Matematický časopis 23 (1973), No. 2

MEASURABILITY OF FUNCTIONS OF TWO VARIABLES

JOZEF DRAVECKÝ and TIBOR NEUBRUNN, Bratislava

1. Introduction

Various results were published about the measurability of functions defined on a product space $X \times Y$ (cf. [4], [5], [6], [7], [10]). It was mostly the real functions or mappings into a metric space that were discussed. The present paper deals with functions defined on a product of two measurable spaces and taking values in a topological space Z. In some results the topology on Z is supposed to be induced by an ordering.

2. Notations and notions

The notion of a measurable space $(X, \mathscr{M})$ and of the product of two measurable spaces $(X, \mathscr{M})$ and $(Y, \mathscr{N})$ (notation $(X \times Y, \mathscr{M} \times \mathscr{N})$) is used in the usual sense (see e. g. [2]).

Given a measurable space $(X, \mathscr{M})$, a function defined on X with values in an ordered set Z is said to be upper (lower) measurable iff for any $a \in Z$ the set $\{x \in X; f(x) < a\}$ ($\{x \in X; f(x) > a\}$) is in $\mathscr{M}$. A function which is both upper and lower measurable is called weakly measurable.

If Z is a topological space not necessarily ordered, then the function f is called measurable iff $f^{-1}[B] \in \mathscr{M}$ for any open set B (and hence for any Borel set) in Z.

Throughout the paper, $(Z, <, \mathscr{G})$ will be referred to as an ordered topological space if and only if $\mathscr{G}$ is the topology indúced by the ordering $<$ of Z (cf. [3], 1. I). Every ordering in this paper is to be understood linear, unless otherwise stated.

3. Remarks on the types of measurability

Consider now an ordered topological space $(Z, <, \mathscr{G})$. As any measurable Z-valued function f is evidently both lower and upper measurable, a natural question arises whether the upper and lower measurability of f imply its measurability in the usual sense. The following example shows that this is not the case.

Example 3.1. Suppose $Z = R \times R$ is the set of all pairs of real numbers. Define the ordering $\prec$ as follows. Let $[x_1, y_1] \prec [x_2, y_2]$ whenever $x_1 < x_2$. In the case $x_1 = x_2$ put $[x_1, y_1] \prec [x_2, y_2]$ if and only if $y_1 < y_2$. Denote by $\mathscr{G}$ the order topology on Z and let $\mathscr{M}$ be the σ-ring generated by the family of all open intervals in Z. We are going to show that the identity function $j(x, y) \mapsto [x, y]$ defined on Z (viewed upon as measurable space $(Z. \mathscr{M})$) into Z (considered as an ordered topological space $(Z, \prec, \mathscr{G})$) is not measurable. The family of all open intervals in Z is of the power of the continuum and hence the generated σ-ring $\mathscr{M}$ is of the same power (cf. [2], page 26, 9C). Now let $\mathscr{F}$ be the set of all functions on the real line R into $\mathscr{U}$ where $\mathscr{U}$ is the usual topology of open sets in R. Evidently card $\mathscr{F} > \mathfrak{c}$, where $\mathfrak{c}$ is the power of the continuum. Given a function $f \in F$, the set $G_f = \bigcup_{t \in R} \{t\} \times f(t)$ is open in Z. Moreover if f and g are distinct elements of F then for some $t \in R$ we have $f(t) = g(t)$, which by definition implies $G_f \neq G_g$. Thus we have established an injection from $\mathscr{F}$ into $\mathscr{G}$ and summarizing we get $\operatorname{card} \mathscr{G} \geq \operatorname{card} \mathscr{F} > \mathfrak{c} = \operatorname{card} \mathscr{M}$. As a consequence there exists an open (and hence Borel) set $G_0 \in \mathscr{G}$ and $j^{-1}[G_0] = G_0 \notin \mathscr{M}$. This shows that j is not measurable in the usual sense although it is evidently both lower and upper measurable.

The proof of the following theorem employs the usual technique used for the real functions and therefore will be omitted.

Theorem 3.1. *Suppose $(X, \mathscr{M})$ is a measurable space and $(Z. <. \mathscr{G})$ an ordered topological space. A sufficient condition for the measurability of a function $f : X \to Z$ to be equivalent with the weak measurability of f is the equality of the σ-ring generated by the sets $\{x; x < a\}$, $\{x; x > a\}$ $(a \in Z)$ to the σ-algebra of all Borel sets in Z, that is the σ-algebra generated by the topology.*

It is not difficult to establish sufficient conditions for a weakly measurable function to be measurable other than the one given in Theorem 3.1. Suppose f is a mapping into an ordered topological space Z. If every open subset G of Z is a Lindelöf space (see [3]) then the σ-ring $\mathscr{B}$ generated by the topology on Z coincides with that generated by its base and by virtue of Theorem 3.1 every weakly measurable function taking values in Z is measurable. In fact any open set G can be covered by a collection $\{B_\gamma; \gamma \in \Gamma\}$ of base elements and under the assumption there is a countable subclass $\{B_n; n = 1, 2, \ldots\}$ of $\{B_\gamma; \gamma \in \Gamma\}$ such that $G = \bigcup \{B_n; n = 1, 2, \ldots\}$ and so G is in the σ-ring $\mathscr{B}_0$ generated by the base. As a consequence $\mathscr{B}_0$ contains all the σ-ring $\mathscr{B}$. The inclusion $\mathscr{B}_0 \subset \mathscr{B}$ is trivial.

In particular, if Z satisfies the second axiom of countability every weakly measurable Z-valued function is measurable since under the assumption any open subset of Z has a countable base and therefore is a Lindelöf space.

The following theorem will throw some more light on the notions of upper, lower and weak measurability.

Theorem 3.2. *Suppose $(Z, <, \mathscr{G})$ is an ordered topological space satisfying the first axiom of countability and let $(X, \mathscr{M})$ be a measurable space with $X \in \mathscr{M}$, that is, let $\mathscr{M}$ be a σ-algebra. Then the following statements are equivalent for a mapping $f: X \to Z$.*

(A) *f is lower measurable.*
(B) *f is upper measurable.*
(C) *f is weakly measurable.*
(D) $\{x \in X; f(x) \geq a\} \in \mathscr{M}$ *for each* $a \in Z$.
(E) $\{x \in X; f(x) \leq a\} \in \mathscr{M}$ *for each* $a \in Z$.

Proof. Since $\mathscr{M}$ is a σ-algebra, the equivalences $\mathrm{A} \Leftrightarrow \mathrm{E}$ and $\mathrm{B} \Leftrightarrow \mathrm{D}$ are obtained immediately from the assertions $\{x; f(x) \leq a\} = X - \{x; f(x) > a\}$ and $\{x; f(x) \geq a\} = X - \{x; f(x) < a\}$, respectively. We are going to prove $\mathrm{A} \Rightarrow \mathrm{D}$. Consider a point $a \in Z$ and a countable base $\{A_n; n = 1, 2, \ldots\}$ of its open neighbourhoods. If a is the first element in Z, then $\{x \in X; f(x) \geq a\} = X \in \mathscr{M}$, which was to be proved. Suppose then a is not the first point in Z. Evidently either

(i) each A_n contains a point $a_n < a$ or
(ii) there is A_n with no such point.

In the former case there holds $\{x; f(x) \geq a\} = \bigcap_n \{x; f(x) > a_n\} \in \mathscr{M}$, being a countable intersection of sets that were assumed measurable. In case (ii), since the topology on Z is induced by ordering, there is a point $b < a$ such that for no $x \in Z$ $b < x < a$ holds true. Then $\{x; f(x) \geq a\} = \{x; f(x) > b\} \in \mathscr{M}$ by assumption A. The proof of $\mathrm{B} \Rightarrow \mathrm{E}$ is analogous. To complete the proof of the Theorem we note that C implies both A and B and that either of the latter two implies the other and hence C by definition.

Remark. As shown by Example 3.1 even under the hypothesis of Theorem 3.2 the measurability of f is not implied by any of the listed conditions.

4. $\mathscr{P}$-systems

In investigating measurability as well as the upper and lower measurability of functions with values in topological spaces the notions defined below will be very useful.

Definition 4.1. *Let $(X, \mathscr{M})$ be a measurable space and let $\mathscr{P} = \{P_n^k; \emptyset \neq P_n^k \in \mathscr{M},$*

$n \in N_k$, $k = 1, 2, \ldots\}$, where N_k is either the set of all positive integers or a set $\{1, 2, \ldots, i_k\}$. $\mathscr{P}$ is called a $\mathscr{P}$-system on X iff $\bigcup \{P_n^k; n \in N_k\} = X$ for each $k = 1, 2, \ldots$.

Definition 4.2. *Suppose $(X, \mathscr{M})$ is a measurable space and $\mathscr{P}$ a $\mathscr{P}$-system on X. Let $(Z, \mathscr{G})$ be a topological space. A function $f\colon X \to Z$ is said to be regular at $x_0 \in X$ relative to $\mathscr{P}$ iff for any $G \in \mathscr{G}$ with $f(x_0) \in G$ there is k_0 such that for $k > k_0$, $x, x_0 \in P_n^k$ implies $f(x) \in G$.*

Definition 4.3. *Let $(X, \mathscr{M})$ be a measurable space and $(X, \mathscr{T})$ a topological space. A $\mathscr{P}$-system $\mathscr{P}$ on X is regular relative to $\mathscr{T}$ iff to any $A \in \mathscr{T}$ and any $x \in A$ there is k_0 such that for $k > k_0$, $x \in P_n^k$ implies $P_n^k \subset A$.*

Lemma 4.1. *Suppose $(X, \mathscr{M})$ is a measurable space. Let $(X, \mathscr{T})$ and $(Z, \mathscr{G})$ be topological spaces and let there be a $\mathscr{P}$-system $\mathscr{P}$ on X, regular relative to $\mathscr{T}$. Then every function $f\colon X \to Z$ which is continuous at a point $x_0 \in X$ is regular relative to $\mathscr{P}$ at x_0.*

Proof. Take $G \in \mathscr{G}$ with $f(x_0) \in G$. Due to the continuity of f there is an open set $A \ni x_0$ such that $f[A] \subset G$. Since $\mathscr{P}$ is a regular $\mathscr{P}$-system there is k_0 such that for $k > k_0$ we have $P_n^k \subset A$ whenever $x_0 \in P_n^k$. But then $x \in P_n^k$ implies $f(x) \in f[P_n^k] \subset f[A] \subset G$, as was to be proved.

Example 4.1. There is a simple way of constructing a regular $\mathscr{P}$-system in a separable metric space (X, ϱ) if the σ-ring $\mathscr{M}$ contains all the open sets of the metric space. The family $\{\Omega_n^k; n = 1, 2, \ldots; k = 1, 2, \ldots\}$ constructed in [7] is an example. It is sufficient even to put $P_n^k = \{x \in X; \varrho(x, x_n) < 1/k\}$, where $\{x_n; n = 1, 2, \ldots\}$ is a dense set in X. Then each P_n^k is open and hence measurable, and since $\{x_n; n = 1, 2, \ldots\}$ is dense in X, $\bigcup \{P_n^k; n = 1, 2, \ldots\} = X$ for every k, too. Therefore the family is indeed a $\mathscr{P}$-system on X. Now if $A \subset X$ is open and $x_0 \in A$, there is $\delta > 0$ such that the sphere $S(x_0, \delta) = \{x \in X; \varrho(x, x_0) < \delta\}$ is a subset of A. Take $k_0 > 2/\delta$. We are going to show that for $k > k_0$, if $P_n^k \in x_0$, then $P_n^k \subset A$. Since under the assumption the members of the $\mathscr{P}$-system are spheres with the diameter $2/k < \delta$ and $x_0 \in P_n^k$, then evidently $x \in P_n^k$ implies $\varrho(x, x_0) < 2/k < \delta$, and consequently $x \in S(x_0, \delta) \subset A$. The $\mathscr{P}$-system is therefore regular relative to the topology induced by the metric.

The following propositions will tell us something more about the existence of a regular $\mathscr{P}$-system.

Lemma 4.2. *Let $(X, \mathscr{T})$ be a topological space having a countable base (that is, satisfying the second axiom of countability) and $(X, \mathscr{M})$ a measurable space with $\mathscr{T} \subset \mathscr{M}$. Then there is a regular $\mathscr{P}$-system on X.*

Proof. Denote by $\{B_t; t = 1, 2, \ldots\}$ a countable base of the topology on X.

We are going to construct a countable family $\{P^k_n; n \in N_k, k = 1, 2, \dots\}$ such that $\bigcup \{P^k_n; n \in N_k\} = X$ for $k = 1, 2, \dots$. Clearly $\mathscr{A}_0 = \{B_n - \bigcup_{i=1}^{n-1} B_i;$ $n = 1, 2, \dots\}$ is a countable family of mutually disjoint sets. Let $\{P^1_n; n \in N_1\}$ be a sequence (finite or infinite as the case may be) consisting of all non-empty sets in $\mathscr{A}_0$. Now suppose that $\{P^k_n; n \in N_k\}$ has been constructed for $k =$ $= 1, 2, \dots, p$. Put $\mathscr{A}_p = \{B_{p+1} \cap P^p_n; n \in N_p\}$ and $\mathscr{B}_p = \{P^p_n - B_{p+1}; n \in N_p\}$. By enumerating all non-empty sets in $\mathscr{A}_p \cup \mathscr{B}_p$ we get a countable family $\{P^{p+1}_n; n \in N_{p+1}\}$ of pairwise disjoint sets. We have to prove that the family $\mathscr{P} = \{P^k_n; n \in N_k, k = 1, 2, \dots\}$ just constructed by induction is a $\mathscr{P}$-system. Each $P^k_n \in \mathscr{P}$ is obviously a non-empty $\mathscr{M}$-measurable set and for $k = 1, 2, \dots$ we have $\bigcup \{P^k_n; n \in N_k\} = \bigcup \{P^{k-1}_n; n \in N_{k-1}\} = \dots = \bigcup \{P^1_n; n \in N_1\} =$ $= \bigcup \{B_t; t = 1, 2, \dots\} = X$. Observe that for any n and $k = 1, 2, \dots$ the set P^k_n is a subset of some P^{k-1}_m and this implies that if $x \in P^{k_1}_n$, $x \in P^{k_2}_m$, $k_1 < k_2$, then $P^{k_2}_m \subset P^{k_1}_n$ due to P^k_i's being mutually disjoint for any fixed k.

Now let A be an open subset of X and let $x_0 \in A$. Since $\{B_t; t = 1, 2, \dots\}$ is a base there is k_0 with $x_0 \in B_{k_0} \subset A$. As $\bigcup \{P^{k_0}_n; n \in N_{k_0}\} = X$ the point x_0 must fall in some $P^{k_0}_m \subset B_{k_0}$. We have already seen that $k > k_0$, $x_0 \in P^k_n$ implies $P^k_n \subset P^{k_0}_m \subset B_{k_0} \subset A$ and therefore $\mathscr{P}$ is regular relative to $\mathscr{T}$.

Lemma 4.3. *Suppose $(X, \mathscr{T})$ is a topological space and $(X, \mathscr{M})$ a measurable space such that $\mathscr{T} \subset \mathscr{M}$.* Let *$\mathscr{P}$ be a regular $\mathscr{P}$-system on X. Then $(X, \mathscr{T})$ satisfies the Suslin condition, that is, any family $\mathscr{S} \subset \mathscr{T}$ of non-empty mutually disjoint sets is countable.*

Proof. We first show that to any element x_0 of a set $A \in \mathscr{T}$ there can be found a $P^k_n \in \mathscr{P}$ to satisfy $x_0 \in P^k_n \subset A$. By virtue of the regularity of $\mathscr{P}$, for $x_0 \in A$ there is a k_0 such that $k > k_0, x_0 \in P^k_n$ implies $P^k_n \subset A$. However to any k there is n with $P^k_n \ni x_0$ and so a P^k_n with the desired properties exists. Now choose a point x_S in every $S \in \mathscr{S}$ and let $\bar{S}$ be a set with $\bar{S} \in \mathscr{P}$ and $x_S \in \bar{S} \subset S$. Since the sets in $\mathscr{S}$ are all mutually disjoint we have $\bar{S}_1 \neq \bar{S}_2$ whenever $S_1 \neq S_2$ and so a one-one mapping is constituted from $\mathscr{S}$ into $\mathscr{P}$, the latter being countable by the definition of the $\mathscr{P}$-system. Then card $\mathscr{S} \leqslant$ $\leqslant$ card $\mathscr{P} \leqslant \aleph_0$ what was to be proved.

Corollary. *A metric space is separable if and only if there is a regular $\mathscr{P}$-system in it.*

Proof. Given a separable metric space, it obviously satisfies the second axiom of countability and the existence of a regular $\mathscr{P}$-system is established by Lemma 4.2. Conversely, let there be a regular $\mathscr{P}$-system $\mathscr{P}$ in a metric space (X, ϱ). If it were not separable then there would exist an uncountable family $\mathscr{B}$ of mutually disjoint open sets in it (see [9]), which contradicts Lemma 4.3.

5. Upper and lower measurability

Theorem 5.1. *Suppose $(X, \mathscr{M})$ is a measurable space and $\mathscr{P}$ a $\mathscr{P}$-system on X. Let $(Y, \mathscr{N})$ be a measurable space and $(Z, <, \mathscr{G})$ an ordered topological space. If a function f on $X \times Y$ into Z has all its x-sections $(x \in X)$ lower $\mathscr{N}$-measurable and if all the y-sections $(y \in Y)$ of f are regular relative to $\mathscr{P}$ at each $x \in X$, then f is a pointwise limit of a sequence $\{f_k; k = 1, 2, \ldots\}$ of lower $\mathscr{M} \times \mathscr{N}$-measurable functions.*

Proof. Let $\mathscr{P} = \{P_n^k\}_{n,k}$ be the above mentioned $\mathscr{P}$-system. Put $Q_1^k = P_1^k$ and for $n \geqslant 2$, $Q_n^k = P_n^k - \bigcup_{i=1}^{n-1} P_i^k$. For any fixed $k = 1, 2, \ldots$ the sets Q_n^k are mutually disjoint relative to n while $\bigcup_n Q_n^k = X$. Moreover, $Q_n^k \in \mathscr{M}$ for all n, k.

Since $Q_n^k \subset P_n^k$ and the sections f^y are regular relative to $\mathscr{P}$ in every $x \in X$, we have for any $x_0 \in X$ and $G \in \mathscr{G}$ with $f^y(x_0) \in G$

$$\underset{k_0}{\exists} \ \underset{k > k_0}{\forall} \ \underset{x \in X}{\forall} \ [(x, x_0 \in Q_n^k) \Rightarrow (f^y(x) \in G)]$$

Choose a point x_n^k in each Q_n^k. Since $X = \bigcup_n Q_n^k$ and $n_1 \neq n_2$ implies $Q_{n_1}^k \cap \cap Q_{n_2}^k = \emptyset$, there exists to any $x \in X$ a unique $n = n(k, x)$ with $x \in Q_n^k$. Define $f_k(x, y) = f(x_n^k, y)$, where x_n^k is the point that had been chosen from $Q_n^k \ni x$. All the functions f_k, $k = 1, 2, \ldots$ are lower $\mathscr{M} \times \mathscr{N}$-measurable. In fact, for any $a \in Z$ we have

$$\{(x, y) \in X \times Y; f_k(x, y) > a\} = \bigcup_n Q_n^k \times \{y \in Y; f(x_n^k, y) > a\},$$

the sets

$$Q_n^k \times \{y \in Y; f(x_n^k, y) > a\}$$

being all in $\mathscr{M} \times \mathscr{N}$.

Now consider $(x, y) \in X \times Y$ and $G \in \mathscr{G}$ with $f(x, y) \in G$. By the regularity of f^y take k_0 such that for $k > k_0$ $f(x_1, y) \in G$, whenever $x_1, x \in P_n^k$. Consequently for $k > k_0$ $f(x_n^k, y) = f_k(x, y) \in G$. This gives $\lim_k f_k(x, y) = f(x, y)$ and completes the proof.

Remark. The proposition obtained from Theorem 5.1 by replacing the word "*lower*" by "*upper*" or by "*weakly*" is evidently true, too.

Lemma 5.1. *Let $(X, \mathscr{M})$ be a measurable space and $(Z, <, \mathscr{G})$ an ordered topological space. Suppose $(Z, \mathscr{G})$ satisfies the first axiom of countability. If a sequence $\{f_i; i = 1, 2, \ldots\}$ of upper (lower) measurable functions defined on X and taking values in Z converges pointwise in the topology $\mathscr{G}$ to a function f, then f is upper (lower) measurable.*

Proof. For the upper measurability of f we have to prove $\{x \in X; f(x) < a\} \in \in \mathscr{M}$ for any $a \in Z$. In the case a is the first element of Z, then $\{x \in X; f(x) < a\} = \emptyset \in \mathscr{M}$. Suppose therefore that a is not the first point in Z and consider a countable base $\{U_n; n = 1, 2, \ldots\}$ of neighbourhoods of a. One of the following two cases may arise.

(i) For some n, U_n contains no point $z < a$. Since the topology is induced by the order this implies the existence of a point $b \in Z$ such that $b < a$ and for no $z \in Z$ we have $b < z < a$. But then

$$\text{(1)} \quad \{x \in X; f(x) < a\} = \{x \in X; f(x) \leqslant b\} = \{x \in X; \exists_{m} \forall_{i>m} f_i(x) \leqslant b\} =$$
$$= \bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} \{x \in X; f_i(x) < a\}$$

(ii) For each U_n there is a point a_n with $U_n \ni a_n < a$. Then the sequence $\{a_n; n = 1, 2, \ldots\}$ converges to a in the topology $\mathscr{G}$. In a similar way as in case (i) it can be shown that

$$\text{(2)} \qquad \{x \in X; f(x) < a\} = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} \{x \in X; f_i(x) < a_n\}.$$

Since the f_i's are upper measurable and $\mathscr{M}$ is a σ-ring, the sets on the right-hand sides of both (1) and (2) are in $\mathscr{M}$, which proves the upper measurability of f. The proof of the lower measurability is analogous and therefore will be omitted.

Corollary. *Under the assumptions of the last Lemma a limit function of a sequence of weakly measurable functions is itself weakly measurable.*

Theorem 5.2. *Suppose $(X, \mathscr{M})$ is a measurable space and $\mathscr{P}$ a $\mathscr{P}$-system on X. Let $(Y, \mathscr{N})$ be a measurable space and $(Z, <, \mathscr{G})$ an ordered topological space. Suppose $(Z, \mathscr{G})$ satisfies the first axiom of countability. If a function $f: X \times Y \to Z$ has all its x-sections lower (upper, weakly) $\mathscr{N}$-measurable and if all its y-sections are regular relative to $\mathscr{P}$ in every $x \in X$, then f is lower (upper, weakly) $\mathscr{M} \times \mathscr{N}$-measurable.*

Proof. By Theorem 5.1, f is a limit of lower (upper, weakly) measurable functions. and by Lemma 5.1 it is itself lower (upper, weakly) measurable, since $(Z, \mathscr{G})$ is assumed to satisfy the first axiom of countability.

Theorem 5.3. *Let $(X, \mathscr{M})$ be a measurable space and $\mathscr{T} \subset \mathscr{M}$ a topology on X with a countable base. Suppose $(Y, \mathscr{N})$ is a measurable space and $(Z, <, \mathscr{G})$ an ordered topological space. Let $f: X \times Y \to Z$ have all its x-sections lower (upper, weakly) measurable and all its y-sections continuous at each $x \in X$. Then f is a limit of a sequence of lower (upper, weakly) $\mathscr{M} \times \mathscr{N}$-measurable* *functions. If, besides, $(Z, \mathscr{G})$ satisfies the first axiom of countability, f is lower (upper, weakly) measurable.*

Proof. By Lemma 4.2 there is a regular $\mathscr{P}$-system on X and by Lemma 4.1 the y-sections of f are regular relative to it. Now we may apply Theorems 5.1 and 5.2 to obtain what was asserted.

6. Measurability

In this section we are going to investigate the measurability of functions mapping a product space $X \times Y$ into a topological space not necessarily ordered.

Theorem 6.1. *Let $(X, \mathscr{M})$ and $(Y, \mathscr{N})$ be measurable spaces, let $\mathscr{P}$ be a $\mathscr{P}$-system on X and $(Z, \mathscr{G})$ a topological space. If a function $f: X \times Y \to Z$ has $\mathscr{N}$-measurable x-sections for $x \in X$ and if its sections f^y for $y \in Y$ are regular relative to $\mathscr{P}$ at each $x \in X$, then there exists a sequence $\{f_k;\ k = 1, 2, \ldots\}$ of $\mathscr{M} \times \mathscr{N}$-measurable functions which converges pointwise to f on $X \times Y$.*

Proof. The proof of Theorem 5.1 can be adapted without difficulty. The functions f_k, $k = 1, 2, \ldots$ are constructed in the same way and their measurability is shown analogously.

The following Theorem provides yet another sufficient condition for a function $f: X \times Y \to Z$ to be a limit of a sequence of measurable functions. The idea of such a condition for the measurability of real-valued functions was suggested in an oral communication by L. Mišík.

Theorem 6.2. *Let $(X, \mathscr{M})$ and $(Y, \mathscr{N})$ be measurable spaces and let $(Z, \mathscr{G})$ be a topological space. Suppose f is a function on $X \times Y$ into Z and such that for $x \in X$ the sections f_x are $\mathscr{N}$-measurable. If there is a countable subfamily $\mathscr{B}$ of $\mathscr{M}$ such that $\mathscr{W} = \{(f^y)^{-1}[G];\ y \in Y,\ G \in \mathscr{G}\}$ is contained in the least totally additive family containing $\mathscr{B}$ (that is $\mathscr{W} \subset \{\bigcup \mathscr{V};\ \mathscr{V} \subset \mathscr{B}\}$), then f is a pointwise limit of a sequence of $\mathscr{M} \times \mathscr{N}$-measurable functions.*

Proof. In view of Theorem 6.1 it is sufficient to show that there is a $\mathscr{P}$-system on X such that for every $y \in Y$ the section f^y is regular relative to it at each $x_0 \in X$. By the same construction as in the proof of Lemma 4.2 starting from the countable family $\mathscr{B} = \{B_t;\ t = 1, 2, \ldots\}$ we obtain a family $\{P_n^k\}_{n,k}$ of $\mathscr{M}$-measurable sets such that for any fixed k we have $\bigcup_n P_n^k = X$ and $P_n^k \cap P_m^k = \emptyset$ for $n \neq m$. Further, if $x_0 \in P_n^{k_1}$, $x_0 \in P_m^{k_2}$, $k_1 < k_2$, then $P_m^{k_2} \subset P_n^{k_1}$. To prove the regularity in $x_0 \in X$ consider an open set $G \ni f^y(x_0)$. Then $x_0 \in (f^y)^{-1}[G] \in \mathscr{W}$ and since $\mathscr{W}$ is "generated" by $\mathscr{B}$, there is a set in $\mathscr{B}$, say B_{k_0}, such that $x_0 \in B_{k_0}$ and $B_{k_0} \subset (f^y)^{-1}[G]$. If $k > k_0$ and $x_0, x \in P_n^k$, then clearly there is m with $x_0, x \in P_m^{k_0}$ and due to the construction of $\{P_n^k\}_{n,k}$

we have $P_m^{k_0} \subset B_{k_0}$. Now $f^y(x) \in f^y[B_{k_0}] \subset f^y[(f^y)^{-1}[G]] = G$, which completes the proof.

Having established sufficient conditions for a function of two variables to be a limit of a sequence of measurable functions, we need a sufficient condition for such a limit function to be itself measurable. A proposition equivalent to the following is proved in [1], page 28.

Lemma 6.1. Let $(X, \mathscr{M})$ *be a measurable space and* $(Z, \mathscr{G})$ *a topological space having the property*

(φ) *To every open set* G *in* $(Z, \mathscr{G})$ *there is a continuous real-valued function* φ *defined on* Z *with* $\varphi(z) \neq 0$ *if and only if* $z \in G$.

Let $g_i \colon X \to Z$, $i = 1, 2, \ldots$ *be* $\mathscr{M}$*-measurable functions and let there for each* $x \in X$ *exist* $g(x) = \lim_i g_i(x)$. *The function* g *thus defined is then also* $\mathscr{M}$*-measurable.*

Example 6.1. Any pseudometrizable topological space has the property (φ). It is enough to consider the function $\varphi(z) = \inf\{\mathrm{d}(z, y); y \in Z - G\}$, where d is a pseudometric on Z.

What has been said in this section implies at once

Theorem 6.3. *If the hypotheses of Theorem 6.1 or of Theorem 6.2 hold and if moreover* Z *has the property* (φ), *then the function* f *is measurable on the product space* $(X \times Y, \mathscr{M} \times \mathscr{N})$.

Theorem 6.4. *Let* $(X, \mathscr{M})$ *and* $(Y, \mathscr{N})$ *be measurable spaces and* $(X, \mathscr{T})$ *a topological space. Let there be a regular* $\mathscr{P}$*-system* $\mathscr{P}$ *on* X. *(This is true in particular if* $\mathscr{T} \subset \mathscr{M}$ *and* $\mathscr{T}$ *has a countable base.) Suppose* $(Z, \mathscr{G})$ *is a topological space. If a function* $f \colon X \times Y \to Z$ *has all its* x*-sections* $\mathscr{N}$*-measurable and all its* y*-sections continuous at every* $x \in X$, *then* f *is a limit of a sequence of* $\mathscr{M} \times \mathscr{N}$*-measurable functions. If, besides,* $(Z, \mathscr{G})$ *has the property* (φ), *then* f *is* $\mathscr{M} \times \mathscr{N}$*-measurable.*

Proof. Obvious.

Considering the problem of plane measurability of a function $f(x, y)$ (that is the problem when the component-spaces are supposed to be real lines with Borel or Lebesgue measurability), the left or right continuity instead of the continuity of the sections of $f(x, y)$ is sometimes assumed (see [5]). In what follows (Theorem 6.5) we show that the idea of $\mathscr{P}$-systems can be applied also in such cases.

Definition 6.1. *A partially ordered metric space* $(X, \mathrm{d}; <)$ *is said to have the property* U *iff*

(1) X *is a partially ordered set and*

(2) *there is a countable dense set* H *in* X *such that for any* $x \in X$ *we have* $x = \lim_n x_n$ *where* $x_n \in H$, $x_n \geqslant x$, $n = 1, 2, \ldots$.

Theorem 6.5. *Let (X, d) be a metric space with the property U. Suppose $(X, \mathscr{M})$ is a measurable space such that $\mathscr{M}$ contains all the open sets in (X, d) and also all the intervals $I_a = \{x \in X; x \leq a\}$, $a \in X$. Let $(Y, \mathscr{N})$ be a measurable space and $(Z, \mathscr{G})$ a topological space with the property (φ). If $f : X \times Y \to Z$ is a function having all its x-sections $\mathscr{N}$-measurable and all its y-sections continuous from the right at each $x \in X$, then f is $\mathscr{M} \times \mathscr{N}$-measurable.*

Remark. Continuity of f^y from the right means that for any $x \in X$ and $G \ni f(x)$, $G \in \mathscr{G}$ there is $\delta > 0$ such that for all $x' \geqslant x$ with $\mathrm{d}(x, x') < \delta$, $f^y(x') \in G$ holds true.

Proof of Theorem 6.5. Consider a sequence $\{z_n; n = 1, 2, \ldots\}$ of all points in H. Put $P_n^k = \{x \in X; \mathrm{d}(x, z_n) < 1/k,\ x \leqslant z_n\}$. Due to the property U we have $\bigcup \{P_n^k; n = 1, 2, \ldots\} = X$ and $\{P_n^k\}_{n,k}$ is easily seen to be a $\mathscr{P}$-system on X. The continuity of f^y from the right implies its regularity relative to $\mathscr{P}$. Applying Theorem 6.3 we get what was to be proved.

The assertions of Theorems 5.3 and 6.4 may hold true even if there is no regular $\mathscr{P}$-system on X. For the space X, take the set of all pairs $[t, i]$ with $0 < t < 1$ and $i \in \{0, 1\}$, with lexicographical ordering and the topology induced by the order. Let $\mathscr{M}$ be the least σ-ring containing all the intervals and also the sets $\{[t, 0];\ 0 < t < 1\}$ and $\{[t, 1]; 0 < t < 1\}$. Let $(Y, \mathscr{N})$ be any measurable space and Z the real line. It is shown in [8] that to any function on $X \times Y$ with (lower) measurable x-sections and continuous y-sections there is a sequence of (lower) measurable functions converging to it. We prove now that in this example no regular $\mathscr{P}$-system exists on X.

If there were a regular $\mathscr{P}$-system $\mathscr{P}$ on X, then to any point, say $[t, 0]$ and any open set containing it, say $U_t = \{x \in X;\ x < [t, 1]\}$, there would be a $P_t \in \mathscr{P}$ with $[t, 0] \in P_t \subset U_t$. This evidently implies that $[t, 0]$ is the greatest element in P_t. The injection from the interval (0,1) into $\mathscr{P}$ just established proves that $\mathscr{P}$ is uncountable, which is a contradiction.

REFERENCES

[1] ДЫНКИН, Е. Б.: Основания теории марковских процессов. Москва 1959.

[2] HALMOS, P. R.: Measure Theory. Princeton 1968.

[3] KELLEY. J. L.: General Topology (Russian translation Общая топология, Москва 1968).

[4] MAHOWALD, M.: On the measurability of functions in two variables. Proc. Amer. Math. Soc. *13*, 1962, 410–411.

[5] MARCZEWSKI, E.–RYLL-NARDZEWSKI, C.: Mesurabilité des fonctions des plusieurs variables, Ann. Soc. polon. Math. *25*, 1951, 145–154.

[6] MICHAEL, J. H.–RENNIE, B. C.: Measurability of functions of two variables. J.Austral. Math. Soc. *1*, 1959, 21–26.

[7] NEUBRUNN, T.: Merateľnosť niektorých funkcií na kartézskych súčinoch. Mat.-fyz. časop. *10*, 1960, 216—221.
[8] NEUBRUNN, T.: Príspevok k merateľným a metrickým priestorom. Habilitačná práca. Bratislava 1963.
[9] NEUBRUNN, T.—SMÍTAL, J.—ŠALÁT, T.: On certain properties characterizing locally separable metric spaces, Časop. pěstov. mat. *92*, 1967, 157—161.
[10] URSELL, H. D.: Some methods of proving measurability, Fundam. math. *32*, 1939, 311—330.

Received December 2, 1971

Katedra matematickej analýzy
Prírodovedeckej fakulty Univerzity Komenského
Bratislava