Bedřich Pondělíček

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A RELATION FOR CLOSURE OPERATIONS ON A SEMIGROUP

BEDŘICH PONDĚLÍČEK, Poděbrady

Let S be a semigroup. The mapping $\mathbf{U}$: $\exp S \to \exp S$ is said to be a $\mathscr{C}$-*closure operation* if $\mathbf{U}$ satisfies the following conditions:

(1) $$\mathbf{U}(\emptyset) = \emptyset;$$

(2) $$A \subset B \subset S \Rightarrow \mathbf{U}(A) \subset \mathbf{U}(B);$$

(3) $$A \subset \mathbf{U}(A) \text{ for each } A \subset S;$$

(4) $$\mathbf{U}(\mathbf{U}(A)) = \mathbf{U}(A) \text{ for each } A \subset S.$$

For $x \in S$ we write simply $\mathbf{U}(x)$ instead of $\mathbf{U}(\{x\})$. A subset A of S will be called $\mathbf{U}$-closed if $\mathbf{U}(A) = A$. The set of all $\mathbf{U}$-*closed* subsets of S will be denoted by $\mathscr{F}(\mathbf{U})$.

In [1] a certain relation for $\mathscr{C}$-closure operations $\mathbf{U}$, $\mathbf{V}$ on S is studied, i. e.

$$A \cap B = AB$$

for every $\mathbf{U}$-closed non-empty subset A of S and for every $\mathbf{V}$-closed non-empty subset B of S.

In this paper we consider semigroups satisfying the relation

(5) $$A \cap B = AB \cap BA$$

for every $\mathbf{U}$-closed non-empty subset A of S and for every $\mathbf{V}$-closed non-empty subset B of S. We denote this fact by $\mathbf{U}\sigma\mathbf{V}$.

Let $\mathscr{C}(S)$ denote the set of all $\mathscr{C}$-closure operations for a semigroup S. It is clear that σ is a symmetric relation on $\mathscr{C}(S)$.

Let $\mathbf{U}$, $\mathbf{V} \in \mathscr{C}(S)$. Then we define $\mathbf{U} \leqq \mathbf{V}$ if and only if $\mathbf{U}(A) \subset \mathbf{V}(A)$ for each $A \subset S$. The ordered set $\mathscr{C}(S)$ is a lattice $(\wedge, \vee)$ and there holds

(6) $$\mathbf{U} \leqq \mathbf{V} \Leftrightarrow \mathscr{F}(\mathbf{V}) \subset \mathscr{F}(\mathbf{U}).$$

(See [1]).

Lemma 1. *Let* $\mathbf{U}_1, \mathbf{V}_1, \mathbf{U}_2, \mathbf{V}_2 \in \mathscr{C}(S)$ *and* $\mathbf{U}_1 \leqq \mathbf{U}_2$, $\mathbf{V}_1 \leqq \mathbf{V}_2$. *If* $\mathbf{U}_1\sigma\mathbf{V}_1$ *then* $\mathbf{U}_2\sigma\mathbf{V}_2$.

The proof follows from (5) and (6).

Let $\varnothing \neq A \subset S$. Put $\mathbf{L}(A) = S^1A = SA \cup A$ and $\mathbf{R}(A) = AS^1 = AS \cup A$. Finally $\mathbf{L}(\varnothing) = \varnothing = \mathbf{R}(\varnothing)$. Clearly $\mathbf{L}, \mathbf{R} \in \mathscr{C}(S)$. Put $\mathbf{M} = \mathbf{L} \vee \mathbf{R}$ and $\mathbf{H} = \mathbf{L} \wedge \mathbf{R}$. Evidently $\mathbf{M}, \mathbf{H} \in \mathscr{C}(S)$. $\mathscr{F}(\mathbf{L})$, $\mathscr{F}(\mathbf{R})$, $\mathscr{F}(\mathbf{M})$ and $\mathscr{F}(\mathbf{H})$, respectively, is the set of all left, right, two-sided and quasi, respectively, ideals of S (including $\varnothing$). We have $\mathbf{M}(A) = S^1AS^1 = SAS \cup AS \cup SA \cup A$ and $\mathbf{H}(A) = \mathbf{L}(A) \cap \mathbf{R}(A)$ for every non-empty subset A of S. (See [1].)

Theorem 1. *Let* $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$. *Then* $\mathbf{U}\sigma\mathbf{V}$ *if and only if* $H \leqq \mathbf{U} \wedge \mathbf{V}$ *and* $x \in \mathbf{U}(x)\mathbf{V}(x) \cap \mathbf{V}(x)\mathbf{U}(x)$ *for every* $x \in S$.

Proof. Let $\mathbf{U}\sigma\mathbf{V}$. Evidently $S \in \mathscr{F}(\mathbf{V})$. If $\varnothing \neq A \in \mathscr{F}(\mathbf{U})$, then $A = A \cap S = AS \cap SA$ and so A is a quasi-ideal of S. Thus $A \in \mathscr{F}(\mathbf{H})$, hence $\mathscr{F}(\mathbf{U}) \subset \mathscr{F}(\mathbf{H})$. It follows from (6) that $\mathbf{H} \leqq \mathbf{U}$. Similarly we obtain that $H \leqq \mathbf{V}$. Thus we have $\mathbf{H} \leqq \mathbf{U} \wedge \mathbf{V}$. By (4) we have $\mathbf{U}(x) \in \mathscr{F}(\mathbf{U})$ and $\mathbf{V}(x) \in \mathscr{F}(\mathbf{V})$ for every x of S. It follows from (3) and (5) that $x \in \mathbf{U}(x) \cap \mathbf{V}(x) = \mathbf{U}(x)\mathbf{V}(x) \cap \mathbf{V}(x)\mathbf{U}(x)$.

Let now $\mathbf{H} \leqq \mathbf{U} \wedge \mathbf{V}$ and let $x \in \mathbf{U}(x)\mathbf{V}(x) \cap \mathbf{V}(x)\mathbf{U}(x)$ for every $x \in S$. If $\varnothing \neq A \in \mathscr{F}(\mathbf{U})$ and $\varnothing \neq B \in \mathscr{F}(\mathbf{V})$, then by (6) $A \in \mathscr{F}(\mathbf{H})$ and $B \in \mathscr{F}(\mathbf{H})$. Hence A, B are quasi-ideals of S. Thus $AB \cap BA \subset AS \cap SA \subset A$ and $AB \cap BA \subset SB \cap BS \subset B$. Hence $AB \cap BA \subset A \cap B$. Let $x \in A \cap B$. Since $x \in A$, hence by (2) we have $\mathbf{U}(x) \subset \mathbf{U}(A) = A$. Similarly $\mathbf{V}(x) \subset B$. Thus $x \in \mathbf{U}(x)\mathbf{V}(x) \cap \mathbf{V}(x)\mathbf{U}(x) \subset AB \cap BA$. Therefore, $A \cap B \subset AB \cap BA$. This implies (5).

Corollary 1. *Let* $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$ *and let* $\mathbf{H} \leqq \mathbf{U} \wedge \mathbf{V}$. *Then the following conditions on* S *are equivalent:*

1. $\mathbf{U}\sigma\mathbf{V}$;
2. $\mathbf{U}(x) \cap \mathbf{V}(y) = \mathbf{U}(x)\mathbf{V}(y) \cap \mathbf{V}(y)\mathbf{U}(x)$ *holds for every* $x, y \in S$,
3. $\mathbf{U}(x) \cap \mathbf{V}(x) = \mathbf{U}(x)\mathbf{V}(x) \cap \mathbf{V}(x)\mathbf{U}(x)$ *holds for every* $x \in S$.

Corollary 2. *Let* $\mathbf{U} \in \mathscr{C}(S)$ *and let* $\mathbf{H} \leqq \mathbf{U}$. *Then the following conditions on* S *are equivalent:*

1. $\mathbf{U}\sigma\mathbf{U}$;
2. $A = A^2$ *holds for every* $\mathbf{U}$*-closed non-empty subset* A *of* S;
3. $\mathbf{U}(x) = \mathbf{U}(x)\mathbf{U}(x)$ *holds for every* $x \in S$,
4. $x \in \mathbf{U}(x)\mathbf{U}(x)$ *holds for every* $x \in S$.

Theorem 2. *The following conditions on a semigroup S are equivalent*:
1. $\mathbf{M}\sigma\mathbf{M}$,
2. *Every two-sided ideal of S is idempotent*;
3. $x \in SxSxS$ *holds for every* $x \in S$.

Proof. $1 \Rightarrow 2$. This follows from Corollary 2.

$2 \Rightarrow 3$. Let every two-sided ideal of S be idempotent. Let $x \in S$. Corollary 2 implies that $x \in \mathbf{M}(x)\mathbf{M}(x) \subset S^1xS^1xS^1$. We shall prove that $x \in SxSxS$. If $x = x^2$, then $x = x^5 \in SxSxS$. If $x = ax^2$ for some $a \in S$, then $x = axax^2 \in \in SxSxS$. Similarly, $x = x^2a$ ($x = xax$, respectively) for some $a \in S$ implies that $x \in SxSxS$. If $x = axbx$ for some a, $b \in S$, then $x = axbaxbx \in SxSxS$. Similarly, $x = xaxb$ for some a, $b \in S$ implies that $x \in SxSxS$. Finally, if $x = ax^2b$ for some a, $b \in S$, then $x = axax^2b^2 \in SxSxS$.

$3 \Rightarrow 1$. Let $x \in SxSxS$ hold for every $x \in S$. Let $x \in S$. Then $x \in SxSxS \subset \subset \mathbf{M}(x)\mathbf{M}(x)$ and so by Corollary 2 $\mathbf{M}\sigma\mathbf{M}$.

Theorem 3. *The following conditions on a semigroup S are equivalent*:
1. $\mathbf{R}\sigma\mathbf{R}$,
2. $\mathbf{R}\sigma\mathbf{M}$,
3. *Every right ideal of S is idempotent*,
4. $x \in xSxS$ *holds for every* $x \in S$.

Proof. $1 \Rightarrow 2$. This follows from Lemma 1.

$2 \Rightarrow 3$. Let $\mathbf{R}\sigma\mathbf{M}$ and let $x \in S$. Theorem 1 implies $x \in \mathbf{R}(x)\mathbf{M}(x) \subset xS^1xS^1 = \mathbf{R}(x)\mathbf{R}(x)$. According to Theorem 1, $\mathbf{R}\sigma\mathbf{R}$. By Corollary 2 it follows that every right ideal of S is idempotent.

$3 \Rightarrow 4 \Rightarrow 1$. This is analogous to the proof of Theorem 2.

Left-right dually we have the following:

Theorem 4. *The following conditions on a semigroup S are equivalent*:
1. $\mathbf{L}\sigma\mathbf{L}$;
2. $\mathbf{M}\sigma\mathbf{L}$;
3. *Every left ideal of S is idempotent*,
4. $x \in SxSx$ *holds for every* $x \in S$.

A semigroup S is called *quasi inverse* (see [2]) if every right ideal of S is idempotent and every left ideal of S is idempotent.

Theorem 5. *The following conditions on a semigroup S are equivalent*:
1. $\mathbf{R}\sigma\mathbf{R}$ *and* $\mathbf{L}\sigma\mathbf{L}$;
2. $\mathbf{R}\sigma\mathbf{M}$ *and* $\mathbf{M}\sigma\mathbf{L}$;
3. $\mathbf{M}\sigma\mathbf{H}$;
4. *S is a quasi inverse semigroup.*

Proof. $1 \Rightarrow 2 \Rightarrow 4 \Rightarrow 1$. This follows from Theorem 3 and from Theorem 4.

$1 \Rightarrow 3$. Let $\mathbf{R}\sigma\mathbf{R}$ and $\mathbf{L}\sigma\mathbf{L}$ hold. Let $x \in S$. Theorem 3 implies that $x \in SxSx$ and so $x \in SxSxSx \subset \mathbf{M}(x)\mathbf{H}(x)$. Similarly, we obtain that $x \in \mathbf{H}(x)\mathbf{M}(x)$ for every $x \in S$. It follows from Theorem 1 that $\mathbf{M}\sigma\mathbf{H}$.

$3 \Rightarrow 2$. This follows from Lemma 1.

Theorem 6. *The following conditions on a semigroup S are equivalent:*

1. $\mathbf{H}\sigma\mathbf{H}$,
2. $\mathbf{R}\sigma\mathbf{H}$;
3. $\mathbf{H}\sigma\mathbf{L}$;
4. $\mathbf{R}\sigma\mathbf{L}$;
5. *S is regular and intraregular,*
6. *Every quasi-ideal of S is idempotent.*

Proof. $1 \Rightarrow 2 \Rightarrow 4$ and $1 \Rightarrow 3 \Rightarrow 4$. This follows from Lemma 1.

$4 \Rightarrow 5$. Let $\mathbf{R}\sigma\mathbf{L}$ and let $x \in S$. Theorem 1 implies that $x \in \mathbf{R}(x)\mathbf{L}(x) \cap \cap \mathbf{L}(x)\mathbf{R}(x) \subset xS^1x \cap S^1x^2S^1$ and so S is a regular and intraregular semigroup.

$5 \Rightarrow 6$. Let S be a regular and intraregular semigroup. Then $x \in xSx \cap Sx^2S$ for any x of S. This implies that $x \in xSxSx$ and so $x \in xSx^2Sx \subset \mathbf{H}(x)\mathbf{H}(x)$. By Corollary 2 we obtain that every quasi-ideal of S is idempotent.

$6 \Rightarrow 1$. This follows from Corollary 2.

If $A \subset S$, $A \neq \varnothing$, then we denote by $\mathbf{P}(A)$ the subsemigroup generated by all elements of A. Put $\mathbf{P}(\varnothing) = \varnothing$. Evidently $\mathbf{P} \in \mathscr{C}(S)$ and $\mathscr{F}(\mathbf{P})$ is the set of all subsemigroups of S (including $\varnothing$). Further $\mathbf{P} \leqq \mathbf{H}$.

Theorem 7. *The following conditions on a semigroup S are equivalent:*

1. $\mathbf{P}\sigma\mathbf{P}$,
2. $\mathbf{R}\sigma\mathbf{P}$;
3. $\mathbf{P}\sigma\mathbf{L}$,
4. *Every element of S is an idempotent and every subsemigroup of S is a quasi--ideal of S.*
5. *Every element of S is an idempotent and $xzy = xy$ for $x, y, z \in S$.*

Proof. $1 \Rightarrow 2$ and $1 \Rightarrow 3$. This follows from Lemma 1.

$2 \Rightarrow 4$. Let $\mathbf{R}\sigma\mathbf{P}$. Theorem 1 implies that $\mathbf{H} \leqq \mathbf{P}$. Since $\mathbf{P} \leqq \mathbf{H}$, hence $\mathbf{H} = \mathbf{P}$ and so $\mathscr{F}(\mathbf{H}) = \mathscr{F}(\mathbf{P})$. Therefore, every subsemigroup of S is a quasi-ideal of S. Since $\mathbf{R}\sigma\mathbf{H}$, hence by Theorem 6 every quasi-ideal of S is idempotent. Let $x \in S$. Then $x \in \mathbf{P}(x) = \mathbf{H}(x) = \mathbf{H}(x)\mathbf{H}(x) = \mathbf{P}(x)\mathbf{P}(x)$. Hence there exists some integer $n > 1$ such that $x = x^n$. It is clear that $\mathbf{P}(x)$ is a cyclic subgroup of S. Let e be an identity of $\mathbf{P}(x)$. Then $x = ex = xe \in \mathbf{H}(e) = \mathbf{P}(e) = \{e\}$ and so $x = e$. Hence, every element x of S is an idempotent.

$3 \Rightarrow 4$. Similarly.

$4 \Rightarrow 5$. Let every element of S be an idempotent and let every subsemigroup of S be a quasi-ideal of S. Then we have $\mathscr{F}(\mathbf{P}) \subset \mathscr{F}(\mathbf{H})$ and so by (6) $\mathbf{H} \leqq \mathbf{P}$. Since $\mathbf{P} \leqq \mathbf{H}$, hence $\mathbf{H} = \mathbf{P}$. We shall prove that $xzy = xy$ for every $x, y, z \in S$. Let $x, y, z \in S$. Put $A = \{x, y\}$. Evidently $\mathbf{H}(A) = \mathbf{P}(A) = \{x, y, xy, yx, xyx, yxy\}$. Since $\mathbf{H}(A)$ is a quasi-ideal of S, hence $xzy \in xS \cap Sy \subset AS \cap SA \subset \mathbf{H}(A)S \cap \cap S\mathbf{H}(A) \subset \mathbf{H}(A)$. If $xzy = x$, then $xzy = xzy^2 = (xzy)y = xy$. If $xzy = y$, then $xzy = x^2zy = x(xzy) = xy$. If $xzy = yx$, then $xzy = x^2zy^2 = x(xzy)y = x(yx)y = (xy)^2 = xy$. If $xzy = xyx$, then $xzy = xzy^2 = (xzy)y = (xyx)y = (xy)^2 = xy$. If $xzy = yxy$, then $xzy = x^2zy = x(xzy) = x(yxy) = (xy)^2 = xy$. Hence, $xzy = xy$ for every $x, y, z \in S$.

$5 \Rightarrow 1$. Let every element of S be an idempotent and let $xzy = xy$ hold for every x, y, z of S. We shall prove that every subsemigroup of S is a quasi-ideal of S. Let A be an arbitrary subsemigroup of S. If $x \in SA \cap AS$, then $x = ue = fv$ for some $e, f \in A$ and for some $u, v \in S$. Thus we have $x = fv = f^2v = f(fv) = fue = fe \in A$. Hence $SA \cap AS \subset A$ and so A is a quasi-ideal of S. Therefore $\mathscr{F}(\mathbf{P}) \subset \mathscr{F}(\mathbf{H})$ and so by (6) $\mathbf{H} \leqq \mathbf{P}$. Evidently $x = x^2 \in \mathbf{P}(x)\mathbf{P}(x)$ for every $x \in S$. Corollary 2 implies that $\mathbf{P}\sigma\mathbf{P}$.

Remark 1. It follows from Theorems in [3] (pp. 108—109) that:

The conditions of Theorem 7 and the following conditions on a semigroup S are equivalent:

6. *Every pair of elements from S is regularly conjugate, i. e. $xyx = x$ for every $x, y \in S$.*

7. *S is anticommutative, i. e. $xy \neq yx$ for every pair of distinct elements x, y from S.*

A $\mathscr{C}$-closure operation $\mathbf{U}$ is said to be a *$\mathscr{Q}$-closure operation* if

$$\mathbf{U}(A) = \bigcup_{x \in A} \mathbf{U}(x) \text{ for each non empty } A \subset S \tag{7}$$

holds. Let $\mathscr{Q}(S)$ denote the set of all $\mathscr{Q}$-closure operations for a semigroup S. Evidently $\mathscr{Q}(S) \subset \mathscr{C}(S)$. It is clear that $\mathbf{L}, \mathbf{R}, \mathbf{M} \in \mathscr{Q}(S)$.

Let $\mathbf{U} \in \mathscr{C}(S)$. We define $\mathbf{U}^* \in \mathscr{Q}(S)$. If $A \subset S$, then $x \in \mathbf{U}^*(A)$ if and only if $\mathbf{U}(x) \cap A \neq \emptyset$. For $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$ we have

$$\mathbf{U} \leqq \mathbf{V} \Rightarrow \mathbf{U}^* \leqq \mathbf{V}^*, \tag{8}$$

$$\mathbf{U}^{**} \leqq \mathbf{U}. \tag{9}$$

(See [1].)

Let $\mathbf{U} \in \mathscr{C}(S)$. We shall introduce the equivalence $\overline{\mathbf{U}}$ on a semigroup S by: for $x, y \in S$, $x\overline{\mathbf{U}}y$ if and only if $\mathbf{U}(x) = \mathbf{U}(y)$. For any element x of S, let $\mathbf{U}_x$ denote the $\overline{\mathbf{U}}$-class of S containing x. (See [4].)

Ir follows from Theorem 4 [4] that

(10) $$\mathbf{U} = \mathbf{U}^* \Rightarrow \mathbf{U}_x \in \mathscr{F}(\mathbf{U}) \text{ for every } x \in S.$$

Theorem 1 [4] implies that

(11) $$A = \bigcup_{x \in A} \mathbf{U}_x \text{ for every non-empty set } A \text{ of } \mathscr{F}(\mathbf{U}^*).$$

Lemma 2. *Every maximal subgroup G of a semigroup S is an $\overline{\mathbf{H}}$-class of S.*

Proof. Let e be an identity of a maximal subgroup G of S. If $x \in G$, then evidently $x \in \mathbf{H}(e)$ and $e \in \mathbf{H}(x)$ and so by (2) and (4) $\mathbf{H}(x) = \mathbf{H}(e)$. Thus we have $x \in \mathbf{H}_e$ and so $G \subset \mathbf{H}_e$. It follows from [5] that $\mathbf{H}_e = \mathbf{R}_e \cap \mathbf{L}_e$ is a subgroup of S. Since G is a maximal subgroup of S, hence $G = \mathbf{H}_e$ which implies that G is an $\overline{\mathbf{H}}$-class.

Theorem 8. *The following conditions on a semigroup S are equivalent:*

1. *$\mathbf{H}^* \sigma \mathbf{U}$ holds for all $\mathbf{U} \in \mathscr{C}(S)$ where $\mathbf{H} \wedge \mathbf{H}^* \leqq \mathbf{U}$;*
2. *$\mathbf{H}^* \sigma \mathbf{U}$ holds for some $\mathbf{U} \in \mathscr{C}(S)$ where $\mathbf{H} \wedge \mathbf{H}^* \leqq \mathbf{U}$;*
3. *$\mathbf{H} \leqq \mathbf{H}^*$;*
4. *$\mathbf{H} = \mathbf{H}^*$;*
5. *S is a union of groups and $G_1 \cup G_2$ is a quasi-ideal of S for every pair of maximal subgroups G_1, G_2 of S;*
6. *S is a union of groups and $G_1 S G_2 \subset G_1 \cup G_2$ holds for every pair of maximal subgroups G_1, G_2 of S.*

Proof. $1 \Rightarrow 2$. Evident.

$2 \Rightarrow 3$. This follows from Theorem 1.

$3 \Rightarrow 4$. Let $\mathbf{H} \leqq \mathbf{H}^*$. By (8) and (9) we have $\mathbf{H}^* \leqq \mathbf{H}^{**} \leqq \mathbf{H}$ and hence $\mathbf{H} = \mathbf{H}^*$.

$4 \Rightarrow 5$. Let $\mathbf{H} = \mathbf{H}^*$. Since $\mathbf{P} \leqq \mathbf{H}$, hence, by (8) we have $\mathbf{P}^* \leqq \mathbf{H}^* = \mathbf{H}$. According to Theorem 8 [4], S is a union of groups. Let G_i ($i = 1, 2$) be maximal subgroups of S. It follows from Lemma 2 that G_i is an $\overline{\mathbf{H}}$-class and so, by (10), $G_i \in \mathscr{F}(\mathbf{H})$. Since $\mathbf{H} = \mathbf{H}^* \in \mathscr{Q}(S)$, hence $G_1 \cup G_2 \in \mathscr{F}(\mathbf{H})$ and so $G_1 \cup G_2$ is a quasi-ideal of S.

$5 \Rightarrow 6$. Let S be a union of groups and let $G_1 \cup G_2$ be a quasi-ideal of S for every pair of maximal subgroups G_1, G_2 of S. Then $G_1 S G_2 \subset (G_1 \cup G_2) S \cap \cap S(G_1 \cup G_2) \subset G_1 \cup G_2$.

$6 \Rightarrow 1$. Let S be a union of groups and let $G_1 S G_2 \subset G_1 \cup G_2$ hold for every pair of maximal subgroups of S. We shall prove that $\mathbf{H} \leqq \mathbf{H}^*$. Let $\varnothing \neq A \in \in \mathscr{F}(\mathbf{H}^*)$. It is known that S is a union of maximal subgroups. Lemma 2 implies that every $\overline{\mathbf{H}}$-class is a maximal subgroup of S. According to (11), A is a union of maximal subgroups of S. Let $x \in AS \cap SA$. Then $x = g_1 s_1 = s_2 g_2$ for some $s_1, s_2 \in S$, for some $g_1 \in G_1 \subset A$ and for some $g_2 \in G_2 \subset A$ where G_1, G_2 are maximal subgroups of S, Let e_i be an identity of a group G_i ($i = 1, 2$). Thus we have $x = g_1 s_1 = e_1 g_1 s_1 = e_1 s_2 g_2 \in G_1 S G_2 \subset G_1 \cup G_2 \subset A$. Therefore $AS \cap \cap SA \subset A$ and so A is a quasi-ideal of S. This means that $A \in \mathscr{F}(\mathbf{H})$. Since $\mathscr{F}(\mathbf{H}^*) \subset \mathscr{F}(\mathbf{H})$, hence, by (6), $\mathbf{H} \leqq \mathbf{H}^*$. Since S is a union of groups, hence S is regular and intraregular. According to Theorem 6, we have $\mathbf{H} \sigma \mathbf{H}$ and so, by Lemma 1, $\mathbf{H}^* \sigma \mathbf{U}$ where $\mathbf{H} \wedge \mathbf{H}^* = \mathbf{H} \leqq \mathbf{U} \in \mathscr{C}(S)$.

Put $\mathbf{O}(A) = A$ for each $A \subset S$. Then $\mathbf{O} \in \mathscr{Q}(S)$, $\mathbf{O} = \mathbf{O}^*$ and for every $\mathbf{U} \in \mathscr{C}(S)$,

$$\mathbf{O} \leqq \mathbf{U} \tag{12}$$

holds.

Theorem 9. *The following conditions on a semigroup S are equivalent:*

1. $\mathbf{O} \sigma \mathbf{U}$ *holds for all* $\mathbf{U} \in \mathscr{C}(S)$;
2. $\mathbf{O} \sigma \mathbf{U}$ *holds for some* $\mathbf{U} \in \mathscr{C}(S)$;
3. $\mathbf{P}^* \sigma \mathbf{U}$ *holds for all* $\mathbf{U} \in \mathscr{C}(S)$;
4. $\mathbf{P}^* \sigma \mathbf{U}$ *holds for some* $\mathbf{U} \in \mathscr{C}(S)$;
5. $\mathbf{H}^* \sigma \mathbf{P}$;
6. *Every non-empty subset of S is a quasi-ideal of S,*
7. *For every $x, y, z \in S$, either $xzy = x$ or $xzy = y$.*

Proof. It is clear that $6 \Leftrightarrow \mathbf{H} = \mathbf{O}$.

$1 \Rightarrow 2$ and $3 \Rightarrow 4$. Evident.

$2 \Rightarrow 6$. It follows from Theorem 1 that $\mathbf{H} \leqq \mathbf{O}$ and so, by (12), $\mathbf{H} = \mathbf{O}$.

$4 \Rightarrow 6$. Theorem 1 implies that $\mathbf{H} \leqq \mathbf{P}^*$ and so $\mathbf{P} \leqq \mathbf{H} \leqq \mathbf{P}^*$. By Lemma 12 [1], we obtain $\mathbf{P} = \mathbf{O}$. This implies $\mathbf{H} \leqq \mathbf{P}^* = \mathbf{O}^* = \mathbf{O}$. Hence, by (12), $\mathbf{H} = \mathbf{O}$.

$5 \Rightarrow 6$. Let $\mathbf{H}^* \sigma \mathbf{P}$. It follows from Theorem 1 that $\mathbf{H} \leqq \mathbf{P}$. Since $\mathbf{P} \leqq \mathbf{H}$, hence $\mathbf{H} = \mathbf{P}$ and so $\mathbf{P}^* \sigma \mathbf{P}$. Hence (by $4 \Rightarrow 6$) $\mathbf{H} = \mathbf{O}$.

$6 \Rightarrow 7$. Let $\mathbf{H} = \mathbf{O}$.Let $x, y, z \in S$. Evidently, $A = \{x, y\}$ is a quasi-ideal of S. Then $xzy \in AS \cap SA \subset A$ and thus we have either $xzy = x$ or $xzy = y$.

$7 \Rightarrow 1$, 3 and 5. Let $xzy \in \{x, y\}$ hold for every $x, y, z \in S$. Then $xyx = x$ for every pair of elements x, y from S. It follows from Remark 1 that $\mathbf{P} \sigma \mathbf{P}$ and $xy = xzy$ for every $z \in S$. This implies that either $xy = x$ or $xy = y$ and so every non-empty subset of S is a subsemigroup of S. Hence $\mathbf{P} = \mathbf{O}$ and so $\mathbf{O} \sigma \mathbf{O}$. It follows from Lemma 1 that $\mathbf{O} \sigma \mathbf{U}$ (for all $\mathbf{U} \in \mathscr{C}(S)$), $\mathbf{P}^* \sigma \mathbf{U}$ (for all $\mathbf{U} \in \mathscr{C}(S)$) and $\mathbf{H}^* \sigma \mathbf{P}$.

Remark 2. It follows from the proof of Theorem 9 that every element of S is an idempotent (see Remark 1). This implies that:

The conditions of Theorem 9 and the following condition on a semigroup S are equivalent:

8. *Every element of S is an idempotent and it satisfies at least one of the conditions of Theorem 8.*

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Katedra matematiky
elektrotechnické fakulty
Českého vysokého učení technického
Poděbrady