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# ON k-THIN SETS AND n-EXTENSIVE GRAPHS

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This article is a sequel to paper [7]. We deal here with the generalised form of a well-known problem from the theory of numbers (originated in 1916 by I. Schur, see [6]); further we give some applications of results in the graph theory (see [4]).

## I

Let $k$, $n$ and $p$ be natural numbers, with $k \geqq 3$.

**Definition 1.** *We say that the set $M$ is a $k$-thin set if from the condition*

$$a_1, a_2, \dots, a_{k-1} \in M$$

*it follows that*

$$a_1 + a_2 + \dots + a_{k-1} \notin M$$

(the numbers $a_i$ need not be different).

**Definition 2.** *The greatest natural number $N$ for which there exist $p$ disjoint $k$-thin sets $S_1, S_2, \dots, S_p$ such that*

$$\{n, n+1, \dots N\} = \bigcup_{i=1}^{p} S_i,$$

*will be denoted by $f(k, n, p)$*(1).

Remark. The $f(k, 1, p)$ is identical to $f(k, p)$ introduced in paper [7].

In the present paper we shall determine the value, resp. the lower estimation of $f(k, n, p)$. Our main result is

**Theorem I.** *For arbitrary natural $k \geqq 3$, $n$ and $p$ we have*

$$f(k, n, p) \geqq kf(k, n, p-1) + (k - n - 1). \tag{1}$$

(1) From results of [5] the existence of $f(k, n, p)$ for an arbitrary natural $k \geqq 3$, $n$ and $p$ follows.

Remark. The special case of this Theorem (case $n = 1$) is proved in paper [7]. The proof of Theorem I is completely analogical to the proof of its special case and therefore we shall not demonstrate it here.

**Corollary 1.** *Let $u$ and $v$ be natural numbers, with $v \geqq u$. We have*

$$f(k, n, v) \geqq k^{v-u} f(k, n, u) + \frac{k - n - 1}{k - 1} (k^{v-u} - 1). \tag{2}$$

Proof of Corollary 1. If we apply the inequality (1) to the number $f(k, n, v)$, we have:

$$f(k, n, v) \geqq kf(k, n, v - 1) + (k - n - 1) \geqq k^2 f(k, n, v - 2) +$$
$$+ k(k - n - 1) + (k - n - 1) \geqq \ldots \geqq k^{v-u} f(k, n, u) +$$
$$+ (k - n - 1)(k^{v-u-1} + k^{v-u-2} + \ldots + k + 1) =$$
$$= k^{v-u} f(k, n, u) + \frac{k - n - 1}{k - 1} (k^{v-u} - 1).$$

**Corollary 2.**

$$f(k, n, p) \geqq \frac{k - 2}{k - 1} (k^p - 1)n + (n - 1). \tag{3}$$

Corollary 2 is a special case of Corollary 1 (case of $u = 1$; obviously $f(k, n, 1) = (k - 1)n - 1$ for arbitrary $k$ and $n$), but we mention it separately due to its great inportance: it gives a lower estimation of $f(k, n, p)$. The estimation (3) is not the best possible and for the case of $n = 1$, $k = 3$ it was already improved (see [1]). For an arbitrary $p \geqq 4$ it is true that

$$f(3, 1, p) \geqq \frac{89 \,.\, 3^{p-4} - 1}{2},$$

which is obviously better than the estimation following from (3):

$$f(3, 1, p) \geqq \frac{3^p - 1}{2}.$$

Again, in the case of $p = 2$ we have in (3) an equation for an arbitrary $n$ and $k \geqq 3$. We state it in the form of a Theorem:

**Theorem II.** *For an arbitrary $n$ and $k \geqq 3$ we have*

$$f(k, n, 2) = \frac{k - 2}{k - 1} (k^2 - 1)n + (n - 1) = (k^2 - k - 1)n - 1.$$

**Remark.** Hence in the case of $p = 2$ our problem is solved completely, since we found the exact value of $f(k, n, 2)$.

**Proof of Theorem II.** From (3) it follows that

$$f(k, n, 2) \geqq (k^2 - k - 1)n - 1.$$

To finish our proof we must show that

$$f(k, n, 2) \leqq (k^2 - k - 1)n - 1.$$

Hence it is suffucient to show that the numbers

$$(4) \qquad n, n + 1, \ldots, (k^2 - k - 1)n$$

cannot be divided into two $k$-thin sets for any $n$ and $k \geqq 3$. We shall use the methods used for the proof of analogical assumptions in paper [8].

We shall prove indirectly. Let us suppose that there exists a division of the numbers (4) into two $k$-thin sets. Let us denote them $A$ and $B$. Without loss of generality we can suppose that $n \in A$. Since the sum of $k - 1$ elements of $A$ cannot belong to $A$, the number $(k - 1)n$ belongs to the set $B$. From analogical considerations it follows that the number

$$(k - 1)^2 n = (k^2 - 2k + 1)n$$

belongs to the set $A$ (this number is smaller than $(k^2 - k - 1)n$). We can write

$$(k^2 - k - 1)n = (k - 2) . n + 1 . (k^2 - 2k + 1)n.$$

The numbers $(k^2 - 2k + 1)n$ and $n$ are from the set $A$, hence the number $(k^2 - k - 1)n$ belongs to the set $B$.

Now we shall distinguish two cases:

a) Let $kn \in A$. We have: $n, kn, (k^2 - 2k + 1)n \in A$, where

$$(k^2 - 2k + 1)n = 1 . n + (k - 2) . kn$$

($kn$ is smaller than $(k^2 - k - 1)n$, since $k \geqq 3$). It is a contradiction, because $A$ is a $k$-thin set.

b) Let $kn \in B$. We have $(k - 1)n, kn, (k^2 - k - 1)n \in B$, where

$$(k^2 - k - 1)n = 1 . (k - 1)n + (k - 2) . kn.$$

It is a contradiction, because $B$ is a $k$-thin set.

From a) and b) it follows that the number $kn$ cannot belong to any of the sets $A$ and $B$; hence the numbers (4) cannot be divided into two $k$-thin sets in any way; q. e. d.

**Remark.** Our method — so simple for the case of $p = 2$ — is already very complicated for the case of $p = 3$. To prove that the numbers $1, 2, \dots, 14$ cannot be divided into three 3-thin sets in any way, we must distinguish 17 cases (we have here $k = 3$, $n = 1$, $p = 3$). In the cases $p \geqq 4$ it is advisable to use computers.

* * *

We give now the second proof of the relation (3) (independent of Theorem I), which gives a good method for the direct division of the numbers

$$(5) \qquad n, n+1, \dots, \frac{k-2}{k-1}(k^p - 1)n + (n-1)$$

into $p$ $k$-thin sets.

Second proof of (3). First we prove relation (3) for the case of $n = 1$, i. e. we prove that the numbers

$$(6) \qquad 1, 2, \dots, \frac{k-2}{k-1}(k^p - 1)$$

can be divided into $p$ $k$-thin sets.

Let us form from the numbers (6) the following sets:

$$F_1 = \{x : x \equiv 1, 2, \dots, (k-2) \pmod{(k-2)k}\},$$

$$F_2 = \{x : x \equiv (k-1), k, \dots, (k-2)k \pmod{(k-2)k^2}\},$$

$$\vdots$$

$$F_m = \left\{x : x \equiv \frac{k-2}{k-1}(k^{m-1} - 1) + 1, \dots, (k-2)k^{m-1} \pmod{(k-2)k^m}\right\},$$

$$\vdots$$

$$F_p = \left\{x : x \equiv \frac{k-2}{k-1}(k^{p-1} - 1) + 1, \dots, (k-2)k^{p-1} \pmod{(k-2)k^p}\right\}.$$

Now we prove that

a) all $F_m$ are $k$-thin sets,

b) every number from (6) belongs to at least one of the sets $F_m$.

a) Let $x_1, x_2, \dots, x_{k-1} \in F_m$ (where $m$ is an arbitrary of the numbers $1, 2, \dots, p$). From the construction of $F_m$ it follows that we can find such numbers $y_1, y_2, \dots, y_{k-1}$ that we have

$$x_1 \equiv y_1 \pmod{(k-2)k^m}, \quad x_2 \equiv y_2 \pmod{(k-2)k^m}, \dots,$$
$$x_{k-1} \equiv y_{k-1} \pmod{(k-2)k^m},$$

where

$$(7) \qquad \frac{k-2}{k-1}(k^{m-1} - 1) + 1 \leqq y_1, y_2, \dots, y_{k-1} \leqq (k-2)k^{m-1}.$$

From (7) it follows:

$$(8) \qquad \sum_{j=1}^{k-1} y_j \leqq (k-1)(k-2)k^{m-1} < (k-2)k^m,$$

$$(9) \qquad \sum_{j=1}^{k-1} y_j \geqq (k-2)(k^{m-1}-1) + (k-1) = (k-2)k^{m-1} + 1.$$

Because of (8) and (9) we have:

$$(k-2)k^{m-1} < \sum_{j=1}^{k-1} y_j < (k-2)k^m.$$

From the last inequality it follows that the number $\sum_{j=1}^{k-1} y_j$ cannot be congruent with any of the numbers of $F_m$ (mod $(k-2)k^m$). The same holds for the number

$$\sum_{j=1}^{k-1} x_j, \text{ since } \sum_{j=1}^{k-1} x_j \equiv \sum_{j=1}^{k-1} y_j \ (\text{mod } (k-2)k^m). \text{ Hence } \sum_{j=1}^{k-1} x_j$$

cannot be equal to any of the numbers of $F_m$ and $F_m$ is a $k$-thin set. Since $m$ was an arbitrary of the numbers $1, 2, \ldots, p$, the proof of part a) is finished.

b) We have to prove that each of the numbers of (6) belongs at least to one of the sets $F_m$. We shall prove it by induction with respect to $p$.

It is very easy to verify that the assertion is valid for $p = 1$.

Let $p > 1$. Let us suppose that the assertion is valid for $p - 1$ (i. e. that the numbers

$$1, 2, \ldots, \frac{k-2}{k-1}(k^{p-1}-1)$$

belong to the sets $F_1, F_2, \ldots, F_{p-1}$). We shall prove that the assertion is valid for $p$, too.

The numbers

$$\frac{k-2}{k-1}(k^{p-1}-1)+1, \ \frac{k-2}{k-1}(k^{p-1}-1)+2, \ldots, (k-2)k^{p-1}$$

obviously belong to the set $F_p$. We must prove yet that each of the numbers

$$(10) \qquad (k-2)k^{p-1}+1, \ (k-2)k^{p-1}+2, \ldots, \frac{k-2}{k-1}(k^p-1),$$

belongs at least to one of the sets $F_m$. Since

$$(k-2)k^{p-1} + \frac{k-2}{k-1}(k^{p-1}-1) = \frac{k-2}{k-1}(k^p-1),$$

every of the numbers (10) can be written in the form

$$(k-2)k^{p-1}+Y, \text{ where } 1 \leqq Y \leqq \frac{k-2}{k-1}(k^{p-1}-1).$$

Because of the inductional assumption every such $Y$ lies at least in one of the sets $F_1, F_2, \ldots, F_{p-1}$. The same is valid for the numbers (10), since they are congruent with the related $Y$ (mod $(k-2)k^{p-1}$), hence also (mod $(k-2)k^s$), where $1 \leqq s \leqq p-1$. A number from (10) belongs therefore into the same set as the $Y$ related to it.

The sets $F_m$ are not disjoint, but we can easy get from them a system of disjoint sets. The proof of (3) for $n = 1$ is finished.

Now we prove (3) for an arbitrary natural $n > 1$, i. e. we prove that the numbers (5) can be divided into $p$ $k$-thin sets.

Let us divide the numbers (5) into $n$-tuples in the following way:

$$a_1 = \{n, n+1, \ldots, 2n-1\},$$
$$a_2 = \{2n, 2n+1, \ldots, 3n-1\},$$
$$\vdots$$
$$a_i = \{in, in+1, \ldots, (in+n-1)\},$$
$$\vdots$$
$$a_{\frac{k-2}{k-1}(k^p-1)} = \left\{\frac{k-2}{k-1}(k^p-1)n, \ldots, \frac{k-2}{k-1}(k^p-1)n+(n-1)\right\}.$$

Let us form from the numbers (5) the sets $G_1, G_2, \ldots, G_p$ in the following way: we put the whole $n$-tuple $a_i$ in the set $G_m$ if and only if $i$ belongs to the set $F_m$ (where $F_m$ are the sets introduced above). Every number from (5) belongs exactly to one $n$-tuple, every $n$-tuple belongs to at least one set $G_m$ (since each of the numbers in (6) belongs to at least one $F_m$), hence each of the numbers in (5) belongs to at least one of the sets $G_m$. We must prove yet that $G_m$ are $k$-thin sets.

Let $x_1, x_2, \ldots, x_{k-1} \in G_m$ ($m$ is an arbitrary of numbers $1, 2, \ldots, p$). Let us denote

$$x_1 + x_2 + \ldots + x_{k-1} = x_0. \tag{11}$$

It is well-known that every number $x_i$ ($i = 0, 1, \ldots, k-1$) can be written in the form:

$$x_i = r_i n + q_i, \tag{12}$$

where $r_i$ and $q_i$ are not negative integers, and

$$q_i \leqq n-1. \tag{13}$$

If we put (12) in (11) we shall have

$$n \sum_{i=1}^{k-1} r_i + \sum_{i=1}^{k-1} q_i = r_0 n + q_0. \tag{14}$$

According to (13) $\sum_{i=1}^{k-1} q_i < (k-1)n$, hence we can write

$$\sum_{i=1}^{k-1} q_i = (k-j)n + q, \tag{15}$$

where

$$2 \leqq j \leqq k, \quad 0 \leqq q \leqq n-1. \tag{16}$$

From (14) and (15) we have the following equation

$$n \sum_{i=1}^{k-1} r_i + (k-j)n + q = r_0 n + q_0. \tag{17}$$

From (13), (16) and (17) it follows that $q_0 = q$, hence

$$\sum_{i=1}^{k-1} r_i + (k-j) = r_0.$$

According to the assumption the numbers $x_1, x_2, \ldots, x_{k-1}$ belong to the set $G_m$, therefore from the construction of $n$-tuples and the sets $G_m$ it follows that the numbers $r_1, r_2, \ldots, r_{k-1}$ belong to the set $F_m$. Further from the construction of the sets $F_m$ the existence of such numbers $t_i (i = 1, 2, \ldots, k-1)$ follows that $r_i \equiv t_i \pmod{(k-2)k^m}$, where

$$(k-1) + (k-2)(k^{m-1}-1) \leqq \sum_{i=1}^{k-1} t_i \leqq (k-1)(k-2)k^{m-1} \tag{18}$$

(see (7), (8) and (9)). From (16) and (18) we get the inequalities:

$$\sum_{i=1}^{k-1} t_i + (k-j) \geqq (k-2)(k^{m-1}-1) + (k-1) + (k-j) > (k-2)k^{m-1},$$

$$\sum_{i=1}^{k-1} t_i + (k-j) \leqq (k-2)(k-1)k^{m-1} + (k-j) \leqq (k-2)k^m.$$

Hence $\sum_{i=1}^{k-1} t_i + (k-j) \notin F_m$, thus the number

$$r_0 = \sum_{i=1}^{k-1} r_i + (k-j) \equiv \sum_{i=1}^{k-1} t_i + (k-j) \pmod{(k-2)k^m}$$

is not from $F_m$ either. But from this it follows that $x_0 = r_0 n + q_0 \notin G_m$. The proof of (3) is completed.

## II

In this part we shall show an application of the above results by the solving of a well-known problem from the graph theory.

**Definition 3.** *Let $n$ and $N$ be arbitrary natural numbers. We shall say that a graph $G$ of $N$ vertices is an $n$-extensive graph if we can denote all vertices of $G$ with numbers $0, 1, \ldots, N-1$ so that two vertices $P_i$ and $P_j$ $(i, j = 0, 1, \ldots, N-1)$ are connected by an edge if and only $|i-j| \geqq n$.*

Remark. Obviously every complete graph is an 1-extensive graph.

**Definition 4.** *Let the natural numbers $n$, $p$ and $k_i \geqq 2$ $(i = 1, 2, \ldots, p)$ be given. We shall denote by $g(n, p; k_1, k_2, \ldots, k_p)$ the greatest natural number $K$ for which all edges of an arbitrary $n$-extensive graph of $K$ vertices can be coloured by $p$ colours so that there does not arise any complete subgraph of $k_i$ vertices, all edges of which are coloured by the same colour $C_i (i = 1, 2, \ldots, p)$*(2).

**Definition 5.** *A complete subgraph, all edges of which are coluored by the same colour $(C_i)$ will be called monochromatic ($C_i$-chromatic).*

Papers [4] and [7] deal with the case of $n = 1$ (i. e. with the case of the complete graph). The results of our paper give a generalisation of the results of [4] and [7].

We shall determine the lower and the upper estimation of the function $g(n, p; k_1, k_2, \ldots, k_p)$.

**Theorem III.** *For an arbitrary natural $n, p$ and $k_i \geqq 3$ $(i = 1, 2, \ldots, p)$ we have*

$$(19) \qquad g(n, p; k_1, k_2, \ldots, k_p) \leqq \leqq \sum_{i=1}^{p} g(n, p; k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_p) + n.$$

Proof. Let $G$ be an $n$-extensive graph of

$$N = \sum_{i=1}^{p} g(n, p; k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_p) + n + 1$$

(2) The existence of the number $K = g(n, p; k_1, k_2, \ldots, k_p)$ for $n = 1$ follows from the article [4]; for $n > 1$ we shall prove it in our paper.

vertices. We shall prove indirectly. Let us suppose that we find such a colouring of all edges of $G$ by $p$ colours that there does not arise any $C_i$-chromatic complete subgraph of $k_i$ vertices ($i = 1, 2, \ldots, p$). Let us denote the vertices of $G$ with numbers $0, 1, \ldots, N-1$ in such a way that two vertices $P_i$ and $P_j$ are connected by an edge if and only if $|i - j| \geqq n$ (it is obviously possible, because $G$ is an $n$-extensive graph of $N$ vertices). The vertex denoted by 0 denote by $V_0$. There exist exactly $n - 1$ vertices which are not connected with $V_0$ by an edge. Let $T_i$ denote the set of this vertices of $G$ which are connected with $V_0$ by an edge of colour $C_i$. Let the number of elements of $T_i$ be $m_i$. Then we have:

$$1 + \sum_{i=1}^{p} m_i + (n-1) = N.$$

From this it follows that we cannot have for every $i(= 1, 2, \ldots, p)$ inequality

$$m_i \leqq g(n, p; k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_p)$$

but there exists at least one $i$ for which

$$m_i > g(n, p; k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_p).$$

Whence it follows that in $T_i$ either there exists a $C_s$-chromatic$(s \neq i)$ complete subgraph of $k_s$ vertices, or there exists a $C_i$-chromatic complete subgraph of $k_i - 1$ vertices. If we give to the later the vertex $V_0$ (which is connected with all vertices of $T_i$ by an edge of colour $C_i$) we shall have a $C_i$-chromatic complete subgraph of $k_i$ vertices. It is a contradiction and the proof of the Theorem is finished.

Remark 1. A special case of Theorem III ($n = 1$, i. e. the case of complete graphs) is proved in paper [4], the methods of which are used in our paper.

Remark 2. Obviously $g(n, p; k_1, \ldots, k_{i-1}, 2, k_{i+1}, \ldots, k_p) = g(n, p - 1; k_1, \ldots, k_{i-1}, k_{i+1}, \ldots k_p)$, therefore by (19) it can be proved by induction (with respect to $p$) that the function $g(n, p; k_1, \ldots, k_p)$ is finite for an arbitrary $n, p$ and $k_i$ ($i = 1, 2, \ldots, p$).

Remark 3. We can state further: by analogical considerations as in the case of $n = 1$ (see [4]) we can prove the inequality

$$(20) \qquad g(n, p; k_1, \ldots, k_p) \leqq \frac{(k_1 + \ldots + k_p)!}{k_1! \ldots k_p!} + n(p + 1)^{(k_1 + \ldots + k_p)}$$

This upper estimation of the function $g(n, p; k_1, \ldots k_p)$ is very rough and probably can be essentialy improved. For the case of $n = 1$ a better estimation is shown in paper [4].

*

Now we shall deal with the lower estimation of the function $g(n, p; k_1, \dots, k_p)$. Determining lower estimations we shall use the results of part I of our article. A connection between the problem of colouring the edges of a graph and the problem of division of numbers into $k$-thin sets was shown first in paper [1].

Further we shall consider only the case $k_1 = k_2 = \dots = k_p = k$. For the sake of simplification we introduce the notation $g(n, p; k, \dots, k) = g(k, n, p)$. Hence $g(k, n, p)$ is the greatest of such natural numbers for which all edges of any $n$-extensive graph of $g(k, n, p)$ vertices can be coloured by $p$ colours so that there does not arise any monochromatic complete subgraph of $k$ vertices.

**Theorem IV.** *For an arbitrary natural* $k(\geqq 3)$, $n$ *and* $p$ *we have*

$$g(k, n, p) \geqq f(k, n, p) + 1. \tag{21}$$

Proof. Let $G$ be an arbitrary $n$-extensive graph of $N = f(k, n, p) + 1$ vertices. Let us form $p$ such $k$-thin sets $I_1, I_2, \dots, I_p$ that each of the numbers $n, n + 1, \dots, f(k, n, p)$ belongs exactly to one of them (existence of such sets follows from the definition of $f(k, n, p)$). Let us denote the vertices of $G$ with the numbers $0, 1, \dots, N - 1$ so that two vertices $P_i$ and $P_j$ are joined by an edge if an only if $|i - j| \geqq n$ (the possibility of such notation follows from the assumption that $G$ is an $n$-extensive graph of $N$ vertices). The edge joining the vertices $P_r$ and $P_s$ is coloured by the colour $C_m (m = 1, 2, \dots, p)$ if and only if $|s - r| \in I_m$. We shall show that this colouring fullfils the demands, i. e. there does not arise any monochromatic complete subgraph of $k$ vertices (Obviously each edge of $G$ is coloured exactly by one colour). We shall prove indirectly. Let us suppose that by this colouring there arises a $C_i$-chromatic $(i = 1, 2, \dots, p)$ complete subgraph with the vertices

$$P_{i_1}, P_{i_2}, \dots, P_{i_k}.$$

We can suppose that

$$i_1 > i_2 > \dots > i_k.$$

$G$ is an $n$-extensive graph, hence we have:

$$n \leqq i_1 - i_2,\ n \leqq i_2 - i_3,\ \dots,\ n \leqq i_{k-1} - i_k,\ n \leqq i_1 - i_k.$$

According to the assumption all edges of this complete subgraph are coloured by the same colour $C_i$ and so we have

$$i_1 - i_2 \in I_i,\ i_2 - i_3 \in I_i,\ \dots,\ i_{k-1} - i_k \in I_i,\ i_1 - i_k \in I_i.$$

Obviously the following is valid

$$(i_1 - i_2) + (i_2 - i_3) + \dots + (i_{k-1} - i_k) = (i_1 - i_k).$$

But this is a contradiction because $I_t$ is a $k$-thin set. The proof of the Theorem is completed.

Remark 1. Special cases of (21) are proved in the papers [1] and [7].

Remark 2. It is easy to verify the following assertion: Let $G$ be a subgraph of an $n$-extensive graph $G'$ of $N = f(k, n, p) + 1$ vertices. All edges of $G$ can be coloured by $p$ colours so that there does not arise any monochromatic complete subgraph of $k$ vertices.

Remark 3. From (3) and (21) we have:

$$g(k, n, p) \geqq \frac{k-2}{k-1}(k^p - 1)n + n.$$

It is a good lower estimation only for the case of a small $k$ (for the case of a great $k$ see [2]).

Remark 4. From (20) and (21) we have the inequality

$$f(k, n, p) \leqq \frac{(pk)!}{(k!)^p} + n(p+1)^{pk} - 1,$$

which gives an upper estimation of $f(k, n, p)$.

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