identified with $L(Y)$. Let $L(Y)$ be given by the tensor $A^\alpha_j(dx^j)_{x_0} \otimes (e_\alpha)_{\varrho(x_0)}$. Let E denote the subspace of $\mathfrak{G}$ determined by $Im\ L(Y)$, i. e. generated by the vectors $E_j = A^\alpha_j e_\alpha$. The mapping $J : G \to Gl(\mathrm{r}), g \mapsto Ad(g^{-1})$ is a representation of G. Let $X_1, X_2 \in \mathfrak{G}$. Then

(10) $$[J_*(X_1)](X_2) = -[X_1, X_2], \text{ see [1], p. 56.}$$

Let (g^α_β) be the matrix of the linear mapping $Ad(g^{-1}) = f_*$, $f(u) = g^{-1}ug$, at the basis $e_\alpha : g^\alpha_\beta$ are some real functions on G. Now using (10), we compute $dg^\alpha_\beta(\varrho)_{x_0}$. Let $X_1 = A^\alpha_j dx^j(v) e_\alpha$, $v \in T_{x_0}(M)$, $v = j^1_0\gamma(t)$. Consequently $X_1 =$

$= j_0^1[\varrho^{-1}(x_0) \, . \, \varrho(\gamma(t))]$, where $\varrho^{-1}(x_0) \, . \, \varrho(\gamma(t))$ denotes the product of $\varrho^{-1}(x_0)$, $\varrho(\gamma(t))$ on G. Hence the linear mapping $J_*(X_1)$ is given by the matrix

$$\frac{d}{dt} g^\alpha_\beta[\varrho^{-1}(x_0)\varrho(\gamma(t))]_{t=0} = \frac{d}{dt} [g^\alpha_\gamma(\varrho(\gamma(t)))]_{t=0} g^\gamma_\beta(\varrho^{-1}(x_0)),$$

as $Ad(ab)^{-1} = Ad(b^{-1})Ad(a^{-1})$. Let $X_2 = e_\delta$. Then (10) yields

$$\frac{d}{dt} [g^\alpha_\beta(\varrho(\gamma(t)))]_{t=0} \, g^\beta_\delta(\varrho^{-1}(x_0))e_\alpha = -[A^\alpha_j dx^j(v)e_\alpha, e_\delta].$$

This implies $\frac{d}{dt} [g^\alpha_\beta(\varrho(\gamma(t)))]_{t=0} \, g^\beta_\delta(\varrho^{-1}(x_0)) = c^\alpha_{\gamma\delta} A^\gamma_j dx^j(v)$, i. e.

$$\frac{d}{dt} [g^\alpha_\beta(\varrho(\gamma(t)))]_{t=0} = c^\alpha_{\gamma\delta} g^\delta_\beta(\varrho(x_0)) A^\gamma_j dx^j(v), \text{ i. e.}$$

(11) $$dg^\alpha_\beta(\varrho(x))_{x_0} = c^\alpha_{\gamma\delta} g^\delta_\beta(\varrho(x_0)) A^\gamma_j (dx^j)_{x_0}.$$

Let $P(M, G, \pi)$ be a principal fibre bundle. Let Γ_p be a distribution on P determining a connection Γ on P. Then $T_p(P) = T_p(P_x) \otimes \Gamma_p$ for any $p \in P$, $\pi p = x$. Denote by H the natural projection $T_p(P) \to \Gamma_p$. Let φ be the fundamental $\mathfrak{G}$-valued form of the connection Γ and let Φ be the curvature form of Γ, i. e. $\Phi = D\varphi = d\varphi H$. Let us recall the relations

(12) $$d\varphi = -1/2[\varphi, \varphi] + \Phi,$$

(13) $$D\Phi = 0 \text{ and}$$

(14) $$d\omega = -[\varphi, \omega] + D\omega,$$

where ω is a $\mathfrak{G}$-valued equivariant horizontal p-form on P. Let Γ_1, Γ_2 be two different connections on P. Let φ_1, φ_2 or Φ_1, Φ_2, or H_1, H_2, be the fundamental forms or the curvature forms, or the natural projedctions of Γ_1, Γ_2, respectively. It is obvious that

(15) $$H_1H_2 = H_1, \quad H_2H_1 = H_2.$$

Denote by $\varphi_{1/2} = \varphi_1 - \varphi_2$, $\varphi_{2/1} = \varphi_2 - \varphi_1$. $\varphi_{1/2}$ and $\varphi_{2/1}$ are equivariant $\mathfrak{G}$-valued horizontal forms on P (see [1]). The form $\varphi_{1/2}$ will be called the fundamental difference form of the pair Γ_1, Γ_2. It is easy to see

(16) $$\varphi_{1/2} = \varphi_1 H_2, \quad \varphi_{2/1} = \varphi_2 H_1.$$

Let Ω be a real or vector valued form on P. The form $d\Omega \, H_s$, $s = 1, 2$, will also be denoted by ${}^sD\Omega$. Now, using (14), we obtain

$$d\varphi_{1/2} = -[\varphi_1, \varphi_{1/2}] + {}^1D\varphi_{1/2},$$
$$d\varphi_{1/2} = -[\varphi_2, \varphi_{1/2}] + {}^2D\varphi_{1/2}.$$

Then

(17) $$^1D\varphi_{1/2} - {}^2D\varphi_{1/2} = [\varphi_{1/2}, \varphi_{1/2}].$$

The form $[\varphi_{1/2}, \varphi_{1/2}]$ will be called the 2-difference form of the pair Γ_1, Γ_2. Using further (15) and (16), (12) implies

$$^2D\varphi_1 = -1/2[\varphi_{1/2}, \varphi_{1/2}] + \Phi_1,$$

$$^1D\varphi_2 = -1/2[\varphi_{1/2}, \varphi_{1/2}] + \Phi_2.$$

Then

(18) $$^2D\varphi_1 - {}^1D\varphi_2 = \Phi_1 - \Phi_2.$$

The form $\Phi_1 - \Phi_2$ will be said to be the 2-difference curvature form of the pair Γ_1, Γ_2. Let dim $(\Gamma_1)_p \cap (\Gamma_2)_p \neq 0$ be constant on P. As $d\varphi_{1/2} = -1/2\,[\varphi_{1/2}, \varphi_1 + \varphi_2] + \Phi_1 - \Phi_2$, the distribution determined by $\varphi_{1/2} = 0$ is integrable if the 2-difference curvature form of the pair Γ_1, Γ_2 vanishes.

Remark. Let Ω be an equivariant $\mathfrak{G}$-valued form on P. If $(\Omega)_u = 0$, then $(\Omega)_{ug} = 0$. Therefore, if $(\Omega)_u = 0$, we can say that Ω vanishes at $\pi u \in M$.

4. It is well known that every connection on P can be identified with a global G-invariant cross-section Γ of the fibered manifold $(J^1(P), P, \beta)$, satisfying $\Gamma(ug) = \Gamma(u)g$ for any $u \in P$, $g \in G$. Let Γ_1, Γ_2 be two different connections on P. We can uniquely construct the jet $R(u) \in J^1_{\pi u}(M, G)_e$, $u \in P$, as follows. Let $\Gamma_1(u) = j^1_{\pi u}\sigma_1$, $\Gamma_2(u) = j^1_{\pi u}\sigma_2$. Denote by $\varrho(x)$ a local mapping of M into G determined by

$$\sigma_2(x) = \sigma_1(x)\varrho(x).$$

We put

$$R(u) = j^1_{\pi u}\varrho(x).$$

Evidently, $\beta R(u) = e \in G$, e is the unit of G. The independence of $R(u)$ from the choice of σ_1 and σ_2 is obvious. Now, $\Gamma_2(u) = j^1_{\pi u}\sigma_1(x)\varrho(x) = \Gamma_1(u)R(u)$, where $\Gamma_1(u)R(u)$ denotes the extension of the action of G on P. In the expressions $g \,.\, R(u)$, $R(u) \,.\, g$, $\Gamma_s(u)g$, we identify g with $j^1_{\pi(u)}(g)$ and the dot denotes the composition on G and its extension.

Lemma 5. *Let $u \in P$, $g \in G$. Then $R(ug) = g^{-1} \,.\, R(u) \,.\, g$.*

Proof. $\Gamma_2(ug) = \Gamma_2(u)g = [\Gamma_1(u)R(u)]g = [\{\Gamma_1(ug)g^{-1}\}R(u)]g = \Gamma_1(ug)\ (g^{-1} \,.\, R(u) \,.\, g)$. Therefore $R(ug) = g^{-1} \,.\, R(u) \,.\, g$.

In the case of $r \geqq dim\ M$, a pair of connections Γ_1, Γ_2 will be called regular or singular, at $x \in M$ if $R(u)$, $\pi u = x$, is regular, or singular, respectively. It is easy to see that a pair of connections Γ_1, Γ_2 is singular if and only if

$$Im\ L(\Gamma_1(u)) \cap Im\ L(\Gamma_2(u)) \neq 0.$$

Further, let Γ be a connection on P, $\Gamma(u) = j^1_{\pi u}\sigma$. Let Ω be a $\mathfrak{G}$-valued q-form on P. Let $v \in T_{\pi u}(B)$, $v = j^1_0\gamma(t)$. Denoting $hv = j^1_0\sigma(\gamma(t))$, we define

$$^h\Omega_u(v_1, \ldots, v_q) = \Omega(hv_1, \ldots, hv_q),\ v_1, \ldots, v_q \in T_{\pi u}(M).$$

Lemma 6. *Let $u \in P$. Then*

$$L(R_1(u)) = {}^{h_2}(\varphi_{1/2})_u. \tag{19}$$

Proof. Let $\Gamma_1(u) = j^1_{\pi u}\sigma_1$, $\Gamma_2(u) = j^1_{\pi u}\sigma_2$, $R_1(u) = j^1_{\pi u}\varrho$. Let $v \in T_{\pi u}(M)$, $v = j^1_0\gamma(t)$. Then $h_2 v = w = j^1_0\sigma_2(\gamma(t)) = j^1_0[\sigma_1(\gamma(t))\varrho(\gamma(t))]$ and thus ${}^{h_2}(\varphi_{1/2})_u(v) = \varphi_{1/2}(w) = \varphi_1(w) = j^1_0\varrho(\gamma(t)) = L(R(u))(v)$. QED.
Put $R_{1s}(u) = j^1_{\pi u}R_1(\sigma_s)$, $s = 1,2$. Analogously to Lemma 6, we have

$$R_{1s}(ug) = g^{-1} . R_{1s}(u) . g. \tag{20}$$

Lemma 7. *$R_{1s}(u) \in \bar{J}^2_{\pi u}(M, G)_e$ is quasi-semi-holonomic with the coefficient 0 and*

$$R_{12}(u) = (R_1^{-1}(u))^{(2)} . R_{11}(u) . (R_1(u))^{(2)}. \tag{21}$$

Proof. The first part is clear. To prove (21), we use the definition of $R_{1s}(u)$ and (20). $R_{12}(u) = j^1_{\pi u}R_1(\sigma_2(x)) = j^1_{\pi u}R_1(\sigma_1(x)\varrho(x)) = j^1_{\pi u}[\varrho^{-1}(x) . R_1(\sigma_1(x)) . \varrho(x)] = (R_1{}^{1}(u))^{(2)} . R_{11}(u) . (R_1(u))^{(2)}$.
Putting further

$$\Gamma_{s_1s_2}(u) = j^1_{\pi u}\Gamma_{s_1}(\sigma_{s_2}(x)),\ s_1, s_2 = 1, 2,$$

we get some connections of the order 2 on P. Γ_{11} or Γ_{22} is the first prolongation of Γ_1, or Γ_2, respectively. They are semih-holonomic, whereas Γ_{12}, Γ_{21} are non-holonomic. It is easy to see

$$\begin{aligned} \Gamma_{21}(u) &= \Gamma_{11}(u)R_{11}(u), \\ \Gamma_{22}(u) &= \Gamma_{11}(u)[R_{11}(u) . (R_1(u))^{(2)}], \\ \Gamma_{12}(u) &= \Gamma_{11}(u)(R_1(u))^{(2)}. \end{aligned} \tag{22}$$

5. Let us consider a trivial principal fibre bundle $R^m \times G$, where the Lie group G acts on $R^m \times G$ by the rule $(x, q)g = (x, qg)$. Let e_α be a basis of the Lie algebra $\mathfrak{G}$ of the left-invariant fields on G, $[e_\beta, e_\gamma] = -c^\alpha_{\beta\gamma}e_\alpha$. Let ω^α be the dual basis of $\mathfrak{G}^*$ to e_α. The manifold $R^m \times G$ is parallelizable. Put $\omega^\alpha = pr_2^*\omega_0^\alpha$, $dt^i = pr_1^*dx^i$. Denote by E_α, E_i the dual basis to ω^α, dt^i . E_α is the fundamental vector field on $R^m \times G$, corresponding to e_α. Let H denote the distribution on $R^m \times G$ determined by

$$\omega^\alpha = A^\alpha_i dt^i,$$

where A_i^α are some real functions on $R^m \times G$. Let us consider a $\mathfrak{G}$-valued form $\varphi = (\omega^\alpha - A_i^\alpha dt^i) \otimes e_\alpha$. Denoting $\Omega = \omega^\alpha \otimes e_\alpha$ and $\Delta = A_i^\alpha dt^i \bigcirc e_\alpha$, we have $\varphi = \Omega - \Delta$. Obviously, $\varphi(E_\alpha) = e_\alpha$. φ is the fundamental form of a connection Γ on $R \times G$ if and only if it is equivariant, i. e. if

$$\varphi R_{g*} = Ad(g^{-1})\varphi. \tag{23}$$

Let $u = (x_0, q) \in R^m \times G$, let $X \in T_u(R^m \times G)$, $X = X_1 + X_2(\omega^\alpha X_2 = 0$, $dt^i(X_1) = 0$, $\alpha = 1, \ldots, r$; $i = 1, \ldots, m)$. As Ω is equivariant, $\varphi R_{g*}(X) - = Ad(g^{-1})\Omega(X_1) - \Delta R_{g*}(X_2)$. Since $Ad(g^{-1})\varphi(X) = Ad(g^{-1})\Omega(X_1) - - Ad(g^{-1})\Delta(X_2)$, (23) is correct if and only if

$$\Delta R_{g*}(X_2) = Ad(g^{-1})\Delta(X_2). \tag{24}$$

Put $X_2 = a^i(E_i)_{(x_0,q)}$. Then $R_{g*}(X_2) = a^i(E_i)_{(x_0,qg)}$. Let $Ad(g^{-1})$ be expressed at the basis e_α by the matrix (g_β^α). Then (24) yields

$$A_i^\alpha(x_0, qg)a^i e_\alpha = g_\beta^\alpha A_i^\beta(x_0, q)a^i e_\alpha.$$

Denoting the restriction of the functions A_i^α to the section $x \mapsto (x, e)$ by $\Gamma_i^\alpha(x)$, (23) is equivalent to

$$A_i^\alpha(x, g) = g_\beta^\alpha(g)\Gamma_i^\beta(x). \tag{25}$$

Putting $g_\beta^\alpha(x, g) = g_\beta^\alpha(g)$ and $\Gamma_i^\alpha(x, g) = \Gamma_i^\alpha(x)$, we have

$$A_i^\alpha(u) = g_\beta^\alpha(u)\Gamma_i^\beta(u), \quad u = (x, g).$$

Now, let Γ_1, Γ_2 be two connections on $P = R^m \times G$. Let $\varphi_s = (\omega^\alpha - - g_\beta^\alpha\, {}^s\Gamma_i^\beta dt^i) \otimes e_\alpha$ be the fundamental forms of Γ_s. Then

$$\varphi_{1/2} = g_\beta^\alpha({}^2\Gamma_i^\beta - {}^1\Gamma_i^\beta)dt^i \otimes e_\alpha. \tag{26}$$

Let $\Gamma_s(u) = j_0^1\sigma_s$, $\pi u = 0$. Since $dg_\beta^\alpha h_s = d[g_\beta^\alpha(\sigma_s)]$, therefore ${}^{h_s}({}^sD\varphi_{1/2})_u = = {}^h(d\varphi_{1/2}H_s)_u = \{d(g_\beta^\alpha(\sigma_s)]_0[{}^2\Gamma_i^\beta(0) - {}^1\Gamma_i^\beta(0)] + g_\beta^\alpha(u)\partial_j({}^2\Gamma_i^\beta - {}^1\Gamma_i^\beta)_0\}dx^j \wedge dx^i \otimes \otimes e_\alpha$.
Using (11) we obtain

$$\begin{aligned} {}^sD\varphi_{1/2} = \{&2c_{\beta\gamma}^\alpha g_\xi^\beta g_\zeta^{\gamma s}\Gamma_j^\xi[{}^2\Gamma_i^\xi - {}^1\Gamma_i^\xi] + \\ &+ g_\beta^\alpha(\partial_{[j}^2\Gamma_{i]}^\beta - \partial_{[j}^1\Gamma_{i]}^\beta)\}dt^j \wedge dt^i \otimes e_\alpha, j < i. \end{aligned} \tag{27}$$

Theorem 1. *Let $P(M, G)$ be a principal fibre bundle. Let Γ_1, Γ_2 be some connections on P. Then*

$${}^{h_s}({}^sD_{\varphi 1/2})_u = -\Delta(R_{1s}(u)).$$

Proof. Since our problem is local, we may suppose that P is the trivial fibre bundle $P = R^m \times G$. Relations (19) and (26) imply that the numbers A_i^α, determining the jet $R_1(u)$ at the basis ω^α, dt^i, are

$$A^\alpha_i(u) = g^\alpha_\beta(u)({}^2\Gamma^\beta_i(u) - {}^1\Gamma^\beta_i(u)). \tag{28}$$

To determine the numbers $A_{ij}(R_{1s}(u)$, we use (11). Let $\pi u = o$. As $R_{1s}(u) = j^1_0 R_1(\sigma_s)$, $A_{ij}(R_{1s}(u))dx^j = d[g^\alpha_\beta(\sigma_s)]_0({}^2\Gamma^\beta_i(o) - {}^1\Gamma^\beta_i(o)) + g^\alpha_\beta(u)\partial_j[{}^2\Gamma^\beta_i - {}^1\Gamma^\beta_i]_0 dx^j = c^\alpha_{\beta\gamma} g^\beta_\xi(u) g^\gamma_\zeta(u) {}^s\Gamma^\xi_j(o)({}^2\Gamma^\zeta_i(o) - {}^1\Gamma^\zeta_i(o)) + g^\alpha_\beta(u)\partial_j[{}^2\Gamma^\beta_i - {}^1\Gamma^\beta_i]_0 dx^j$. Therefore $A^\alpha_{[ij]} = 2c^\alpha_{\beta\gamma} g^\beta_\xi(u) g^\gamma_\zeta(u) {}^s\Gamma^\xi_j(o)[{}^2\Gamma^\zeta_i(o) - {}^1\Gamma^\zeta_i(o)] + g^\alpha_\beta(u)(\partial^2_{[j}\Gamma^\beta_{i]} - \partial^1_{[j}\Gamma^\beta_{i]})_0$.
But $R_{1s}(u)$ is quasi-semi-holonomic with the coefficient *0*. That is why (9) yields

$$-\Delta(R_{1s}(u) = -A[_{[ij]}dx^i \wedge dx^j \otimes e_\alpha], \quad i < j.$$

Comparing with (27), we complete the proof.

As $R_{11}(u)$ and $R_{12}(u)$ are elements of the group $\tilde{J}^2_{\pi u}(M, G)_e$, Lemmas 2 and 4 imply

$$\Delta(R_{12} \,.\, R_{11}^{1}) = \Delta(R_{12}) + \Delta(R_{11}^{-1}) = \Delta(R_{12}) - \Delta(R_{11}) = \Delta(R_{11}^{-1} \,.\, R_{12}).$$

Now, Theorem 1 and relation (17) yield

Theorem 2. *Let $u \in P$. Then*

$$\Delta(R_{12} \,.\, R_{11}^{1})_u = {}^{h_1}({}^1D\varphi_{1/2})_u - {}^{h_2}({}^2D\varphi_{1/2})_u = {}^{h_2}[\varphi_{1/2}, \varphi_{1/2}]_u.$$

Putting further ${}^2R(u) = R_{11}(u) \,.\, (R_1(u))^{(2)}$, we have $\Gamma_{22}(u) = \Gamma_{11}(u)^2\mathrm{R}(u)$. It is easy to see that ${}^2R(u)$ is semi-holonomic.

Theorem 3. *Let $u \in P$. Then*

$$\Delta({}^2R(u)) = {}^{h_2}(\Phi_2)_u - {}^{h_1}(\Phi_1)_u. \tag{29}$$

Proof. (29) can be proved by direct computation. However, Kolař [3] showed: $\Delta(\Gamma_{22}(u)) = u_*{}^{h_2}(\Phi_2)_u$, $\Delta(\Gamma_{11}(u)) = u_*{}^{h_1}(\Phi_1)_u$, where u is the mapping $G \to P_{\pi u}$, $u(g) = ug$. Since $\Gamma_{11}(u)$ and $\Gamma_{22}(u)$ are semi-holonomic and $\Gamma_{11}(u)^2R(u)$ is the extension of the action $P \times G \to P$, Lemmas 2 and 3 imply directly (29).

6. Let Φ be a Lie grupoid over M. Let $a, b : \Phi \to N$ denote the right and left unit projections. Let $I : M \to \Phi$ denote the natural inclusion of the manifold of units into the groupoid. A non-holonomic or semi-holonomic or holonomic infinitesimal connection of the order $r \geqq 1$ in Φ is a C^∞ map $\Gamma : M \to \tilde{J}^r(M, \Phi)$. or $M \to \bar{J}^r(M, \Phi)$, or $M \to J^r(M, \Phi)$, respectively, satisfying

$$\beta\Gamma = I,\ j^r a\Gamma(x) = j^r_x[x],\ j^r b\Gamma(x) = j^r_x \quad \text{(see [6])},$$

for all $x \in M$, where $j^r a$ is the r-jet of a and j^r_x is the jet of the identity mapping on M. For $r = 1$ this corresponds to the above introduced connection on any of the principal fibre bundles determined by Φ. Conversely, the principal fibre bundle $P(M, G)$ determines the grupoid $\Phi = P \times P/G$ and the connection on P determines the connection on Φ. Denote by $G(\Phi)$ the isotropy group bundle, i. e.

$$G_x = \{\Theta \in \Phi : a\Theta = b\Theta = x\}.$$

Let Γ_1, Γ_2 be two connectios in Φ. Put

$$\Gamma_{1s}(x) = j^1_x\Gamma_1 \, . \, (\Gamma_s(x))^{(2)}, \; \Gamma_{2s}(x) = j^1_x\Gamma_2 \, . \, (\Gamma_s(x))^{(2)} \quad \text{(see [6])},$$

where the dot denotes the composition in Φ as well as its extension. $\Gamma_{s_1s_2}$ is a 2-connection in Φ. Put

$$R_{1s}(x) = \Gamma^{-1}_{1s}(x) \, . \, \Gamma_{2s}(x), \; {}^2R(x) = \Gamma^{1}_{11}(x) \, . \, \Gamma_{22}(x).$$

$R_{1s}(x) \in \tilde{D}^2(G(\Phi))$ is quasi-semi-holonomic with the coefficient 0 and ${}^2R(x)$ is semi-holonomic. The pair of the connections Γ_1, Γ_2 will be said to be quasi--holonomic with respect to Γ_s, or quasi-holonomic, or holonomic at $x \in M$ if $R_{1s}(x)$, or $R^{-1}_{11}(x) \, . \, R_{12}(x)$, or ${}^2R(x)$ is quasi-holonomic, or quasi-holonomic, or holonomic, respectively. Now, Theorems 1, 2, 3 give.

Theorem 4. *The pair of the connections Γ_1, Γ_2 is quasi-holonomic with respect to Γ_s or quasi-holonomic or holonomic at $x \in M$ if and only if the form ${}^sD\varphi_{1\,2}$ or the 2-difference form of the pair Γ_1, Γ_2, or the 2-difference curvature form of the pair Γ_1, Γ_2, respectively, vanishes at $x \in M$.*

REFERENCES

[1] BISHOP, R. L., — CRITTENDEN, R. I.: Geometry of manifolds (Russian). Moscow 1967.

[2] EHRESMANN, C.: Extension du calcul des jets aux jets non-holonomes, CSAS Paris, *239*, 1762—1764, 1954.

[3] KOLÁŘ, I.: On the torsion of spaces with connection, Czechosl. Math. J. *21*, 96, 124—136, 1971.

[4] KOLÁŘ, I.: Higher order torsions of spaces with Cartan connection. Cahiers de topo. et géo. diff. Vol. XII, *2*, 137—145, 1971.

[5] VIRSÍK, I.: Non-holonomic connections on vector bundles, Czechosl. Math. J. *17* (92), 1967, 108—147.

[6] VIRSÍK, I.: On the holonomity of higher order connections, Cahiers de top. et géo. diff., Vol. XII. *2*. 197—212, 1971.

Received December 6. 1972

Katedra matematiky a deskriptívnej geometrie
Vysokej školy dopravnej
Marxa—Engelsa 25
010 88 Žilina