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Leibniz Rule

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LEIBNIZ RULE

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Preface

The classical rule for high order derivations of a product of functions has a certain analogue in the more general case of normed modules. The general Leibniz rule can be expressed as some morphism of functors (the rule in [1] is not valid). These functors map the category of bounded polylinear mappings into the category of polylinear mappings. The first functor is a functor of multiplication. The second functor is the composition of a certain extending functor from the category of bounded polylinear mappings into itself with the first functor. Basic algebraic properties of the extending functor are described in [3].

The terminology is taken from [2] and [4]. Differential calculus is used in a more general form than in [2].

Notations

R is a normed commutative associative ring with unit which contains the field of real numbers as a subring;

p, q are non-negative integers.

The other notations in this paper are the same as in [3], but we shall consider normed right R-modules and bounded R-polylinear mappings.

$\mathscr{U}$ is the additive category of right R-modules and R-linear mappings;

$Upolimap_n$ is the additive category which is in [3] denoted by $Polimap_n$;

U is a non-empty open set of A;

F_U^p is the additive functor from $\mathscr{A}$ into $\mathscr{U}$ defined as follows:

1. $F_U^p(E)$ is the right R-module of all continuously differentiable mappings up to the order p from U into E (see [2]),
2. $\xi F_U^p(\varphi) = \xi \circ \varphi$ for every $\mathscr{A}$-morphism φ and $\xi \in F_U^p(E)$;

M_U^p is the additive functor from $Polimap_n$ into $Upolimap_n$ defined as follows:

1. for every $Polimap_n$-object $X : E_1 \oplus \ldots \oplus E_n \to E$ and for each $(\xi_1, \ldots, \xi_n) \in F^p_U(E_1) \oplus \ldots \oplus F^p_U(E_n)$, $u \in U$, we have $u((\xi_1, \ldots, \xi_n)M^p_U(X)) = (u\xi_1, \ldots, u\xi_n)X$ ($M^p_U(X)$ is an R-polylinear mapping from $F^p_U(E_1) \oplus \ldots \oplus F^p_U(E_n)$ into $F^p_U(E)$),
2. for every $Polimap_n$-morphism $(\varphi_1, \ldots, \varphi_n, \varphi)$, $M^p_U(\varphi_1, \ldots, \varphi_n, \varphi) = (F^p_U(\varphi_1), \ldots, F^p_U(\varphi_n), F^p_U(\varphi))$;

D^p is the symbol of the p-th derivation;

$\Theta^{p,q}_U$ is the morphism from F^{p+q}_U into $F^q_U \circ Pl^p_A$ defined by the relation $u(\xi\Theta^{p,q}_U(E)) = (uD^0\xi, \ldots, uD^p\xi)$ for every $\mathscr{A}$-object E, $\xi \in F^{p+q}_U(E)$ and $u \in U$.

The morphisms $\Delta^{p,q}_U$

1. Theorem. *Let $X : E_1 \oplus \ldots \oplus E_n \to E$ be a $Polimap_n$-object. Then $(\Theta^{p,q}_U(E_1), \ldots, \Theta^{p,q}_U(E_n), \Theta^{p,q}_U(E))$ is a $Upolimap_n$-morphism from $M^{p+q}_U(X)$ into $M^q_U(Lex^p_A(X))$.*

Proof. If $p = 0$, the proposition holds. Let it hold for p. For each $(\xi_1, \ldots, \xi_n) \in F^{p+q+1}_U(E_1) \oplus \ldots \oplus F^{p+q+1}_U(E_n)$ and $u \in U$, we have

$$(u(((\xi_1, \ldots, \xi_n)M^{p+q+1}_U(X))\Theta^{p+1,q}_U(E)))^r = uD^r((\xi_1, \ldots, \xi_n)M^{p+q+1}_U(X)) =$$
$$= (u(((\xi_1, \ldots, \xi_n)M^{p+q}_U(X))\Theta^{p+q}_U(E)))^r =$$
$$= (u((\xi_1\Theta^{p,q}_U(E_1), \ldots \xi_n\Theta^{p,q}_U(E_n))M^q_U(Lex^p_A(X))))^r =$$
$$= ((u(\xi_1\Theta^{p,q}_U(E_1)), \ldots, u(\xi_n\Theta^{p,q}_U(E_n)))Lex^p_A(X))^r =$$
$$= (((uD^0\xi_1, \ldots, uD^p\xi_1), \ldots, (uD^0\xi_n, \ldots, uD^p\xi_n))\, Lex^p_A(X))^r =$$
$$= (((uD^0\xi_1, \ldots, uD^{p+1}\xi_1), \ldots, (uD^0\xi_n, \ldots, uD^{p+1}\xi_n))\, Lex^{p+1}_A(X))^r =$$
$$= ((u(\xi_1\Theta^{p+1,q}_U(E_1)), \ldots, u(\xi_n\Theta^{p+1,q}_U(E\check{z})))Lex^{p+1}_U(X))^r =$$
$$= (u((\xi_1{}^{p+1,q}_U(E_1), \ldots, \xi_n\Theta^{p+1,q}_U(E_n))M^q_U(Lex^{p+1}_A(X))))^r,$$

where $r = 0, \ldots, p$. For every A-object E, $\xi \in F^{p+q+1}_U(E)$, $u \in U$ and $a \in A$, we have

$$a(uD^1(\xi\Theta^{p,q+1}_U(E))) = (a(uD^1\xi), \ldots, a(uD^{p+1}\xi));$$

this follows from [2] 8.1.5. For every $Polimap_n$-object $X : E_1 \oplus \ldots \oplus E_n \to E$, $(\xi_1, \ldots, \xi_n) \in F^{q+1}_U(E_1) \oplus \ldots \oplus F^{q+1}_U(E_n)$, $u \in U$ and $a \in A$, we have

$$aD^1((\xi_1, \ldots, \xi_n)M^{q+1}_U(X)) = \sum_{i=1}^{n} (u\xi_1, \ldots, a(uD^1\xi_i), \ldots, u\xi_n)X;$$

this follows from [2] 8.1.4, 8.2.1. Therefore

$$a(u(((\xi_1, \ldots, \xi_n)M_U^{p+q+1}(X))\Theta_U^{p+1,q}(E)))^{p+1} =$$

$$= a(uD^{p+1}((\xi_1, \ldots, \xi_n)M_U^{p+q+1}(X)))) =$$

$$= (a(uD^1(((\xi_1, \ldots, \xi_n)M_U^{p+q+1}(X))\Theta_U^{p,q+1}(E))))^p =$$

$$= (a(uD^1((\xi_1\Theta_U^{p,q+1}(E_1), \ldots, \xi_n\Theta_U^{p,q+1}(E_n))\ M_U^{q+1}(Lex_A^p(X))))^p =$$

$$= \sum_{i=1}^{n} ((u(\xi_1\Theta_U^{p,q+1}(E_1)), \ldots, a(uD^1(\xi_i\Theta_U^{p,q+1}(E_i))), \ldots,$$

$$u(\xi_n\Theta_U^{p,q+1}(E_n)))Lex_A^p(X))^p =$$

$$= \sum_{i=1}^{n} (((uD^0\xi_1, \ldots, uD^p\xi_1), \ldots, (a(uD^1\xi_i), \ldots,$$

$$a(uD^{p+1}\xi_i)), \ldots, (uD^0\xi_n, \ldots, uD^p\xi_n))Lex_A^p(X))^p =$$

$$= a(((uD^0\xi_1, \ldots, uD^{p+1}\xi_1), \ldots, (uD^0\xi_n, \ldots, uD^{p+1}\xi_n))\ Lex_A^{p+1}(X))^{p+1} =$$

$$= a(((u(\xi_1\Theta_U^{p+1,q}(E_1)), \ldots, u(\xi_n\Theta_U^{p+1,q}(E_n)))\ Lex_A^{p+1}(X))^{p+1} =$$

$$= a(u(\xi_1\Theta_U^{p+1,q}(E_1), \ldots, \xi_n\Theta_U^{p+1,q}(E_n))M_U^q(Lex_A^{p+1}(X)))^{p+1}$$

for each $a \in A$.

2. Definition. *The Upolimap$_n$-morphism* $(\Theta_U^{p,q}(E_1), \ldots, \Theta_U^{p,q}(E_n),\ \Theta_U^{p,q}(E))$ *will be denoted by* $\Delta_U^{p,q}(X)$.

3. Theorem. $\Delta_U^{p,q}$ *is a morphism from* M_U^{p+q} *into* $M_U^q \circ Lex_A^p$.

The proof is clear.

4. Note. Theorem 3 expresses the general Leibniz rule.

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