Zdena Riečanová

A Note on Absolutely Measurable Sets

Matematický časopis, Vol. 18 (1968), No. 4, 339--345

Persistent URL: http://dml.cz/dmlcz/126869

Matematický časopis 18 (1968), No. 4

A NOTE ON ABSOLUTELY MEASURABLE SETS

ZDENA RIEČANOVÁ, Bratislava

On a non-empty set X there are given two systems **C** and **U** of subsets of X fulfilling a certain system cf axioms. Let us take all outer measures γ defined on the smallest hereditary σ-ring $\mathbf{H}(\mathbf{C})$ over **C** such that all sets of **C** are γ-measurable. The sets of $\mathbf{H}(\mathbf{C})$ that are γ-measurable for all such outer measures γ are called absolutely measurable. We show (Theorem 2, Theorem 4 and Theorem 5) athat every absolutely measurable set E can be approximated (according to any examined outer measure) by sets of **C** from within and by sets of **U** from outside, under the assumption that either γ is finite on **C**, or there exist $U_n \in \mathbf{U}$ such that $\bigcup_{n=1}^{\infty} U_n \supset E$, $\gamma(U_n) < \infty$, $n = 1, 2, \ldots$

From this result we can get particularly the regularity of absolutely measurable sets according to all Carathéodory outer measures in a metric space ([2], Theorem 12, p. 11) and the regularity of absolutely measurable sets according to all Carathéodory outer measures in a locally compact Hausdorff topological space ([3], p. 203).

If X is a non-empty set of elements and **A** is an arbitrary system of subsets of X, then we denote by $\mathbf{R}(\mathbf{A})$ the smallest ring, by $\mathbf{S}(\mathbf{A})$ the smallest σ-ring, by $\mathbf{H}(\mathbf{A})$ the smallest hereditary σ-ring containing the system **A**. We shall use according to [1] the notions ring, σ-ring, hereditary σ-ring, as well as measure and outer measure. If X is a topological space and $A \subset X$, then by $\bar{A}$ we denote the smallest closed set containing the set A.

Let X be any non-empty set of elements. Let **C** and **U** be any systems of subsets of X fulfilling the following conditions:

V_1. $\emptyset \in \mathbf{C}$, $\emptyset \in \mathbf{U}$.

V_2. If $U_i \in \mathbf{U}$ for $i = 1, 2, \ldots$, then $\bigcup_{i=1}^{\infty} U_i \in \mathbf{U}$.

V_3. If $C_1, C_2 \in \mathbf{C}$, then $C_1 \cup C_2 \in \mathbf{C}$.

V_4. For any two sets U and C such that $U \in \mathbf{U}$, $C \in \mathbf{C}$ we have $U - C \in \mathbf{U}$, $C - U \in \mathbf{C}$.

V_5. For any set $C \in \mathbf{C}$ there exist sets $U \in \mathbf{U}$, $C_1 \in \mathbf{C}$ such that $C \subset U \subset C_1$.

V_6. $\mathbf{U} \subset \mathbf{S}(\mathbf{C})$.

V_7. For any set $C \in \boldsymbol{C}$ there exist sets $U_i \in \boldsymbol{U}$ ($i = 1, 2, \dots$) such that $C = \bigcap_{i=1}^{\infty} U_i$.

By γ we shall denote in the present paper an outer measure defined on $\boldsymbol{H}(\boldsymbol{C})$. In agreement with [1] we shall call a set $E \in \boldsymbol{H}(\boldsymbol{C})$ γ-measurable if $\gamma(A) = \gamma(A \cap E) + \gamma(A - E)$ for any $A \in \boldsymbol{H}(\boldsymbol{C})$.

Definition 1. *Put* $\mathscr{A}_1 = \{\gamma$: *all sets of* $\boldsymbol{C}$ *are* γ*-measurable*$\}$, $\mathscr{A}_2 = \{\gamma$: *all sets of* $\boldsymbol{C}$ *are* γ*-measurable and* γ *is finite on* $\boldsymbol{C}\}$.

Note 1. The following assertions can be easily proved:

1. $\boldsymbol{H}(\boldsymbol{C}) = \boldsymbol{H}(\boldsymbol{U})$, $\boldsymbol{S}(\boldsymbol{C}) = \boldsymbol{S}(\boldsymbol{U})$.

2. $\gamma \in \mathscr{A}_1$ if and only if each set $U \in \boldsymbol{U}$ is γ-measurable.

3. $\gamma \in \mathscr{A}_1$ if and only if $\gamma(A \cup B) = \gamma(A) + \gamma(B)$ for any pair of sets A, $B \in \boldsymbol{H}(\boldsymbol{C})$ for which there exists $C \in \boldsymbol{C}$ such taht $B \subset C$ and $A \cap C = \emptyset$.

4. $\gamma \in \mathscr{A}_1$ if and only if $\gamma(A \cup B) = \gamma(A) + \gamma(B)$ for any pair of sets A, $B \in \boldsymbol{H}(\boldsymbol{C})$ for which there exists $U \in \boldsymbol{U}$ such that $B \subset U$ and $A \cap U = \emptyset$.

Example 1. If X is a metric space, $\boldsymbol{C}$ the system of all closed subsets of X and $\boldsymbol{U}$ the system of all open subsets of X, then $\mathscr{A}_1$ is the set of all Carathéodory outer measures in X.

We easily find out that the systems $\boldsymbol{C}$ and $\boldsymbol{U}$ fulfil the conditions $V_1 - V_7$ and the system $\boldsymbol{H}(\boldsymbol{C})$ coincides with the system of all subsets of X. An outer measure in X is Carathéodory (i. e. $\gamma(A \cup B) = \gamma(A) + \gamma(B)$ for any $A, B \subset X$ such that $\varrho(A, B) > 0$) if and only if every closed subset of X is γ-emasurable. The proof of this assertion see e. g. in [6].

Example 2. If X is a locally compact Hausdorff topological space, $\boldsymbol{C}$ the system of all compact G_δ subsets of X and $\boldsymbol{U}$ the system of all open sets belonging to $\boldsymbol{S}(\boldsymbol{C})$, then $\mathscr{A}_1$ is the set of all Carathéodory outer measures defined on the smallest hereditary σ-ring over the system of all compact subsets of X.

We easily find out that the systems $\boldsymbol{C}$ and $\boldsymbol{U}$ fulfil the conditions V_1—V_7. If $\boldsymbol{C}_1$ is the system of all compact subsets of X, then $\boldsymbol{H}(\boldsymbol{C}_1) = \boldsymbol{H}(\boldsymbol{C})$. An outer measure on $\boldsymbol{H}(\boldsymbol{C}_1)$ is Carathéodory (i. e. $\gamma(A \cup B) = \gamma(A) + \gamma(B)$ for any A, $B \in \boldsymbol{H}(\boldsymbol{C}_1)$ such that there exist open U, V such that $\bar{A} \subset U$, $\bar{B} \subset V$, $U \cap V = \emptyset$) if and only if each set of $\boldsymbol{C}$ is γ-measurable.

The proof of the assertion that each set of $\boldsymbol{C}$ is measurable according to any Carathéodory outer measure see in [4]. On the contrary we shall show that if γ is an outer measure on $\boldsymbol{H}(\boldsymbol{C}_1)$ such that each set of $\boldsymbol{C}$ is γ-measurable, then γ is a Carathéodory outer measure on $\boldsymbol{H}(\boldsymbol{C}_1)$.

Let A, $B \in \boldsymbol{H}(\boldsymbol{C}_1)$ be such sets, that there are open sets U, V for which $\bar{A} \subset U$, $\bar{B} \subset V$, $U \cap V = \emptyset$. The set A is evidently σ-bounded, i. e. there exist compact sets K_n ($n = 1, 2, \dots$) such that $A \subset \bigcup_{n=1}^{\infty} K_n$. We have $\bar{A} \cap$

$\cap K_n \subset \bar{A} \subset U$ for any n. Because X is a locally compact Hausdorff topological space, there are open sets $U_n \in \mathbf{S}(\mathbf{C})$ such that $\bar{A} \cap K_n \subset U_n \subset U$. Put $O = \bigcup_{n=1}^{\infty} U_n$. Clearly $O \in \mathbf{S}(\mathbf{C})$ i. e. O is γ-measurable and hence

$$\gamma(A \cup B) = \gamma[(A \cup B) \cap O] + \gamma[(A \cup B) - O] = \gamma(A) + \gamma(B).$$

Definition 2. *Let γ be an outer measure on $\mathbf{H}(\mathbf{C})$,* a *set $E \in \mathbf{H}(\mathbf{C})$ is called:*
a. *outer regular according to γ, if $\gamma(E) = \inf\{\gamma(U): E \subset U \in \mathbf{U}\}$;*
b. *inner regular according to γ, if $\gamma(E) = \sup\{\gamma(C): E \supset C \in \mathbf{C}\}$;*
c. *regular according to γ, if it is at the same time outer regular and inner regular according to γ.*

Definition 3. *Put*
$\mathbf{B}_1 = \{E: E \in \mathbf{H}(\mathbf{C}), E$ is γ-measurable according to each $\gamma \in \mathscr{A}_1\}$,
$\mathbf{B}_2 = \{E: E \in \mathbf{H}(\mathbf{C}), E$ *is γ-measurable according to each* $\gamma \in \mathscr{A}_2\}$.

Theorem 1. $\mathbf{B}_1 = \mathbf{B}_2$.

Proof. Clearly $\mathbf{B}_1 \subset \mathbf{B}_2$. Let $E \notin \mathbf{B}_1$. Then there exists $\gamma \in \mathscr{A}_1$ such that E is not γ-measurable. It means that there is a set $A \in \mathbf{H}(\mathbf{C})$ such that $\gamma(A) < \gamma(A \cap E) + \gamma(A - E)$. Put $\gamma^*(B) = \gamma(B \cap A)$ for any $B \in \mathbf{H}(\mathbf{C})$. Clearly $\gamma^* \in \mathscr{A}_2$. We have

$$\gamma^*(E) = \gamma(E \cap A) < \gamma(A) - \gamma(A - E) = \gamma^*(A) - \gamma^*(A - E),$$

hence

$$\gamma^*(A) < \gamma^*(E) + \gamma^*(A - E) = \gamma^*(E \cap A) + \gamma^*(A - E).$$

It means that the set E is not γ^*measurable, i. e. $E \notin \mathbf{B}_2$. Therefore $\mathbf{B}_1 = \mathbf{B}_2$.

Definition 4. *A set $E \in \mathbf{H}(C)$ is called absolutely measurable, if $E \in \mathbf{B}_1 = \mathbf{B}_2$.*

Theorem 2. *Every absolutely measurable set is regular according to each outer measure $\gamma \in \mathscr{A}_2$.*

First we shall prove a few lemmas.

Lemma 1. *If $U_1, U_2 \in \mathbf{U}$ and there is a set $C \in \mathbf{C}$ such that $U_2 \subset C$, then $U_1 \cap U_2 \in \mathbf{U}$.*

Proof. $U_1 \cap U_2 = U_2 - (C - U_1)$. According to V_4 we have $C - U_1 \in \mathbf{C}$ and $U_2, U_2 - (C - U_1) \in \mathbf{U}$. Hence $U_1 \cap U_2 \in \mathbf{U}$.

Lemma 2. *If $E \in \mathbf{H}(\mathbf{C})$, then there are sets $C_n \in \mathbf{C}$ $(n = 1, 2, \ldots)$ such that $E \subset \bigcup_{n=1}^{\infty} C_n$.*

Proof. Let $\mathbf{B} = \{A : A \subset \bigcup_{n=1}^{\infty} C_n,\ C_n \in \mathbf{C},\ n = 1, 2, \ldots\}$. $\mathbf{B}$ is evidently a hereditary system, closed under countable unions, hence $\mathbf{B}$ is a hereditary σ-ring. Further $\mathbf{C} \subset \mathbf{B}$. Hence $\mathbf{H}(\mathbf{C}) \subset \mathbf{B}$.

Lemma 3. *If $\gamma \in \mathscr{A}_2$, then all sets of $\mathbf{C}$ are outer regular according to γ.*

Proof. Let $\gamma \in \mathscr{A}_2$ and $C \in \mathbf{C}$. According to V_5 there exist sets $U \in \mathbf{U}$, $D \in \mathbf{C}$ such that $C \subset U \subset D$. According to V_7 there are sets $U_n \in \mathbf{U} (n = 1, 2, \ldots)$ such that $C = \bigcap_{n=1}^{\infty} U_n$. Hence $C = \bigcap_{n=1}^{\infty} (U_n \cap U)$. According to Lemma 1 we have $U_n \cap U \in \mathbf{U}$ for $n = 1, 2, \ldots$ Further $\gamma(U_n \cap U) \leqq \gamma(D) < \infty$ $(n = 1, 2, \ldots)$. Put $V_n = U_n \cap U$, $W_n = \bigcap_{k=1}^{n} V_n$. Then $W_n \in \mathbf{U}$, $W_n \supset W_{n+1}$, $\gamma(W_n) < \infty$, $n = 1, 2, \ldots$ and $C = \bigcap_{n=1}^{\infty} W_n$.

Because the restriction of the function γ on $\mathbf{S}(\mathbf{C})$ is a measure on $\mathbf{S}(\mathbf{C})$ and by V_6 we have $\mathbf{U} \subset \mathbf{S}(\mathbf{C})$, then

$$\lim_{n \to \infty} \gamma(W_n) = \gamma(\bigcap_{n=1}^{\infty} W_n) = \gamma(C) < \infty.$$

Choose an arbitrary $\varepsilon > 0$. Then there is a positive integer n_0 such that $\gamma(W_{n_0}) < \gamma(C) + \varepsilon$. Thus we showed that $\gamma(C) = \inf\{\gamma(U) : C \subset U \in \mathbf{U}\}$.

Lemma 4. *If $\gamma \in \mathscr{A}_2$, then all sets of $\mathbf{S}(\mathbf{C})$ are regular according to γ.*

Proof. Let $\gamma \in \mathscr{A}_2$. Let $\bar{\gamma}$ be the function defined for $A \in \mathbf{S}(\mathbf{C})$ by the equality $\bar{\gamma}(A) = \gamma(A)$. Clearly $\bar{\gamma}$ is a measure on $\mathbf{S}(\mathbf{C})$, $\bar{\gamma}(C) < \infty$ for $C \in \mathbf{C}$ and according to Lemma 3, all sets of $\mathbf{C}$ are outer regular according to $\bar{\gamma}$. By theorem 8, [5] all sets of $\mathbf{S}(\mathbf{C})$ are regular according to $\bar{\gamma}$ and hence according to γ.

Lemma 5. *Let $\gamma \in \mathscr{A}_2$. For any $A \in \mathbf{H}(\mathbf{C})$ put*

$$\gamma_0(A) = \inf \{\gamma(U) : A \subset U \subset \mathbf{U}\}.$$

Then $\gamma_0 \in \mathscr{A}_2$.

Proof. Let $\gamma \in \mathscr{A}_2$. Then the function $\bar{\gamma}$ defined for $A \in \mathbf{S}(\mathbf{C})$ by the equality $\bar{\gamma}(A) = \gamma(A)$ is a measure on $\mathbf{S}(\mathbf{C})$. Let γ^* be the outer measure induced by $\bar{\gamma}$ on $\mathbf{H}(\mathbf{S}(\mathbf{C})) = \mathbf{H}(\mathbf{C})$. For $E \in \mathbf{H}(\mathbf{C})$ we have

$$\gamma^*(E) = \inf \{\gamma(F): E \subset F \in \mathbf{S}(\mathbf{C})\}.$$

Let us show that $\gamma_0(E) = \gamma^*(E)$ for any $E \in \mathbf{H}(\mathbf{C})$. $\mathbf{U} \subset \mathbf{S}(\mathbf{C})$ by V_6 and hence $\gamma_0(E) \geqq \gamma^*(E)$. By the help of the regulartiy (with respect to γ) of the sets of $\mathbf{S}(\mathbf{C})$ of a finite measure, we easily prove that $\gamma^*(E) = \gamma_0(E)$.

We showed that γ_0 is the outer measure induced by a measure on $\mathbf{S}(\mathbf{C})$. Hence all sets of $\mathbf{S}(\mathbf{C})$ are γ_0-measurable, therefore $\gamma_0 \in \mathscr{A}_1$. If $C \in \mathbf{C}$, then $\gamma_0(C) = \gamma^*(C) = \gamma(C) < \infty$, i. e. $\gamma_0 \in \mathscr{A}_2$.

Lemma 6. *Let $\gamma \in \mathscr{A}_2$. Let for $A \in \boldsymbol{H}(\boldsymbol{C})$*

$$\gamma_0(A) = \inf\{\gamma(U): A \subset U \in \boldsymbol{U}\}.$$

Then, if E is γ_0-measurable and $\gamma_0(E) < \infty$, E is outer regular according to γ.

Proof. According to Lemma 5, $\gamma_0 \in \mathscr{A}_2$. Let E be γ_0-measurable and $\gamma_0(E) < < \infty$. Choose an arbitrary $\varepsilon > 0$. Then from the definition of γ_0 it follows that there is $U \in \boldsymbol{U}$ such that $U \supset E$ and

$$\gamma_0(E) + \varepsilon > \gamma(U) = \gamma_0(U) = \gamma_0(U \cap E) + \gamma_0(U - E) \geqq \gamma_0(E) + + \gamma(U - E),$$

i. e. $\varepsilon > \gamma(U - E)$ and the assertion is evident.

Proof of Theorem 2. Let $\gamma \in \mathscr{A}_2$ and let $E \in \boldsymbol{H}(\boldsymbol{C})$ be any absolutely measurable set. First we show that E is outer regular according to γ.

According to Lemma 2, there are sets $C_n \in \boldsymbol{C}$ $(n = 1, 2, \ldots)$ such that $E \subset \bigcup_{n=1}^{\infty} C_n$. Put $E_n = E \cap C_n$ $(n = 1, 2, \ldots)$. Then E_n are absolutely measurable sets for $n = 1, 2, \ldots$ and we have $E = \bigcup_{n=1}^{\infty} E_n$.

Let γ_0 be the outer measure on $\boldsymbol{H}(\boldsymbol{C})$ defined for $A \in \boldsymbol{H}(\boldsymbol{C})$ by the equality $\gamma_0(A) = \inf\{\gamma(U): A \subset U \in \boldsymbol{U}\}$. According to Lemma 5 we have $\gamma_0 \in \mathscr{A}_2$ and hence E_n is γ_0-measurable. Futher we have $\gamma_0(E_n) \leqq \gamma_0(C_n) < \infty$ for $n = 1, 2, \ldots$ According to Lemma 6 the set E_n is outer regular according to γ and hence $E = \bigcup_{m=1}^{\infty} E_n$ is outer regular according to γ.

Let us show now that E is also inner regular according to γ, If there is $C \in \boldsymbol{C}$ such that $E \subset C$, then $C - E$ is absolutely measurable set and according to the first part of proof, $C - E$ is outer regular according to γ. Choose an arbitrary $\varepsilon > 0$. Then there is $U \in \boldsymbol{U}$, $U \supset C - E$ such that $\gamma(C - E) + \varepsilon > > \gamma(U)$. Thus $\gamma(U) - \gamma(C - E) < \varepsilon$. Further we have $C - U \subset E$ and according to V_4 we have $C - U \in \boldsymbol{C}$. We get

$$\gamma(E) - \gamma(C - U) = \gamma[E - (C - U)] \leqq \gamma[U - (C - E)] = = \gamma(U) - \gamma(C - E) < \varepsilon.$$

Now let E be any absolutely measurable set. According to Lemma 2 and V_3 there are sets $C_n \in \boldsymbol{C}$, $C_n \subset C_{n+1}$ $(n = 1, 2, \ldots)$ such that $E \subset \bigcup_{n=1}^{\infty} C_n$. Hence we have $\gamma(E) = \lim_{n\to\infty} \gamma(E \cap C_n)$. If $c < \gamma(E)$, then there is n_0 such that $c < \gamma(E \cap C_{n_0})$ and by the previous the set $E \cap C_{n_0}$ is inner regular. Hence there exists a set $C \in \boldsymbol{C}$ such $C \subset E \cap C_{n_0} \subset E$ and $c < \gamma(C)$. Hence E is inner regular according to γ.

Note 2. As a corollary we get from Theorem 2 the theorem on regularity of absolutely measurable sets according to any Carathéodory outer measure in a locally compact Huasdorff topological space ([3], p. 203).

Theorem 3. *Every absolutely measurable set in a metric space X is regular according to any Carathéodory outer measure in X that is finite on the system of all bounded closed sets.*

Proof. Let X be a metric space. Let $\mathbf{C}$ be the system of all closed bounded subsets of X. Let $\mathbf{U}$ be the system of all open subsets of X. The systems $\mathbf{C}$ and $\mathbf{U}$ clearly fulfil the conditions $V_1 - V_5$, V_7. If A is any closed set, then $A = \bigcup_{n=1}^{\infty} (A \cap C_n^x)$, where $C_n^x = \{y : \varrho(x, y) \leqq n\}$, and x is an arbitrary but fixed element of X. Hence $A \in \mathbf{S}(\mathbf{C})$ and also every open set belongs to $\mathbf{S}(\mathbf{C})$. Hence V_6 is proved. The assertion of Theorem 3 follows from Theorem 2.

Theorem 4. *Every absolutely measurable set E_0 is inner regular according to each $\gamma \in \mathscr{A}_1$ for which $\gamma(E_0) < \infty$.*

Proof. Put $\gamma^*(E) = \gamma(E \cap E_0)$ for $E \in \mathbf{H}(\mathbf{C})$. Then clearly $\gamma^* \in \mathscr{A}_2$, hence E_0 is regular according to γ^*. Further

$$\gamma(E_0) = \gamma^*(E_0) = \sup\{\gamma^*(C) : E_0 \supset C \in \mathbf{C}\} = \sup\{\gamma(C) : E_0 \supset C \in \mathbf{C}\}$$

i. e. E_0 is regular according to γ.

Note 3. From Theorem 4 the theorem on the inner regularity of absolutely measurable sets in metric space follows ([2], theorem 12, p. 11). That is why we choose only in Theorem 4 a metric space for X, the system of all closed subsets of X for $\mathbf{C}$ and the system of all open subsets of X for $\mathbf{U}$. See also Example 1.

Note 4. The assertion does not hold that every absolutely measurable set E_0 is outer regular according to each $\gamma \in \mathscr{A}_1$ such that $\gamma(E_0) < \infty$.

Let $X = (-\infty, \infty)$ be the metric space with usual topology. For $E \subset X$ we define $\gamma(E)$ as the number of elements of the set E if E is finite and $\gamma(E) = \infty$ if E is infinite. Let $\mathbf{C}$ be the system of all closed subsets of X, $\mathbf{U}$ be the system of all open subsets of X. Then evidently $\mathbf{C}$ and $\mathbf{U}$ fulfill the conditions $V_1 - V_7$. It is easy to show that $\gamma \in \mathscr{A}_1$. Clearly no finite subset is outer regular according to γ.

Theorem 5. *Every absolutely measurable set E is outer regular with respect to each $\gamma \in \mathscr{A}_1$ such that there is $U \in \mathbf{U}$, $U \supset E$ and $\gamma(U) < \infty$.*

Proof. Let $E \in \mathbf{B}_1$. Let $\gamma \in \mathscr{A}_1$ be such that there is $U \in \mathbf{U}$, $U \supset E$, $\gamma(U) < \infty$. For $A \in \mathbf{H}(\mathbf{C})$ define

$$\gamma_0(A) = \gamma(A \cap U).$$

Clearly $\gamma_0 \in \mathscr{A}_2$. According to Theorem 2 the set E is regular according to γ_0 and hence also according to γ.

Theorem 6. *Every absolutely measurable set is regular according to any outer measure $\gamma \in \mathscr{A}_1$ such that for any $C \in \mathbf{C}$ there is a sequence of sets $U_n \in \mathbf{U}$, $\gamma(U_n) < \infty$ $(n = 1, 2, \ldots)$ and $C \subset \bigcup_{n=1}^{\infty} U_n$.*

Proof. Let $\gamma \in \mathscr{A}_1$ have the mentioned property. Let $E \in \mathbf{B}_1$. Then there is an ascedent sequence of sets $U_n \in \mathbf{U}$ such that $\gamma(U_n) < \infty$ $(n = 1, 2, \ldots)$ and $E \subset \bigcup_{n=1}^{\infty} U_n$. According to Theorem 4 and Theorem 5 the sets $E \cap U_n$ are regular according to γ for $n = 1, 2, \ldots$ Clearly $E = \bigcup_{n=1}^{\infty} (E \cap U_n)$ and $\gamma(E) = \lim_{n \to \infty} \gamma(E \cap U_n)$. The regularity of the set E follows immediately from this fact.

REFERENCES

[1] Халмош П. Р., *Теория меры*, Москва 1953.

[2] Mickle E. J., Rado T., *On density theorems for outer measures*, Rozpr. mat., Warszawa 1960.

[3] Риечанова З., *Заметка о множествах регулярных по внешней мере Каратэодори*, Mat.-fyz. časop. *15* (1965), 200—205.

[4] Риечанова З., *О внейшей мере Каратэодори*, Mat.-fyz. časop. *12* (1962), 246-252.

[5] Риечанова З., *О регулярности меры*, Mat. časop. *17* (1967), 38-47.

[6] Saks S., *Theory of the Integral*, New York 1937.

Received February 1, 1967.

Katedra matematiky a deskriptívnej geometrie
Slovenskej vysokej školy technickej,
Bratislava