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ON THE ABEL-TYPE SERIES

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1.

This paper is divided into seven parts. The first of them forms an introduction. The next part presents a proof of the combinatorial identity

$$\sum_{i=0}^{n} \frac{(\alpha + i)^i}{i!} \frac{(\beta - i)^{n-i}}{(n - i)!} = \sum_{\nu=0}^{n} \frac{(\alpha + \beta)^\nu}{\nu!} \tag{1.1}$$

and the object of the third part is to show how from the latter Abel's generalized binomial formula [1]

$$(x + \alpha)^n = \sum_{i=0}^{n} \binom{n}{i} \alpha(\alpha - \beta i)^{i-1}(x + \beta i)^{n-i} = \tag{1.2}$$

$$= x^n + \binom{n}{1}\alpha(x + \beta)^{n-1} + \binom{n}{2} \alpha(\alpha - 2\beta)(x + 2\beta)^{n-2} + \ldots$$

can be deduced.

The relation (1.1) has not been unknown, on the contrary, it was proved with the help of Abel's identity by Birnbaum and Pyke in their article [2].

The different examples in the fourth part serve to illustrate how the various relations of the Abel-type may be established with the help of identity (1.1).

Because for $\alpha = 0$ the equation (1.2) gives the identity $x^n = x^n$, we may assume $\alpha \neq 0$. Further if we substitute $-\beta$ for β, the formula (1.2) may be written as follows

$$\sum_{i=0}^{n} \frac{1}{\alpha + \beta i} \frac{(x - \beta i)^{n-i}}{(n - i)!} \frac{(\alpha + \beta i)^i}{i!} = \frac{1}{\alpha} \frac{(x + \alpha)^n}{n!}. \tag{1.3}$$

But after replacing in general

$$\frac{n^k}{k!} \quad \text{by} \quad \binom{n}{k} = \frac{(n)_k}{k!}$$

we see that the Abel formula in the form (1.3) is similar to the identity

$$(1.4) \qquad \sum_{s=0}^{n} \frac{1}{q+sd} \binom{p-sd}{n-s} \binom{q+sd}{s} = \frac{1}{q} \binom{p+q}{n}$$

which is a special case of the Hagen's combinatorial formula no. 17 [3].

Thus (1.3) may be proved in a way analogous to the method which I have used to prove (1.4) in paper [4]. The proof which is of some importance for what follows, is contained in the fifth part.

In concluding her paper [5], O. Engelberg mentions as the result of certain probabilities the following binomial identity

$$(1.5) \quad \sum_{i=0}^{b-1} \frac{1}{1+\mu i} \binom{(1+\mu)(b-i)-2}{b-i-1} \binom{(1+\mu)i}{i} = \frac{1}{1+\mu} \binom{(1+\mu)b}{b}$$

which was proved by the help of usual combinatorial methods by H. Gould in [4].

But with the help of the formula

$$(1.6) \qquad \sum_{s=0}^{n} \frac{1}{r+sd'} \binom{p-sd}{n-s} \binom{q+sd}{s} = \frac{1}{r} \binom{p+q}{n} +$$

$$+ \sum_{k=1}^{n} \frac{(rd-qd')(rd-(q-1)d')\dots(rd-(q-k+1)d')}{r(r+d')\dots(r+kd')} \binom{p+q-k}{n-k}$$

which I have derived in the quoted paper [4], we obtain this relation immediately.

There arises now the question of the existence of a generalization of the Abel's formula, similar to (1.6). Such formula is proved in the sixth part and takes the form

$$(1.7) \qquad \sum_{i=0}^{n} \frac{1}{\delta+\beta' i} \frac{(\gamma-\beta i)^{n-i}}{(n-i)!} \frac{(\alpha+\beta i)^{i}}{i!} =$$

$$= \sum_{\nu=0}^{n} \frac{(\beta\delta-\alpha\beta')^{\nu}}{\delta(\delta+\beta')\dots+(\delta+\nu\beta')} \frac{(\alpha+\gamma)^{n-\nu}}{(n-\nu)!} .$$

The example in the last part of this paper shows the applicability of the preceding formula.

Finally, it should be noted that the sums on the left of the relations (1.6) and (1.7) are special cases of the more general expressions

$$\sum_{s=0}^{n} \frac{1}{r + sd''} \binom{p - sd}{n - s} \binom{q + sd'}{s}$$

and

$$\sum_{i=0}^{n} \frac{1}{\delta + \beta'' i} \frac{(\gamma - \beta i)^{n-i}}{(n - i)!} \frac{(\alpha + \beta' i)^{i}}{i!}$$

respectively, yet we are not in the position of stating anything about them.

In fact, it would be interesting to try to evaluate the sums

$$\sum_{s=0}^{n} \binom{p - sd}{n - s} \binom{q + sd'}{s} \quad \text{or} \quad \sum_{i=0}^{n} \frac{(\gamma - \beta i)^{n-i}}{(n - i)!} \frac{(\alpha + \beta' i)^{i}}{i!}$$

of which the first is a generalization of the sum in the well-known Jensen's identity

$$\sum_{s=0}^{n} \binom{p - sd}{n - s} \binom{q + sd}{s} = \sum_{\nu=0}^{n} \binom{p + q - \nu}{n - \nu} d^{\nu}$$

while the second is a generalisation of the expression in the equation

$$\sum_{i=0}^{n} \frac{(\gamma - \beta i)^{n-i}}{(n - i)!} \frac{(\alpha + \beta i)^{i}}{i!} = \sum_{\nu=0}^{n} \frac{(\alpha + \gamma)^{\nu}}{\nu!} \beta^{n-\nu}$$

which is derived from the identity (1.1) by a simple transformation.

The principal results of this paper have been communicated at the Conference on Graph Theory and Combinatorial Analysis held in Smolenice in June 1966.

2.

We write

(2.1) $$J_1(\alpha, \beta; n) = \sum_{i=0}^{n} \binom{n}{i} (\alpha + i)^{i} (\beta - i)^{n-i} =$$

$$= n! \sum_{i=0}^{n} \frac{(\alpha + i)^i}{i!} \frac{(\beta - i)^{n-i}}{(n - i)!} = n!\, K_1 (\alpha, \beta; n)$$

where α, β may be any numbers (real or complex).

When we carry out the following operations, 1° replacing $(\alpha + i)$ by $(\alpha + \beta) - (\beta - i)$; 2° using the binomial formula to evaluate $[\alpha + \beta - (\beta - i)]^i$; 3 changing the order of summation according to the rule

$$\sum_{i\,0}^{n} \sum_{\nu\,0}^{i} = \sum_{\nu\,0}^{n} \sum_{i\,\nu}^{n};$$

4° using the well-known formula

$$\binom{n}{i}\binom{i}{\nu} = \binom{n}{\nu}\binom{n-\nu}{i-\nu};$$

5 transforming the inner sum by inserting the substitution $i - \nu - \mu$; 6 and finally using another well-known relation

$$\sum_{\mu-0}^{k} (-1)^{\mu} (a - \mu)^k \binom{k}{\mu} = k!$$

where a is an arbitrary number, we obtain successively

$$J_1(\alpha, \beta; n) = \sum_{i=0}^{n} \binom{n}{i} [\alpha + \beta - (\beta - i)]^i (\beta - i)^{n-i} =$$

$$= \sum_{i\,0}^{n} \binom{n}{i} (\beta - i)^{n-i} \sum_{\nu-0}^{i} \binom{i}{\nu} (\alpha + \beta)^{\nu} (-1)^{i-\nu} (\beta - i)^{i-\nu} =$$

$$= \sum_{\nu\,0}^{n} (\alpha + \beta)^{\nu} \sum_{i\,\nu}^{n} \binom{n}{i}\binom{i}{\nu} (-1)^{i-\nu} (\beta - i)^{n-\nu} =$$

$$= \sum_{\nu\,0}^{n} \binom{n}{\nu} (\alpha + \beta)^{\nu} \sum_{i-\nu}^{n} \binom{n-\nu}{i-\nu} (-1)^{i-\nu} [\beta - \nu - (i - \nu)]^{n-\nu} =$$

$$= \sum_{\nu=0}^{n} \binom{n}{\nu} (\alpha + \beta)^{\nu} \sum_{\mu=0}^{n-\nu} \binom{n-\nu}{\mu} (-1)^{\mu} (\beta - \nu - \mu)^{n-\nu} =$$

$$= \sum_{\nu=0}^{n} \binom{n}{\nu} (\alpha + \beta)^{\nu} (n - \nu)! = n! \sum_{\nu=0}^{n} \frac{(\alpha + \beta)^{\nu}}{\nu!} .$$

Returning to the original denotation we get thus the following results

(2.2) $$J_1(\alpha, \beta; n) =$$

$$= \sum_{i=0}^{n} \binom{n}{i} (\alpha + i)^{i} (\beta - i)^{n-i} = n! \sum_{\nu=0}^{n} \frac{(\alpha + \beta)^{\nu}}{\nu!}$$

and

(2.3) $$K_1(\alpha, \beta; n) =$$

$$= \sum_{i=0}^{n} \frac{(\alpha + i)^{i}}{i!} \frac{(\beta - i)^{n-i}}{(n - i)!} = \sum_{\nu=0}^{n} \frac{(\alpha + \beta)^{\nu}}{\nu!} .$$

Putting now $-\alpha$ instead of α and $-\beta$ instead of β, we see that

(2.4) $$J_2(\alpha, \beta, n) =$$

$$= \sum_{i=0}^{n} \binom{n}{i} (\alpha - i)^{i} (\beta + i)^{n-i} = n! \sum_{\nu=0}^{n} (-1)^{n-\nu} \frac{(\alpha + \beta)^{\nu}}{\nu!} .$$

and

(2.5) $$K_2(\alpha, \beta; n) =$$

$$= \sum_{i=0}^{n} \frac{(\alpha - i)^{i}}{i!} \frac{(\beta + i)^{n-i}}{(n - i)!} = \sum_{\nu=0}^{n} (-1)^{n-\nu} \frac{(\alpha + \beta)^{\nu}}{\nu!}$$

3.

We now turn our attention to Abel's extension of the binomial formula

(3.1) $$(x + \alpha)^{n} = \sum_{i=0}^{n} \binom{n}{i} \alpha (\alpha - \beta i)^{i-1} (x + \beta i)^{n-i} ,$$

where α, β, x are arbitrary real or complex numbers and n is a non-negative integer.

To prove this relation by applying the results of the preceding chapter, we put down

$$\gamma = \frac{\alpha}{\beta}, \quad \delta = \frac{x}{\beta};$$

we can suppose $\beta \neq 0$, because for $\beta = 0$ Abel's equation is reduced to the binomial formula. Then the sum in (3.1) may be written in the form ·

$$\beta^n \sum_{i=0}^{n} \binom{n}{i} \gamma(\gamma - i)^{i-1} (\delta + i)^{n-i}.$$

But this expression equals further

$$\beta^n \sum_{i=0}^{n} \binom{n}{i} [(\gamma - i) + i] (\gamma - i)^{i-1} (\delta + i)^{n-i} =$$

$$= \beta^n \left\{ \sum_{i=0}^{n} \binom{n}{i} (\gamma - i)^i (\delta + i)^{n-i} + \right.$$

$$\left. + n \sum_{i=1}^{n} \binom{n-1}{i-1} [\gamma - 1 - (i - 1)]^{i-1} [\delta + 1 + (i - 1)]^{n-1-(i-1)} \right\} =$$

$$= \beta^n \left\{ J_2(\gamma, \delta; n) + n \sum_{\mu=0}^{n-1} \binom{n-1}{\mu} (\gamma - 1 - \mu)^{\mu} (\delta + 1 + \mu)^{n-1-\mu} \right\} =$$

$$= \beta^n \{J_2(\gamma, \delta; n) + nJ_2(\gamma - 1, \delta + 1; n - 1)\} =$$

$$= \beta^n n! \{K_2(\gamma, \delta; n) + K_2(\gamma - 1, \delta + 1; n - 1)\} =$$

$$= \beta^n n! \left\{ \sum_{\nu=0}^{n} (-1)^{n-\nu} \frac{(\gamma + \delta)^{\nu}}{\nu!} - \sum_{\nu=0}^{n-1} (-1)^{n-\nu} \frac{(\gamma + \delta)^{\nu}}{\nu!} \right\} =$$

$$= \beta^n n! \frac{(\gamma + \delta)^n}{n!} = \beta^n \left(\frac{\alpha + x}{\beta} \right)^n = (\alpha + x)^n.$$

Thus, we have proved that Abel's identity (3.1) is a consequence of one of the identities of the foregoing part.

4.

We shall now show how the various relations of the Abel-type can be established with the use of the identities (2.2), ..., (2.5).

1. *If n is a positive integer greater than* 1, *show that* [6]

$$\sum_{i=1}^{n=1} \binom{n}{i} i^{n-i-1} (n-i)^{i-1} = 2n^{n-2}.$$

Proof. If S represents the sum, we get

$$S = S_1 + S_2,$$

where

$$S_1 = \sum_{i=1}^{n=1} \binom{n-1}{i-1} i^{n-i-1}(n-i)^{i-1}$$

and

$$S_2 = \sum_{i=1}^{n-1} \binom{n-1}{i} i^{n-i-1} (n-i)^{i-1} .$$

But putting $n - i = i'$, we find that

$$S_1 = \sum_{i=1}^{n-1} \binom{n-1}{n-i} i^{n-i-1} (n-i)^{i-1} =$$

$$= \sum_{i'-1}^{n-1} \binom{n-1}{i'} (n-i')^{i'-1} i'^{n-i'-1} = S_2$$

so that

$$S = 2S_2.$$

Now because $n > 1$ we can add into S_2 the term with $i = 0$, so that

$$nS_2 = \sum_{i-0}^{n-1} \binom{n-1}{i} (n-i+i)(n-i)^{i-1} i^{n-i-1} =$$

$$= \sum_{i\ 0}^{n-1} \binom{n-1}{i} (n-i)^i\, i^{n-i-1} +$$

$$+ (n \quad 1) \sum_{i\ 1}^{n\ 1} \binom{n-2}{i-1} [(n-1-(i-1)]^{i-1} [1+(i-1)]^{n-2-(i-1)} =$$

$$-(n-1)!\, K_2(n, 0; n-1) +$$

$$+ (n-1) \sum_{i'-0}^{n\ 2} \binom{n-2}{i'} (n-1-i')^{i'} (1+i')^{n-2\ i'} =$$

$$(n-1)!\, \{K_2(n, 0; n-1) + K_2(n-1, 1; n-2)\} =$$

$$-(n-1)! \left\{ \sum_{\nu\ 0}^{n-1} (-1)^{n-1-\nu} \frac{n^\nu}{\nu!} - \sum_{\nu\ 0}^{n-2} (-1)^{n-1-\nu} \frac{n^\nu}{\nu!} \right\} =$$

$$-(n-1)! \frac{n^{n-1}}{(n-1)!} = n^{n\ 1}.$$

Thus S_2 $n^{n\ 2}$ and $S = 2n^{n-2}$.

2. *If n is a positive integer, show that* [7]

$$\sum_{p\ 1}^{n} (-1)^p \binom{n}{p} (p+1)^{p-1} p^{n-p} - (-1)^n.$$

Proof. The identity may be written in the form

$$\sum_{p\ 0}^{n} \binom{n}{p} (1+p)^{p-1} (-p)^{n-p} = 1.$$

Further the sum equals

$$\sum_{p\ 0}^{n} \binom{n}{p} (1+p-p)(1+p)^{p-1} (-p)^{n-p} -$$

$$= \sum_{p=0}^{n} \binom{n}{p} (1 + p)^p (-p)^{n-p} -$$

$$- n \sum_{p=1}^{n} \binom{n-1}{p-1} (2 + p - 1)^{p-1}[-1 - (p - 1)]^{n-1-(p-1)} =$$

$$= n!K_1(1,0; n) - n \sum_{p=0}^{n-1} \binom{n-1}{p} (2 + p)^p (-1 - p)^{n-2-p} =$$

$$= n!\{K_1(1, 0; n) - K_1(2, -1; n - 1)\} =$$

$$= n! \left\{ \sum_{\nu=0}^{n} \frac{1}{\nu!} - \sum_{\nu=0}^{n-1} \frac{1}{\nu!} \right\} = n! \frac{1}{n!} = 1.$$

3. *Show that for all α, β and integer $n > 0$* [2] (see also [8])

$$(\alpha - 1)(\beta - n) \sum_{i=0}^{n} \frac{1}{1+i} \binom{n}{i} (\alpha + i)^i (\beta - i)^{n-i-1} =$$

$$= \frac{1}{n+1} [(\alpha + \beta)^n (\alpha + \beta - n - 1) - (\beta + 1)^n (\beta - n)].$$

Proof. The expression on the left of this identity can be successively transformed as follows

$$(\alpha - 1) \sum_{i-0}^{n} \left\{ \beta - i - (n - i) \right\} \frac{1}{1+i} \binom{n}{n-i} (\alpha + i)^i (\beta - i)^{n-i-1} =$$

$$= (\alpha - 1) \left\{ \sum_{i=0}^{n} \frac{1}{1+i} \binom{n}{i} (\alpha + i)^i (\beta - i)^{n-i} - \right.$$

$$\left. - n \sum_{i\ 0}^{n-1} \frac{1}{1+i} \binom{n-1}{i} (\alpha + i)^i (\beta - i)^{n-1-i} \right\} =$$

$$= \sum_{i=0}^{n} \{\alpha + i - (i+1)\} \frac{1}{1+i} \binom{n}{i} (\alpha + i)^i (\beta - i)^{n-i} -$$

$$- n \sum_{i=0}^{n-1} \{\alpha + i - (i+1)\} \frac{1}{1+i} \binom{n-1}{i} (\alpha + i)^i (\beta - i)^{n-1-i} =$$

$$= \sum_{i=0}^{n} \frac{1}{1+i} \binom{n}{i} (\alpha + i)^{i+1} (\beta - i)^{n-i} -$$

$$- n \sum_{i=0}^{n-1} \frac{1}{1+i} \binom{n-1}{i} (\alpha + i)^{i+1} (\beta - i)^{n-i-1} -$$

$$- \sum_{i=0}^{n} \binom{n}{i} (\alpha + i)^i (\beta - i)^{n-i} + n \sum_{i=0}^{n-1} \binom{n-1}{i} (\alpha + i)^i (\beta - i)^{n-1-i}.$$

Now, with regard to (2.3) the two last terms give

$$- (\alpha + \beta)^n$$

so that the sum which we examine equals

$$\frac{1}{n+1} \sum_{i=0}^{n} \binom{n+1}{i+1} (\alpha + i)^{i+1} (\beta - i)^{n-i} -$$

$$- \sum_{i=0}^{n-1} \binom{n}{i+1} (\alpha + i)^{i+1} (\beta - i)^{n-i-1} - (\alpha + \beta)^n =$$

$$= \frac{1}{n+1} \sum_{i=1}^{n+1} \binom{n+1}{i} (\alpha - 1 + i)^i (\beta + 1 - i)^{n+1-i} -$$

$$- \sum_{i=1}^{n} \binom{n}{i} (\alpha - 1 + i)^i (\beta + 1 - i)^{n-i} - (\alpha + \beta)^n.$$

Adding and subtracting the expression

$$\frac{1}{n+1}(\beta+1)^{n+1}+(\beta+1)^n,$$

we get finally

$$\frac{1}{n+1}J_1(\alpha-1,\beta+1;n+1)-J_1(\alpha-1,\beta+1;n)-$$

$$-\frac{(\beta+1)^n}{n+1}(\beta-n)-(\alpha+\beta)^n=\frac{(\alpha+\beta)^{n+1}}{n+1}-(\alpha+\beta)^n-$$

$$-\frac{(\beta+1)^n}{n+1}(\beta-n)=\frac{1}{n+1}[(\alpha+\beta)^n(\alpha+\beta-n-1)-(\beta+1)^n(\beta-n)]$$

which is the desired result.

Note. The identities [9]

$$\sum_{k\,0}^{n}\binom{n}{k}(n-k)^{n-k}(k-1)^{k-1}=-(n-1)^n$$

and [10]

$$\sum_{k\,0}^{n}\binom{n}{k}(1+k)^{k-1}(n+1-k)^{n-k}=(n+2)^n$$

may be proved in the same manner.

5.

Let us assume $\alpha \neq 0$ and put $-\beta$ instead of β. Then Abel's formula may be written in the form

$$S(\alpha,x;n,\beta)=\sum_{i=0}^{n}\frac{1}{\alpha+\beta i}\frac{(x-\beta i)^{n-i}}{(n-i)!}\frac{(\alpha+\beta i)^i}{i!}=$$

(5.1)

$$=\frac{1}{\alpha}\frac{x^n}{n!}+\sum_{i=1}^{n}\frac{1}{i}\frac{(x-\beta i)^{n-i}}{(n-i)!}\frac{(\alpha+\beta i)^{i-1}}{(i-1)!}=\frac{1}{\alpha}\frac{(x+\alpha)^n}{n!}.$$

As indicated in the introductory part, this relation may be proved in the same way as I have proved the analogous identity (1.4) in my paper [4], that is, by induction.

(5.1) is clearly correct when $n = 0$. Now, starting with the hypothesis that this formula is valid for n, we will show that

$$S(\alpha, x; n+1, \beta) =$$

$$= \sum_{i=0}^{n+1} \frac{1}{\alpha + \beta i} \frac{(x - \beta i)^{n+1-i}}{(n+1-i)!} \frac{(\alpha + \beta i)^i}{i!} = \frac{1}{\alpha} \frac{(x+\alpha)^{n+1}}{(n+1)!}.$$

For this purpose we write

$$\frac{1}{\alpha} \frac{(x+\alpha)^{n+1}}{(n+1)!} = \frac{1}{\alpha} \frac{(x+\alpha)^n}{n!} \frac{x+\alpha}{n+1} = \frac{x+\alpha}{n+1} S(\alpha, x; n, \beta)$$

$$= \frac{x+\alpha}{n+1} \left\{ \frac{1}{\alpha} \frac{x^n}{n!} + \sum_{i=1}^{n} \frac{1}{i} \frac{(x - \beta i)^{n-i}}{(n-i)!} \frac{(\alpha + \beta i)^{i-1}}{(i-1)!} \right\}$$

and after some small modifications we obtain further

$$\frac{1}{\alpha} \frac{(x+\alpha)^{n+1}}{(n+1)!} = \frac{1}{\alpha} \frac{x^{n+1}}{(n+1)!} +$$

$$+ \frac{1}{n+1} \left\{ \frac{x^n}{n!} + \sum_{i=1}^{n} \frac{(x - \beta i) + (\alpha + \beta i)}{i} \frac{(x - \beta i)^{n-i}}{(n-i)!} \frac{(\alpha + \beta i)^{i-1}}{(i-1)!} \right\} =$$

$$= \frac{1}{\alpha} \frac{x^{n+1}}{(n+1)!} + \frac{1}{n+1} \left\{ \frac{x^n}{n!} + \right.$$

$$+ \sum_{i=1}^{n} \frac{n+1-i}{i} \frac{(x - \beta i)^{n+1-i}}{(n+1-i)!} \frac{(\alpha + \beta i)^{i-1}}{(i-1)!} +$$

$$\left. + \sum_{i=1}^{n} \frac{(x - \beta i)^{n-i}}{(n-i)!} \frac{(\alpha + \beta i)^i}{i!} \right\}.$$

The first term and the second sum in the parentheses yield

$$\sum_{i=0}^{n} \frac{(x - \beta i)^{n-i}}{(n - i)!} \frac{(\alpha + \beta i)^i}{i!}$$

and the first sum can be divided into two parts:

$$(n + 1) \sum_{i=1}^{n} \frac{1}{i} \frac{(x - \beta i)^{n+1-i}}{(n + 1 - i)!} \frac{(\alpha + \beta i)^{i-1}}{(i - 1)!}$$

and

$$- \sum_{i=1}^{n} \frac{(x - \beta i)^{n+1-i}}{(n + 1 - i)!} \frac{(\alpha + \beta i)^{i-1}}{(i - 1)!} =$$

$$= - \sum_{i=0}^{n-1} \frac{(x - \beta - \beta i)^{n-i}}{(n - i)!} \frac{(\alpha + \beta + \beta i)^i}{i!}.$$

Furthermore, if we add

$$\frac{[\alpha + \beta(n + 1)]^n}{n!}$$

to the first sum and subtract the same expression from the other sum we obtain

$$(n + 1) \sum_{i=1}^{n+1} \frac{1}{i} \frac{(x - \beta i)^{n+1-i}}{(n + 1 - i)!} \frac{(\alpha + \beta i)^{i-1}}{(i - 1)!}$$

or

$$- \sum_{i=0}^{n} \frac{(x - \beta - \beta i)^{n-i}}{(n - i)!} \frac{(\alpha + \beta + \beta i)^i}{i!}$$

respectively.

Finally, applying all these results we get

$$\frac{1}{\alpha} \frac{(x + \alpha)^{n+1}}{(n + 1)!} = \frac{1}{\alpha} \frac{x^{n+1}}{(n + 1)!} +$$

$$+ \sum_{i=1}^{n+1} \frac{1}{i} \frac{(x - \beta i)^{n+1-i}}{(n + 1 - i)!} \frac{(\alpha + \beta i)^{i-1}}{(i - 1)!} +$$

$$+\frac{1}{n+1}\left\{\sum_{i=0}^{n}\frac{(x-\beta i)^{n-i}}{(n-i)!}\,\frac{(\alpha+\beta i)^{i}}{i!}-\right.$$

$$\left.-\sum_{i=0}^{n}\frac{(x-\beta-\beta i)^{n-i}}{(n-i)!}\,\frac{(\alpha+\beta+\beta i)^{i}}{i!}\right\}=$$

$$=S(\alpha, x; n+1, \beta)+\frac{\beta^n}{n+1}\left\{K_1\left(\frac{\alpha}{\beta}, \frac{x}{\beta}; n\right)-K_1\left(\frac{\alpha+\beta}{\beta}, \frac{x-\beta}{\beta}; n\right)\right\}.$$

But the symbols in the parentheses represent the same values, so that

$$\frac{1}{\alpha}\,\frac{(x+\alpha)^{n+1}}{(n+1)!}=S(\alpha, x; n+1, \beta)$$

and this is the required equation.

Abel's formula is thus established.

6.

Now we shall give the proof of the identity (1.7), mentioned in the first part.

For this purpose we introduce the denotation

$$S(\alpha, \gamma, \delta; n, \beta, \beta')=\sum_{i=0}^{n}\frac{1}{\delta+\beta' i}\,\frac{(\gamma-\beta i)^{n-i}}{(n-i)!}\,\frac{(\alpha+\beta i)^{i}}{i!}=\frac{1}{\delta}\,\frac{\gamma^n}{n!}+$$

$$+\sum_{n=1}^{n}\frac{\alpha+\beta i}{(\delta+\beta' i)i}\,\frac{(\gamma-\beta i)^{n-i}}{(n-i)!}\,\frac{(\alpha+\beta i)^{i-1}}{(i-1)!}.$$

By breaking up the function

$$\frac{\alpha+\beta i}{(\delta+\beta' i)i}$$

into partial fractions we get

$$\frac{\alpha+\beta i}{(\delta+\beta' i)i}=\frac{\beta\delta-\alpha\beta'}{\delta}\,\frac{1}{\delta+\beta' i}+\frac{\alpha}{\delta}\,\frac{1}{i},$$

so that

$$S(\alpha, \gamma, \delta; n, \beta, \beta') = \frac{1}{\delta} \frac{\gamma^n}{n!} + \frac{\beta\delta - \alpha\beta'}{\delta} \sum_{i=1}^{n} \frac{1}{\delta + \beta' i} \frac{(\gamma - \beta i)^{n-i}}{(n-i)!} \frac{(\alpha + \beta i)^{i-1}}{(i-1)!} +$$

$$+ \frac{\alpha}{\delta} \sum_{i-1}^{n} \frac{1}{i} \frac{(\gamma - \beta i)^{n-i}}{(n-i)!} \frac{(\alpha + \beta i)^{i-1}}{(i-1)!}.$$

Now, putting $(i + 1)$ instead of i in the first sum and calculating the second sum from the relation (5.1), we obtain

$$S(\alpha, \gamma, \delta; n, \beta, \beta') = \frac{1}{\delta} \frac{(\alpha + \gamma)^n}{n!} +$$

$$+ \frac{\beta\delta - \alpha\beta'}{\delta} \sum_{i\ 0}^{n-1} \frac{1}{\delta + \beta' + \beta' i} \frac{(\gamma - \beta - \beta i)^{n-1-i}}{(n-1-i)!} \frac{(\alpha + \beta + \beta i)^i}{i!} =$$

$$= \frac{1}{\delta} \frac{(\alpha + \gamma)^n}{n!} + \frac{\beta\delta - \alpha\beta'}{\delta} S(\alpha + \beta, \gamma - \beta, \delta + \beta' : n - 1, \beta, \beta').$$

We have thus a recurrence formula for the required sum, whence it follows that

$$S(\alpha, \gamma, \delta; n, \beta, \beta') = \frac{1}{\delta} \frac{(\alpha + \gamma)^n}{n!} + \frac{\beta\delta - \alpha\beta'}{\delta(\delta + \beta')} \frac{(\alpha + \gamma)^{n-1}}{(n-1)!} +$$

$$+ \frac{(\beta\delta - \alpha\beta')^2}{\delta(\delta + \beta')(\delta + 2\beta')} \frac{(\alpha + \gamma)^{n-2}}{(n-2)!} + \ldots + \frac{(\beta\delta - \alpha\beta')^n}{\delta(\delta + \beta') \ldots (\delta + n\beta')}.$$

But this is the combinatorial identity (1.7)

$$\sum_{i-0}^{n} \frac{1}{\delta + \beta' i} \frac{(\gamma - \beta i)^{n-i}}{(n-i)!} \frac{(\alpha + \beta i)^i}{i!} =$$

$$= \sum_{\nu=0}^{n} \frac{(\beta\delta - \alpha\beta')^\nu}{\delta(\delta + \beta')(\delta + 2\beta') \ldots (\delta + \nu\beta')} \frac{(\alpha + \gamma)^{n-\nu}}{(n-\nu)!}$$

7.

For want of a more convenient example we shall show the applicability of the preceding formula by proving once more the Birnbaum-Pyke's binomial identity of part 5.

$$(\alpha - 1)(\beta - n)\sum_{i=0}^{n} \frac{1}{1+i}\binom{n}{i}(\alpha + i)^i (\beta - i)^{n-i-1} =$$

$$= -\frac{1}{n+1}\left[(\alpha+\beta)^n(\alpha+\beta-n-1) - (\beta+1)^n(\beta-n)\right].$$

For this purpose we write the sum on the left of this equation in the form

$$n!(\alpha - 1)(\beta - n)\sum_{i=0}^{n} \frac{1}{(1+i)(\beta - i)} \frac{(\beta - i)^{n-i}}{(n-i)!} \frac{(\alpha + i)^i}{i!}$$

Since

$$\frac{1}{(1+i)(\beta-i)} = \frac{1}{1+\beta}\left(\frac{1}{1+i} + \frac{1}{\beta - i}\right)$$

we obtain, after substituing the above result,

$$\frac{(\alpha-1)(\beta-n)n!}{1+\beta}\left\{\sum_{i=0}^{n}\frac{1}{1+i}\frac{(\beta-i)^{n-i}}{(n-i)!}\frac{(\alpha+i)^i}{i!} + \right.$$

$$\left. + \sum_{i=0}^{n}\frac{1}{\beta-i}\frac{(\beta-i)^{n-i}}{(n-i)!}\frac{(\alpha+i)^i}{i!}\right\}.$$

But the first sum in the parentheses is in our denotation equal to $S(\alpha, \beta, 1; n, 1, 1)$ and in the second sum we put $(n - i)$ instead of i. Then we obtain

$$\frac{(\alpha-1)(\beta-n)n!}{1+\beta}\left\{S(\alpha, \beta, 1; n, 1, 1) + \right.$$

$$\left. + \sum_{i=0}^{n}\frac{1}{\beta-n+i}\frac{(\alpha+n-i)^{n-i}}{(n-i)!}\frac{(\beta-n+i)^i}{i!}\right\} =$$

$$\frac{(\alpha - 1)(\beta-n)n!}{1+\beta}\{S(\alpha, \beta, 1; n, 1, 1) + S(\beta - n, \alpha + n, \beta - n; n, 1, 1)\} =$$

$$-\frac{(\alpha-1)(\beta-n)n!}{1+\beta}\left\{\frac{1}{\beta-n}\frac{(\alpha+\beta)^n}{n!} + \sum_{\nu=0}^{n}\frac{(1-\alpha)^\nu}{(\nu+1)!}\frac{(\alpha+\beta)^{n-\nu}}{(n-\nu)!}\right\} =$$

$$\frac{\alpha-1}{1+\beta}(\alpha+\beta)^n - \frac{\beta-n}{(1+\beta)(n+1)}\sum_{\nu=0}^{n}\binom{n+1}{\nu+1}(1-\alpha)^{\nu+1}(\alpha+\beta)^{n-\nu} =$$

$$= \frac{\alpha - 1}{1 + \beta} (\alpha + \beta)^n - \frac{\beta - n}{(n + 1)(1 + \beta)} \sum_{\nu=1}^{n+1} \binom{n + 1}{\nu} (1 - \alpha)^\nu (\alpha + \beta)^{n+1-\nu} =$$

$$= \frac{\alpha - 1}{1 + \beta} (\alpha + \beta)^n - \frac{\beta - n}{(n + 1)(1 + \beta)} \left\{ (1 + \beta)^{n+1} - (\alpha + \beta)^{n+1} \right\} =$$

$$= - \frac{1}{n + 1} (1 + \beta)^n (\beta - n) + \frac{1}{n + 1} (\alpha + \beta)^n (\alpha + \beta - n - 1)$$

which is the correct result.

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