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Matematický časopis, Vol. 23 (1973), No. 3, 270--284

Persistent URL: http://dml.cz/dmlcz/126883

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PREFACE

The classical Leibnitz rule for higher derivations of a product of mappings is not valid in the case of Banach spaces. The general Leibnitz rule can be expressed by using certain extensions of polylinear mappings. These extensions and their fundamental algebraic properties are described in this paper. The general Leibnitz rule will be described in another paper (the rule in [1] is not valid).

Polylinear mappings are objects of some category. The extensions are functors from the category into itself and they generate a sequence $T^0, T^1, \ldots, T^\infty$ where T^0 is the identical functor. There is a canonical epimorphism from T^{p+1} into T^p, $p = 0, 1, \ldots$. The functors $T^0, T^1, \ldots, T^\infty$ are coherent and preserve identities of polylinear mappings. For example, if X is a commutative associative algebra then $T^0(\mathrm{X}), T^1(\mathrm{X}), \ldots, T^\infty(\mathrm{X})$ are commutative associative algebras. Analogical results can be obtained for other classes of algebras.

Two sequences of functors $T^0, T^1, \ldots, T^\infty$ are studied in this paper. The first sequence $Lex_A^0, Lex_A^1, \ldots, Lex_A^\infty$ is based upon some R-module A. The second sequence $Lex^0, Lex^1, \ldots, Lex^\infty$ is isomorphic to $Lex_R^0, Lex_R^1, \ldots, Lex_R^\infty$. The functors Lex_A^r preserve symmetry and antisymmetry, $r = 0, 1, \ldots, \infty$.

Homological and other aspects of the extensions are not considered (see [2], [3] and [5]).

The terminology is taken from [4].

NOTATIONS

R is a commutative associative ring with a unit;
1 is the unit of R;
$\mathscr{A}$ is the additive category of right R-modules and R-linear mappings;
1_E is the identical $\mathscr{A}$-morphism of an $\mathscr{A}$-object E;
A is an $\mathscr{A}$-object;

p is a non-negative integer;

r is a non-negative integer or ∞;

$\circ$ is the symbol of the composition of mappings, morphisms and functors;

L_A^p is the additive functor from $\mathscr{A}$ into $\mathscr{A}$ defined as follows:

1. L_A^0 is the identical functor,
2. $L_A^{p+1}(E) = \text{Hom}\,(A, L_A^p(E))$ for every $\mathscr{A}$-object E, and $\xi L_A^{p+1}(\varphi) = \xi \circ L_A^p(\varphi)$ for every $\mathscr{A}$-morphism φ: $\mathrm{E} \to \mathrm{F}$ and $\xi \in L_A^{p+1}(E)$;

${}^sL_A^p(E)$ is the right R-submodule of all symmetrical elements of $L_A^p(E)$ i. e.

1. ${}^sL_A^0(E) = L_A^0(E)$, ${}^sL_A^1(E) = L_A^1(E)$,
2. $\xi \in {}^sL_A^{p+2}(E)$ iff $v\xi \in {}^sL_A^{p+1}(E)$ and $vw\xi = wv\xi$ for each $v, w \in A$;

${}^aL_A^p(E)$ is the right R-submodule of all antisymmetric elements of $L_A^p(E)$, i. e.

1. ${}^aL_A^0(E) = L_A^0(E)$, ${}^aL_A^1(E) = L_A^1(E)$,
2. $\xi \in {}^aL_A^{p+2}(E)$ iff $v\xi \in {}^aL_A^{p+1}(E)$ and $vv\xi = 0$ for each $v \in A$;

$\oplus$ is the symbol of the direct product;

Pl_A^p is the additive functor from $\mathscr{A}$ into $\mathscr{A}$ defined as follows:

1. $Pl_A^p(E) = L_A^0(E) \oplus \ldots \oplus L_A^p(E)$ for every $\mathscr{A}$-object E,
2. $Pl_A^p(\varphi) = L_A^0(\varphi) \oplus \ldots \oplus L_A^p(\varphi)$ for every $\mathscr{A}$-morphism φ;

Pl_A^∞ is the additive functor from $\mathscr{A}$ into $\mathscr{A}$ defined as follows:

1. $Pl_A^\infty(E) = L_A^0(E) \oplus L_A^1(E) \oplus \ldots$ for every $\mathscr{A}$-object E,
2. $Pl_A^\infty(\varphi) = L_A^0(\varphi) \oplus L_A^1(\varphi) \oplus \ldots$ for every $\mathscr{A}$-morphism φ;

Π_A^p is the epimorphism from Pl_A^{p+1} into Pl_A^p defined by the relation $(\xi^0, \ldots, \xi^{p+1})\,\Pi_A^p(E) = (\xi^0, \ldots, \xi^p)$ for every $\mathscr{A}$-object E and $(\xi^0, \ldots, \xi^{p+1}) \in Pl_A^{p+1}(E)$;

Pl^p is the additive functor from A into A defined as follows:

1. $Pl^p(E) = \underbrace{E \oplus \ldots \oplus E}_{p+1}$ for every $\mathscr{A}$-object E,
2. $Pl^p(\varphi) = \underbrace{\varphi \oplus \ldots \oplus \varphi}_{p+1}$ for every $\mathscr{A}$-morphism φ;

Pl^∞ is the additive functor from A into A defined as follows:

1. $Pl^\infty(E) = E \oplus E \oplus \ldots$ for every $\mathscr{A}$-object E,
2. $Pl^\infty(\varphi) = \varphi \oplus \varphi \oplus \ldots$ for every $\mathscr{A}$-morphism φ;

Π^p is the epimorphism from Pl^{p+1} into Pl^p defined by the relation $(\xi^0, \ldots, \xi^{p+1})\Pi^p(E) = (\xi^0, \ldots, \xi^p)$ for every $\mathscr{A}$-object E and $(\xi^0, \ldots, \xi^{p+1}) \in Pl^{p+1}(E)$;

I^p is the isomorphism from L_R^p into L^p defined as follows:

1. I^0 is the identical morphism,
2. $\xi I^{p+1}(E) = (1\xi)I^p(E)$ for every $\mathscr{A}$-object E and $\xi \in L_R^{p+1}(E)$;

K^p is the isomorphism from Pl_R^p into Pl^p defined by the relation $(\xi^0, \ldots, \xi^p)K^p(E) = (\xi^0 I^0(E), \ldots, \xi^p I^p(E))$ for every $\mathscr{A}$-object E and $(\xi^0, \ldots, \xi^p) \in Pl^p(E)$;

K^∞ is the isomorphism from Pl_R^∞ into Pl^∞ defined by the relation $(\xi^0, \xi^1, \ldots)K^\infty(E) = (\xi^0 I^0(E), \xi^1 I^1(E), \ldots)$ for every $\mathscr{A}$-object E and $(\xi^0, \xi^1, \ldots) \in Pl_R^\infty(E)$;

n is a positive integer;

$\mathrm{Hom}(E_1, \ldots, E_n; E)$ is the right R-module of all polylinear mappings from $E_1 \oplus \ldots \oplus E_n$ into E where $E_1, \ldots, E_n, E$ are $\mathscr{A}$-objects;

Polimap$_n$ is the category defined as follows:

1. X is a *Polimap*$_n$-object iff X is a polylinear mapping from $E_1 \oplus \ldots \oplus E_n$ into E where $E_1, \ldots, E_n, E$ are $\mathscr{A}$-objects,
2. for every *Polimap*$_n$-objects $X : E_1 \oplus \ldots \oplus E_n \to E$, $Y : F_1 \oplus \ldots \oplus F_n \to F$, Hom (X, Y) is a right R-submodule of $\mathrm{Hom}(E_1, F_1) \oplus \ldots \oplus \mathrm{Hom}(E_n, F_n) \oplus \mathrm{Hom}(E, F)$ and $(\varphi_1, \ldots, \varphi_n, \varphi) \in \mathrm{Hom}(X, Y)$ iff the diagram

$$\begin{array}{ccc} E_1 \oplus \ldots \oplus E_n & \xrightarrow{\varphi_1 \oplus \ldots \oplus \varphi_n} & F_1 \oplus \ldots \oplus F_n \\ X \downarrow & & \downarrow Y \\ E & \xrightarrow[\varphi]{} & F \end{array}$$

   commutes,

3. $(\varphi_1, \ldots, \varphi_n, \varphi) \circ (\chi_1, \ldots, \chi_n, \chi) = (\varphi_1 \circ \chi_1, \ldots, \varphi_n \circ \chi_n, \varphi \circ \chi)$ for every *Polimap*$_n$-morphisms $(\varphi_1, \ldots, \varphi_n, \varphi) : X \to Y$, $(\chi_1, \ldots, \chi_n, \chi) : Y \to Z$;

σ is a permutation of the set $\{1, \ldots, n\}$;

σ^* is the functor from *Polimap*$_n$ into *Polimap*$_n$ defined as follows:

1. for every *Polimap*$_n$-object $X : E_1 \oplus \ldots \oplus E_n \to E$, $\sigma_*(X)$ is a polylinear mapping from $E_{1\sigma} \oplus \ldots \oplus E_{n\sigma}$ into E and $(\xi_1, \ldots, \xi_n)\sigma_*(X) = (\xi_{1\tau}, \ldots, \xi_{n\tau})X$, where $\tau = \sigma^{-1}$, for each $(\xi_1, \ldots, \xi_n) \in E_{1\sigma} \oplus \ldots \oplus E_{n\sigma}$,
2. $\sigma_*(\varphi_1, \ldots, \varphi_n, \varphi) = \varphi_{1\sigma}, \ldots, \varphi_{n\sigma}, \varphi)$ for every *Polimap*$_n$-morphism $(\varphi_1, \ldots, \varphi_n, \varphi)$;

$m_1, \ldots, m_n$ are positive integers;

$[X_1, \ldots, X_n]X$ is the composition of the *Polimap*$_i$-objects $X_i : E_{i,1} \oplus \ldots \oplus E_{i,m_i} \to E_i$, $i = 1, \ldots, n$, with the *Polimap*$_n$-object $X : E_1 \oplus \ldots \oplus E_n \to E$, i. e. $[X_1, \ldots, X_n]X$ is a *Polimap*$_{m_1 + \ldots + m_n}$ object and

$(\xi_{1,1}, \ldots, \xi_{1,m_1}, \ldots, \xi_{n,1}, \ldots, \xi_{n,m_n})([X_1, \ldots, X_n]X) =$
$= ((\xi_{1,1}, \ldots, \xi_{1,m_1})X_1, \ldots, (\xi_{n,1} \ldots, \xi_{n,m_n})X_n)X$
for each $(\xi_{1,1}, \ldots, \xi_{1,m_1}, \ldots, \xi_{n,1}, \ldots, \xi_{n,m_n}) \in$
$E_{1,1} \oplus \ldots \oplus E_{1,m_1} \oplus \ldots \oplus E_{n,1} \oplus \ldots \oplus E_{n,m_n}$.

PART I

1. The functors Lex^r_A

1.1. Definition. *Let $X : E_1 \oplus \ldots \oplus E_n \to E$ be a Polimap$_n$-object. By $Lex^p_A(X)$ we shall denote the polylinear mapping from $Pl^p_A(E_1) \oplus \ldots \oplus Pl^p_A(E_n)$ into $Pl^p_A(E)$ defined as follows:*

1. $Lex^0_A(X) = X$,
2. *for each* $(\xi_1, \ldots, \xi_n) \in Pl^{p+1}_A(E_1) \oplus \ldots \oplus Pl^{p+1}_A(E_n)$ *and* $a \in A$, *we have*
$((\xi_1, \ldots, \xi_n)Lex^{p+1}_A(X))^q =$
$= (((\xi^0_{,1} \ldots, \xi^p_1), \ldots, (\xi^0_n, \ldots, \xi^p_n))Lex^p_A(X))^q$
$a((\xi_1, \ldots, \xi_n)Lex^{p+1}_A(X))^{p+1} =$
$= \sum_{i=l}^{n} (((\xi^0_1, \ldots, \xi^p_1), \ldots, (a\xi^1_i, \ldots, a\xi^{p+1}_i), \ldots,$
$(\xi^0_n, \ldots, \xi^p_n))Lex^p_A(X)^p),$
where $q = 0, \ldots, p$.

1.2. Theorem *If $(\varphi_1, \ldots, \varphi_n, \varphi) : X \to Y$ is a Polimap$_n$-morphism, then $(Pl^p_A(\varphi_1), \ldots, Pl^p_A(\varphi_n), Pl^p_A(\varphi))$ is a Polimap$_n$-morphism from $Lex^p_A(X)$ into $Lex^p_A(Y)$.*

Proof. If $p = 0$, the proposition holds. Let it hold for p. Let $X : E_1 \oplus \oplus \ldots \oplus E_n \to E$, $Y : F_1 \oplus \ldots \oplus F_n \to F$. For each $(\xi_1, \ldots, \xi_n) \in Pl^{p+1}_A(E_1) \oplus \oplus \ldots \oplus Pl^{p+1}_A(E_n)$ and $a \in A$, we have
$((\xi_1 Pl^{p+1}_A(\varphi_1), \ldots, \xi_n Pl^{p+1}_A(\varphi_n))Lex^{p+1}_A(Y))^q =$
$= (((\xi^0_1, \ldots, \xi^p_1)Pl^p_A(\varphi_1), \ldots, (\xi^0_n, \ldots, \xi^p_n)Pl^p_A(\varphi_n)) \times$
$Lex^p_A(Y))^q =$
$= (((\xi^0_1, \ldots, \xi^p_1), \ldots, (\xi^0_n, \ldots, \xi^p_n))Lex^p_A(X)) \times$
$Pl^p_A(\varphi))^q =$
$= (((\xi_1, \ldots, \xi_n)Lex^{p+1}_A(X))Pl^{p+1}_A(\varphi))^q$

$a((\xi_1 Pl^{p+1}_A(\varphi_1), \ldots, \xi_n Pl^{p+1}_A(\varphi_n))Lex^{p+1}_A(Y))^{p+1} =$
$= \sum_{i=1}^{n} (((\xi^1_0 L^0_A(\varphi_1), \ldots, \xi^p_1 L^p_A(\varphi_1)), \ldots,$
$(a(\xi^1_i L^1_A(\varphi_i)), \ldots, a(\xi^{p+1}_i L^{p+1}_A(\varphi_i))), \ldots,$
$(\xi^0_n L^0_n(\varphi_n), \ldots, \xi^p_n L^p_A(\varphi_n)))Lex^p_A(Y))^p =$

$$= \sum_{i=1}^{n} (((\xi_1^0 L_A^0(\varphi_1), \ldots, \xi_1^p L_A^p(\varphi_1)), \ldots,$$

$$((a\xi_i^1)L_A^0(\varphi_i), \ldots, (a\xi_i^{p+1})L_A^p(\varphi_i)), \ldots,$$
$$(\xi_n^0 L_A^0(\varphi_n), \ldots, \xi_n^p L_A^p(\varphi_n)))Lex_A^p(Y))^p =$$

$$= \sum_{i=1}^{n} (((\xi_1^0, \ldots, \xi_1^p)Pl_A^p(\varphi_1), \ldots,$$

$$(a\xi_i^1, \ldots, a\xi_i^{p+1})Pl_A^p(\varphi_i), \ldots,$$

$$(\xi_n^0, \ldots, \xi_n^p)Pl_A^p(\varphi_n))\, Lex_A^p(Y))^p =$$

$$= \sum_{i=1}^{n} ((((\xi_1^0, \ldots, \xi_1^p), \ldots, (a\xi_i^1, \ldots, a\xi)_i^{p+1}, \ldots,$$

$$(\xi_n^0, \ldots, \xi_n^p))Lex_A^p(X))Pl_A^p(\varphi))^p =$$

$$= \sum_{i=1}^{n} (((\xi_1^0, \ldots, \xi_1^p), \ldots, (a\xi_i^1, \ldots, a\xi_i^{p+1}), \ldots,$$

$$(\xi_n^0, \ldots, \xi_n^p))Lex_A^p(X))^p L_A^p(\varphi) =$$

$$= (a((\xi_1, \ldots, \xi_n)Lex_A^{p+1}(X))^{p+1})L_A^p(\varphi) =$$

$$= a(((\xi_1, \ldots, \xi_n)Lex_A^{p+1}(X))^{p+1}L_A^{p+1}(\varphi)) =$$

$$= a(((\xi_1, \ldots, \xi_n)Lex_A^{p+1}(X))Pl_A^{p+1}(\varphi))^{p+1},$$

where $q = 0, \ldots, p$.

1.3. Definition. *The Polimap$_n$-morphism* $(Pl_A^p(\varphi_1), \ldots, Pl_A^p(\varphi n), Pl_A^p(\varphi))$ *will be denoted by* $Lex_A^p(\varphi_1, \ldots, \varphi_n, \varphi)$.

1.4. Theorem. Lex_A^p *is a functor from Polimap$_n$ into Polimap$_n$.*
The proof is obvious.

1.5. Definition. *Let* $X : E_1 \oplus \ldots \oplus E_n \to E$ *be a Polimap$_n$-object. By* $Lex_A^\infty(X)$ *will be denoted the polylinear mapping from* $Pl_A^\infty(E_1) \oplus \ldots \oplus Pl_A^\infty(E_n)$ *into* $Pl_A^\infty(E)$ *defined as follows:*

$$((\xi_1, \ldots, \xi_n(Lex_A^\infty(X))^q =$$

$$= (((\xi_1^0, \ldots, \xi_1^q), \ldots, (\xi_n^0, \ldots, \xi_n^q))Lex_A^q(X))^q,$$
where $q = 0, 1, \ldots$.

1.6. Theorem. *If* $(\varphi_1, \ldots, \varphi_n, \varphi) : X \to Y$ *is a Polimap$_n$-morphism, then* $(Pl_A^\infty(\varphi_1), \ldots, Pl_A^\infty(\varphi_n), Pl_A^\infty(\varphi))$ *is a Polimap$_n$-morphism from* $Lex_A^\infty(X)$ *into* $Lex_A^\infty(Y)$.

Proof. It follows immediately from Theorem 1.2.

1.7. Definition. *The Polimap$_n$-morphism* $(Pl_A^\infty(\varphi_1), \ldots, Pl_A^\infty(\varphi_n), Pl_A^\infty(\varphi))$ *will be denoted by* $Lex_A^\infty(\varphi_1, \ldots, \varphi, \varphi)$.

1.8. Theorem. Lex_A^∞ *is a functor from Polimap$_n$ into Polimap$_n$.*
The proof is obvious.

2. The morphisms P_A^p

2.1. Theorem. *If* $X : E_1 \oplus \ldots \oplus E_n \to E$ *is a Polimap$_n$-object, then* $(\Pi_A^p(E_1), \ldots, \Pi_A^p(E_n), \Pi_A^p(E))$ *is a Polimap$_n$-morphism from* $Lex_A^{p+1}(X)$ *into* $Lex_A^p(X)$.

2.2 Definition. *The Polimap$_n$-morphism* $(\Pi_A^p(E_1), \ldots, \Pi_A^p(E_n), \Pi_A^p(E))$ *will be denoted by* $P_A^p(X)$.

2.3. Theorem. P_A^p *is a morphism from* Lex_A^{p+1} *into* Lex_A^p.
The proof is ovious.
2.4. Note. Lex_A^∞ can be regarded as the projective limit of the sequence

$$Lex_A^0 \xleftarrow{P_A^0} Lex_A^1 \xleftarrow{P_A^1} \ldots .$$

3. Coherence of Lex_A^r

3.1. Theorem. *Let* $X_i : E_{i,1} \oplus \ldots \oplus E_{i,m_i} \to E_i$ *be a Polimap$_i$-object,* $i = 1, \ldots, n$. *Let* $X : E_1 \oplus \ldots \oplus E_n \to E$ *be a Polimap$_n$-object. Then*

$$Lex_A^p([X_1, \ldots, X_n]X) = [Lex_A^p(X_1), \ldots, Lex_A^p(X_n)]Lex_A^p(X).$$

Proof. If $p = 0$, the proposition holds. Let it hold for p. For each $(\xi_{1,1}, \ldots, \xi_{1,m_1}, \ldots, \xi_{n,1}, \ldots, \xi_{n,m_n}) \in Pl_A^{p+1}(E_{1,1}) \oplus \ldots \oplus Pl_A^{p+1}(E_{1,m_1}) \oplus \ldots \oplus Pl_A^{p+1}(E_{n,1}) \oplus \ldots \oplus Pl_A^{p+1}(E_{n,m_n})$ and $a \in A$, we have

$$((\xi_{1,1}, \ldots, \xi_{1,m_1}, \ldots, \xi_{n,1}, \ldots, \xi_{n,m_n}) \times$$
$$Lex_A^{p+1}([X_1, \ldots, X_n]X))^q =$$
$$= ((\xi_{1,1}, \ldots, \xi_{1,m_1}, \ldots, \xi_{n,1}, \ldots, \xi_{n,m_n}) \times$$
$$([Lex_A^{p+1}(X_1), \ldots, Lex_A^{p+1}(Xn)]Lex_A^{p+1}(X)))^q$$

$$a((\xi_{1,1}, \ldots, \xi_{1,m_1}, \ldots, \xi_{n,1}, \ldots, \xi_{n,m_n}) \times$$
$$Lex_A^{p+1}([X_1, \ldots, X_n]X))^{p+1} =$$

$$=\sum_{i=1}^{n}\sum_{k=1}^{mi}(((\xi_{1,1}^{0}, \ldots, \xi_{1,1}^{p}), \ldots,$$

$$(\xi_{1,m_1}^{0}, \ldots, \xi_{1,m_1}^{p}), \ldots, (\xi_{i,1}^{0}, \ldots, \xi_{i,1}^{p}), \ldots,$$

$$(a\xi_{i,k}^{p}, \ldots, a\xi_{i,k}^{p+1}), \ldots, (\xi_{i,m_i}^{0}, \ldots, \xi_{i,m_i}^{p}), \ldots,$$

$$(\xi_{n,1}^{0}, \ldots, \xi_{n,1}^{p}), \ldots, (\xi_{n,m_n}^{0}, \ldots, \xi_{n,m_n}^{p}))$$

$$Lex_A^p([X_1, \ldots, X_n]X))^p =$$

$$=\sum_{i=1}^{n}\sum_{k=1}^{m_i}(((\xi_{1,1}^{0}, \ldots, \xi_{1,1}^{p}), \ldots,$$

$$(\xi_{1,m_1}^{0}, \ldots, \xi_{1,m_1}^{p}), \ldots, (\xi_{i,1}^{0}, \ldots, \xi_{i,1}^{p}), \ldots.,$$

$$(a\xi_{i,k}^{1}, \ldots, a\xi_{i,\ k}^{p+1}), \ldots, (\xi_{i\ ,m_i}^{0}, \ldots, \xi_{i\ ,m_i}^{p}), \ldots,$$

$$\xi_{n,1}^{0}, \ldots, \xi_{n,1}^{p}), \ldots, (\xi_{n,m_n}^{0}, \ldots, \xi_{n,m_n}^{p}))$$

$$[Lex_A^p(X_1), \ldots, Lex_A^p(X_n)]Lex_A^p(X))^p =$$

$$=\sum_{i=1}^{n}\sum_{k=1}^{m_i}((((\xi_{1,1}^{0}, \ldots, \xi_{1,1}^{p}, \ldots,$$

$$(\xi_{1\ ,m_1}^{0}, \ldots, \xi_{1,m_1}^{p}))Lex_A^p(X_1), \ldots,$$

$$((\xi_{i,1}^{0}, \ldots, \xi_{i,1}^{p}), \ldots, (a\xi_{i,k}^{1}, \ldots, a\xi_{i,\ k}^{p+1}), \ldots,$$

$$(\xi_{i\ ,m_i}^{0}, \ldots, \xi_{i,m_i}^{p}))Lex_A^p(X_i), \ldots,$$

$$((\xi_{n,1}^{0}, \ldots, \xi_{n,1}^{p}), \ldots, (\xi_{n,m_n}^{0}, \ldots, \xi_{n,m_n}^{p}))$$

$$Lex_A^p(X_n))Lex_A^p(X))^p =$$

$$=\sum_{i=1}^{n}((((\xi_{1,1}^{0}, \ldots, \xi_{1,1}^{p}), \ldots, (\xi_{1,m_1}^{0}, \ldots, \xi_{1,m_1}^{p}))$$

$$Lex_A^p(X_1), \ldots, \sum_{k=1}^{m_i}((\xi_{i,1}^{0}, \ldots, \xi_{i\ ,1}^{p}), \ldots,$$

$$(a\xi_{i\ ,k}^{1}, \ldots, a\xi_{i,\ k}^{p+1}), \ldots, (\xi_{i,m_i}^{0}, \ldots, \xi_{i\ ,m_i}^{p}))$$

$$Lex_A^p(X_i), \ldots, ((\xi_{n,1}^{0}, \ldots, \xi_{n,1}^{p}, \ldots,$$

$$(\xi_{n,m_n}^{0}, \ldots, \xi_{n,m_n}^{p}))Lex_A^p(X_n))Lex_A^p(X))^p =$$

$$=\sum_{i=1}^{n}((((\xi_{1,1}^{0}, \ldots, \xi_{1,1}^{p}, \ldots, (\xi_{1,m_1}^{0}, \ldots, \xi_{1,m_1}^{p}))$$

$$Lex_A^p(X_1), \ldots, (a((\xi_{i,1}, \ldots, \xi_{i,m_i})Lex_A^{p+1}(X_i))^1, \ldots,$$

$$a((\xi_{i,1}, \ldots, \xi_{i,m_i})Lex_A^{p+1}(X_i))^{p+1}), \ldots,$$

$$((\xi^0_{n,1}, \ldots, \xi^p_{n,1}), \ldots, (\xi^0_{n,m_n}, \ldots, \xi^p_{n,m_n}))\times$$

$$Lex_A^p(X_n))Lex_A^p(X))^p =$$

$$= a(((\xi_{1,1}, \ldots, \xi_{1,m_1})Lex_A^{p+1}(X_1), \ldots,$$

$$(\xi_{n,1}, \ldots, \xi_{n,m_n})Lex_A^{p+1}(X_n))Lex_A^{p+1}(X))^{p+1},$$

where $q = 0, \ldots, p$.

3.2. Theorem. *The diagram*

$$\begin{array}{ccc} Polimap_n & \xrightarrow{Lex_A^p} & Polimap_n \\ \sigma_* \downarrow & & \downarrow \sigma_* \\ Polimap_n & \xrightarrow[Lex_A^p]{} & Polimap_n \end{array}$$

commutes.

The Proof is accomplished by induction.

3.3. Theorem. *Let $E_1, \ldots, E_n, E$ be $\mathscr{A}$-objects. Then the mapping $X \to Lex_A^p(X)$, $X \in \mathrm{Hom}(E_1, \ldots, E_n; E)$, is an $\mathscr{A}$-morphism from* $\mathrm{Hom}(E_1, \ldots, E_n; E)$ *into* $\mathrm{Hom}(Pl_A^p(E_1), \ldots, Pl_A^p(E_n); Pl_A^p(E))$.

The proof by induction.

3.4. Theorem. $Lex_A^p(X) = Pl_A^p(X)$ *for every $Polimap_1$-object $X : E_1 \to E$.*

Proof. If $p = 0$, the proposition holds. Let it hold for p. For each $\xi_1 \in Pl_A^{p+1}(E_1)$ and $a \in A$, we have

$$(\xi_1 Lex_A^{p+1}(X))^q = ((\xi_1^0, \ldots, \xi_1^p)Lex_A^p(X))^q =$$

$$= ((\xi_1^0, \ldots, \xi_1^p)Pl_A^p(X))^q = (\xi_1 Pl_A^{p+1}(X))^q$$

$$a(\xi_1 Lex_A^{p+1}(X))^{p+1} = ((a\xi_1^1, \ldots, a\xi_1^{p+1})Lex_A^p(X))^p =$$

$$= ((a\xi_1^1, \ldots, a\xi_1^{p+1})Pl_A^p(X))^p = (a\xi_1^{p+1})L_A^p(X) =$$

$$= a(\xi_1^{p+1}L_A^{p+1}(X)) = a(\xi_1 Pl_A^{p+1}(X))^{p+1},$$

where $q = 0, \ldots, p$.

3.5. Theorem. *Let $X : E_1 \oplus \ldots \oplus E_{n+1} \to E$ be a $Polimap_{n+1}$-object. Let $\hat{X} : E_1 \oplus \ldots \oplus \hat{E}_i \oplus \ldots \oplus E_{n+1} \to E$ be a $Polimap_n$-object. Suppose that*

$e_j \in E_j$, $j = 1, \ldots n + 1$. *If* $(\xi_1, \ldots, e_i, \ldots, \xi_{n+1})X = (\xi_1, \ldots, \hat{\xi}_i, \ldots \xi_{n+1})\hat{X}$ *for each* $(\xi_1, \ldots, \xi_{n+1}) \in E_1 \oplus \ldots \oplus E_{n+1}$, *then*

$$(\xi_1, \ldots, (e_i, \underbrace{0, \ldots, 0}_{p+1}), \ldots, \xi_{n+1}) Lex_A^p(X) =$$

$$= (\xi_1, \ldots, \xi_i, \ldots, \xi_{n+1}) Lex_A^p(X)$$

for each $(\xi_1, \ldots, \xi_{n+1}) \in Pl_A^p(E_1) \oplus \ldots \oplus Pl_A^p(E_{n+1})$.

The proof by induction.

3.6. Note. It is clear that the foregoing propositions are valid for Lex_A^∞, too.

3.7. Application. If $X : E \oplus E \to E$ is an associative algebra with the unit e, then $Lex_A^r(X)$ is an associative algebra with the unit $(e, \underbrace{0, \ldots, 0}_{p+1})$.

Proof. $[X, 1_E]X = [1_E, X]X$, because X is associative. By Theorem 3.1 and Theorem 3.4, we have

$$[Lex_A^r(X), 1_{Pl_r^A(E)}]Lex_A^r(X) = [Lex_A^r(X), Lex_A^r(1_E)]Lex_A^r(X) =$$
$$= Lex_A^r([X, 1_E]X) = Lex_A^r([1_E, X]X) =$$
$$= [Lex_A^r(1_E), Lex_A^r(X)]Lex_A^r(X) = [1_{Pl_r^A(E)}, Lex_A^r(X)]Lex_A^r(X).$$

By Theorem 3.4 and Theorem 3.5, we have

$$((e, 0, \ldots, 0), \xi) Lex_A^r(X) = \xi\, Lex_A^r(1_E) = \xi$$
$$(\xi, (e, 0, \ldots, 0)) Lex_A^r(X) = \xi\, Lex_A^r(1_E) = \xi$$

for each $\xi \in Pl_A^r(E)$.

3.8. Application. If $X : E \oplus E \to E$ is a Lie algebra then $Lex_A^r(X)$ is a Lie algebra.

Proof. We show that $(\xi, \xi)Lex_A^r(X) = 0$ for each $\xi \in E$. If $r = 0$, the proposition holds. Let it hold for $r = p$. For each $\xi \in E$ and $a \in A$, we have

$$((\xi, \xi)Lex_A^{p+1}(X))^q = ((\xi, \xi)Lex_A^p(X))^q = 0$$

$$a((\xi, \xi)Lex_A^{p+1}(X))^{p+1} = ((a\xi, \xi)Lex_A^p(X) + (\xi, a\xi)Lex_A^p(X))^p = 0,$$

where $q = 0, \ldots, p$. The proposition evidently holds for $r = \infty$, too.

Let π: $\{1, 2, 3\} \to \{1, 2, 3\}$ be the cyclic permutation $1\pi = 2$, $2\pi = 3$, $3\pi = 1$. The Jacobi identity

$$[1_E, X]X + \pi_*([1_E, X]X) + \pi_*(\pi_*([1_E, X]X)) = 0$$

and Theorems 3.1—3.4 imply the relation

$[1_{Pl^r_A(E)}, Lex^r_A(X)]Lex^r_A(X) + \pi_*([1_{Pl^r_A(E)}, Lex^r_A(X)]Lex^r_A(X)) +$

$+ \pi_*(\pi_*([1_{Pl^A_r(E)}, Lex^r_A(X)]Lex^r_A(X))) =$

$= Lex^r_A([1_E, X]X + \pi_*([1_E, X]X) + \pi_*(\pi_*([1_E, X]X))) = 0.$

3.9. Note. Using Theorems 3.1—3.5 we can obtain analogical results for other algebras.

PART II

1. Functors Lex^r

1.1. Definition. *Let $X : E_1 \oplus \ldots \oplus E_n \to E$ be a $Polimap_n$-object. By $Lex^p(X)$ we shall denote the polylinear mapping from $Pl^p(E_1) \oplus \ldots \oplus Pl^p(E_n)$ into $Pl^p(E)$ defined as follows:*

1. $Lex^0(X) = X$,

2. *for each* $(\xi_1, \ldots, \xi_n) \in Pl^{p+1}(E_1) \oplus \ldots \oplus Pl^{p+1}(E_n)$, *we have*

$((\xi_1, \ldots, \xi_n)Lex^{p+1}(X))^q =$

$= (((\xi_1^0, \ldots, \xi_1^p), \ldots, ((\xi_n^0, \ldots, \xi_n^p))Lex^p(X))^q$

$((\xi_1, \ldots, \xi_n)Lex^{p+1}(X))^{p+1} =$

$$= \sum_{i=1}^{n} (((\xi_1^0, \ldots, \xi_1^p), \ldots, (\xi_i^1, \ldots, \xi_i^{p+1}), \ldots, (\xi_n^0, \ldots, \xi_n^p))Lex^p(X))^p$$

where $q = 0, \ldots, p$.

1.2. Theorem. *If $(\varphi_1, \ldots, \varphi_n, \varphi) : X \to Y$ is a $Polimap_n$-morphism, then $(Pl^p(\varphi_1), \ldots, Pl^p(\varphi_n), Pl^p(\varphi))$ is a $Polimap_n$-morphism from $Lex^p(X)$ into $Lex^p(Y)$.*

Proof. If $p = 0$, the proposition holds. Let it hold for p. Ler $X : E_1 \oplus \oplus \ldots \oplus E_n \to E$, $Y : F_1 \oplus \ldots \oplus F_n \to F$. For each $(\xi_1, \ldots, \xi_n) \in Pl^{p+1}(E_1) \oplus \oplus \ldots \oplus Pl^{p+1}(E_n)$, we have

$((\xi_1 Pl^{p+1}(\varphi_1), \ldots, \xi_n Pl^{p+1}(\varphi_n))Lex^{p+1}(Y))^q =$

$= (((\xi_1, \ldots, \xi_n)Lex^{p+1}(X))Pl^{p+1}(\varphi))^q$

$((\xi_1 Pl^{p+1}(\varphi_1), \ldots, \xi_n Pl^{p+1}(\varphi_n))Lex^{p+1}(Y))^{p+1} =$

$$= \sum_{i=1}^{n} (((\xi_1^0\varphi_1, \ldots, \xi_1^p\varphi_1), \ldots, (\xi_i^1\varphi_i, \ldots, \xi_i^{p+1}\varphi_i), \ldots,$$

$(\xi_n^0\varphi_n, \ldots, \xi_n^p\varphi_n))Lex^p(\mathbf{X}))^p =$

$$= \sum_{i=1}^{n} (((\xi_1^0, \ldots, \xi_1^p)Pl^p(\varphi_1), \ldots, (\xi_i^1, \ldots, \xi_i^{p+1})Pl^p(\varphi_i), \ldots,$$

$(\xi_n^0, \ldots, \xi_n^p)Pl^p(\varphi_n))Lex^p(\mathbf{X}))^p =$

$$= \sum_{i=1}^{n} ((((\xi_1^0, \ldots, \xi_1^p), \ldots, (\xi_i^1, \ldots, \xi_i^{p+1}), \ldots,$$

$$(\xi_n^0, \ldots, \xi_n^p))Lex^p(X))Pl^p(\varphi))^p =$$

$$= \sum_{i=1}^{n} (((\xi_1^0, \ldots, \xi_1^p), \ldots, (\xi_i^1, \ldots, \xi_i^{p+1}), \ldots,$$

$$(\xi_n^0, \ldots, \xi_n^p))Lex^p(X))^p\varphi =$$

$$= (((\xi_1, \ldots, \xi_n)Lex^{p+1}(X))Pl^{p+1}(\varphi))^{p+1},$$

where $q = 0, \ldots, p$.

1.3. Definition. *The Polimap$_n$-morphism* $(Pl^p(\varphi_1), \ldots, Pl^p(\varphi_n), Pl^p(\varphi))$ *will be denoted by* $Lex^p(\varphi_1, \ldots, \varphi_n, \varphi)$.

1.4. Theorem *Lex^p is a functor from Polimap$_n$ into Polimap$_n$.*

The proof is obvious.

1.5. Definition. *Let* $X : E_1 \oplus \ldots \oplus E_n \to E$ *be a Polimap$_n$-object. By* $Lex^\infty(X)$ *will be denoted the polylinear mapping from* $Pl^\infty(E_1) \oplus \ldots \oplus Pl^\infty(E_n)$ *into* $Pl^\infty(E)$ *defined as follows:*

$((\xi_1, \ldots, \xi_n)Lex^\infty(X))^q = (((\xi_1^0, \ldots, \xi_0^q), \ldots,$

$(\xi_n^0, \ldots, \xi_n^q))Lex^q(X))^q,$

where $q = 0, 1, \ldots$.

1.6. Theorem. *If* $(\varphi_1, \ldots, \varphi_n, \varphi) : X \to Y$ *is a Polimap$_n$-morphism, then* $(Pl^\infty(\varphi_1), \ldots, Pl^\infty(\varphi_n), Pl^\infty(\varphi))$ *is a Polimap$_n$-morphism from* $Lex^\infty(X)$ *into* $Lex^\infty(Y)$.

Proof. It follows immediately from Theorem 1.2.

1.7. Definition. *The Polimap$_n$-morphism* $(Pl^\infty(\varphi_1), \ldots, Pl^\infty(\varphi_n), Pl^\infty(\varphi))$ *will be denoted by* $Lex^\infty(\varphi_1, \ldots, \varphi_n, \varphi)$.

1.8. Theorem. *Lex^∞ is a functor from Polimap$_n$ into Polimap$_n$.*

The proof is obvious.

2. Morphisms P^p

2.1. Theorem. *If* $X : E_1 \oplus \ldots \oplus E_n \to E$ *is a* $Polimap_n$*-object, then* $(\Pi^p(E_1), \ldots, \Pi^p(E_n), \Pi^p(E))$ *is a* $Polimap_n$*-morphism from* $Lex^{p+1}(X)$ *into* $Lex^p(X)$.

2.2. Definition. *The* $Polimap_n$*-morphism* $(\Pi^p(E_1), \ldots, \Pi^p(E_n), \Pi^p(E))$ *will be denoted by* $P^p(X)$.

2.3. Theorem. P^p *is a morphism from* Lex^{p+1} *into* Lex^p.

The proof is obvious.

2.4. Note. Lex^∞ can be regarded as the projective limit of the sequence

$$Lex^0 \xleftarrow{P^0} Lex^1 \xleftarrow{P^1} \ldots .$$

3. Coherence of Lex^r

Analogical results as in I.3 can be obtained for Lex^r, too. The proofs are similar.

PART III

1. Symmetry of Lex_A^r

1.1. Theorem *Let* $X : E_1 \oplus \ldots \oplus E_n \to E$ *be a* $Polimap_n$*-object* $(\xi_1, \ldots, \xi_n) \in \in Pl_A^{p+2}(E_1) \oplus \ldots \oplus Pl_A^{p+2}(E_n)$ *and* $v, w \in A$. *Then*

$$vw((\xi_1, \ldots, \xi_n)Lex_A^{p+2}(X))^{p+2} =$$

$$= \sum_{i<j}(((\xi_1^0, \ldots, \xi_1^p), \ldots, (v\xi_i^1, \ldots, v\xi_i^{p+1}), \ldots,$$

$$(w\xi_j^1, \ldots, w\xi_j^{p+1}), \ldots, (\xi_n^0, \ldots, \xi_n^p))Lex_A^p(X) +$$

$$+ ((\xi_1^0, \ldots, \xi_1^p), \ldots, (w\xi_i^1, \ldots, w\xi_i^{p+1}), \ldots,$$

$$(v\xi_j^1, \ldots, v\xi_j^{p+1}), \ldots, (\xi_n^0, \ldots, \xi_n^p))Lex_A^p(X))^p +$$

$$+ \sum_{k=1}^{n}(((\xi_1^0, \ldots, \xi_1^p), \ldots, (vw\xi_k^2, \ldots, vw\xi_k^{p+2}), \ldots,$$

$$(\xi_n^0, \ldots, \xi_n^p))Lex_A^p(X))^p .$$

The proof is clear

1.2. Theorem. *Let* $X : E_1 \oplus \ldots \oplus E_n \to E$ *be a* $Polimap_n$*-object. If* $\xi_i \in$

$\in {}^sL_A^0(E_i) \oplus \ldots \oplus {}^sL_A^p(E_i), i = 1, \ldots, n$, *then* $(\xi_1, \ldots, \xi_n)Lex_A^p(X) \in {}^sL_A^0(E) \oplus$ $\oplus \ldots \oplus {}^sL_A^p(E)$.

Proof. If $p = 0, 1$, the proposition holds. Let it hold for $p, p + 1$. The validity of the proposition for $p + 1, p + 2$ follows immediately from Theorem 1.1.

1.3. Theorem. *Let* $X : E_0 \oplus \ldots \oplus E_0 \to E$ *be an antisymmetric Polimap*$_n$*-object. If* $\xi_i \in {}^aL_A^0(E_0) \oplus \ldots \oplus {}^aL_A^p(E_0), i = 1, \ldots, n$, *then* $(\xi_1, \ldots, \xi_n)Lex_A^p(X)$ $\in {}^aL_A^0(E) \oplus \ldots \oplus {}^aL_A^p(E)$.

The proof is analogous.

1.4. Note. It is clear that the foregoing propositions are valid for Lex_A^∞, too.

2. Isomorphisms Λ^r

2.1. Theorem. *If* $X : E_1 \oplus \ldots \oplus E_n \to E$ *is a Polimap*$_n$*-object then* $(K^r(E_1), \ldots, K^r(E_n), K^r(E))$ *is a Polimap*$_n$*-isomorphism from* $Lex_R^r(X)$ *into* $Lex^r(X)$.

Proof. If $r = 0$, the proposition holds. Let it hold for $r = p$. For each $(\xi_1, \ldots, \xi_n) \in Pl_R^{p+1}(E_1) \oplus \ldots \oplus Pl_R^{p+1}(E_n)$, we have

$$((\xi_1K^{p+1}(E_1), \ldots, \xi_nK^{p+1}(E_n))Lex^{p+1}(X))^q =$$

$$= (((\xi_1^0I^0(E_1), \ldots, \xi_1^pI^p(E_1)), \ldots, ((\xi_n^0I^0(E_n), \ldots,$$

$$\xi_n^pI^p(E_n)))Lex^p(X))^q =$$

$$= (((\xi_1^0, \ldots, \xi_1^p)K^p(E_1), \ldots, (\xi_n^0, \ldots, \xi_n^p)K^p(E_n))\times$$

$$Lex^p(X))^q =$$

$$= ((((\xi_1^0, \ldots, \xi_1^p), \ldots (\xi_n^0, \ldots, \xi_n^p))Lex_R^p(X))K^p(E))^q =$$

$$= (((\xi_1^0, \ldots, \xi_1^p), \ldots, (\xi_n^0 \ldots, \xi_n^p))Lex_R^p(X))^qI^q(E) =$$

$$= (((\xi_1, \ldots, \xi_n)Lex_R^{p+1}(X))K^{p+1}(E))^q$$

$$((\xi_1K^{p+1}(E_1), \ldots, \xi_nK^{p+1}(E_n))Lex^{p+1}(X))^{p+1} =$$

$$= \sum_{i=1}^{n} (((\xi_1^0I^0(E_1), \ldots, \xi_1^pI^p(E_1)), \ldots, (\xi_i^1I^1(E_i), \ldots,$$

$$\xi_i^{p+1}I^{p+1}(E_i)), \ldots, (\xi_n^0I^0(E_n), \ldots, \xi_n^pI^p(E_n)))Lex^p(X))^p =$$

$$= \sum_{i=1}^{n} (((\xi_1^0I^0(E_1), \ldots, \xi_1^pI^p(E_1)), \ldots, ((1\xi_i^1)I^0(E_i), \ldots,$$

$$(1\xi_i^{p+1})I^p(E_i)), \ldots, (\xi_n^0I^0(E_n), \ldots,$$

$$\xi_n^p I^p(E_n)))Lex^p(X))^p =$$

$$= \sum_{i=1}^{n}(((\xi_1^0, \ldots, \xi_1^p)K^p(E_1), \ldots, (1\xi_i^1, \ldots, 1\xi_i^{p+1})K^p(E_i), \ldots,$$

$$(\xi_n^0, \ldots, \xi_n^p)K^p(E_n))Lex^p(X))^p =$$

$$= \sum_{i=1}^{n}((((\xi_1^0, \ldots, \xi_1^p, \ldots, (1\xi_i^1, \ldots, 1\xi_i^{p+1}), \ldots,$$

$$(\xi_n^0, \ldots, \xi_n^p)Lex_R^p(X))K^p(E))^p =$$

$$= \sum_{i=1}^{n}(((\xi_1^0, \ldots, \xi_1^p), \ldots, (1\xi_i^1, \ldots, 1\xi_i^{p+1}), \ldots,$$

$$(\xi_n^0, \ldots, \xi_n^p))Lex_R^p(X))^p I^p(E) =$$

$$= (1((\xi_1 \ldots, \xi_n)Lex_R^{p+1}(X))^{p+1})I^p(E) =$$

$$= ((\xi_1, \ldots, \xi_n)Lex_R^{p+1}(X))I^{p+1}(E) =$$

$$= (((\xi_1, \ldots, \xi_n)Lex_R^{p+1}(X))K^{p+1}(E))^{p+1},$$

where $q = 0, \ldots, p$. Since $K^{p+1}(E_1), \ldots, K^{p+1}(E_n)$, $K^{p+1}(E)$ are isomorphisms, $(K^{p+1}(E_1), \ldots, K^{p+1}(E_n), K^{p+1}(E))$ is a *$Polimap_n$*-isomorphism.

The proposition evidently holds for $r = \infty$, too.

2.2. Definition. *The $Polimap_n$-isomorphism* $(K^r(E_1), \ldots, K^r(E_n), K^r(E))$ *will be denoted by* $\Lambda^r(X)$.

2.3. Theorem. *Λ^r is an isomorphism from Lex_R^r into Lex^r and the diagram*

$$\begin{array}{ccccc}
Lex_R^0 & \xleftarrow{P_R^0} & Lex_R^1 & \xleftarrow{P_R^1} & \ldots \\
\downarrow{\scriptstyle \Lambda^0} & & \downarrow{\scriptstyle \Lambda^1} & & \\
Lex^0 & \xleftarrow{P^0} & Lex^1 & \xleftarrow{P^1} & \ldots
\end{array}$$

commutes.

The proof is obvious.

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Received June 19, 1972

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