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NOTE ON THE ŠEVRIN RADICAL IN SEMIGROUPS

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J. Bosák [1] showed on examples that the radical defined by Ševrin [2] can be distinct from the set of all nilpotent elements and from the radicals with respect to an ideal J. In this note it is shown that the Ševrin radical has similar properties as the Clifford radical, the Schwarz radical and the McCoy radical (see [3]).

Definition. *Let S be a semigroup and J a two-sided ideal of S. Let I be such an ideal of S that every subsemigroup $S' \subseteq S$, generated by a finite number of elements of I, is nilpotent with respect to J (i. e. for a positive integer n we have $(S')^n \subseteq J$). Then I is called a locally nilpotent ideal with respect to J. The union of all locally nilpotent ideals with respect to J will be called the Ševrin radical with respect to J and it will be denoted by $L(J)$.*

Lemma 1. *Let I_1 be a locally nilpotent ideal with respect to J_1 and I_2 a locally nilpotent ideal with respect to J_2. Then $I_1 \cap I_2$ is a locally nilpotent ideal with respect to $J_1 \cap J_2$.*

Proof. If we take a finite number of elements of $I_1 \cap I_2$, then the semigroup A generated by these elements is nilpotent with respect to J_1 and J_2, i. e. there exist positive integers n_1 and n_2 such that $A^{n_1} \subseteq J_1$ and $A^{n_2} \subseteq J_2$. Let $n = \max \{n_1, n_2\}$. Then $A^n \subseteq J_1$, $A^n \subseteq J_2$, i. e. $A^n \subseteq J_1 \cap J_2$, q. e. d.

Lemma 2. *Let J_1 and J_2 be ideals of S and $J_1 \subseteq J_2$. Then $L(J_1) \subseteq L(J_2)$.*

Proof. If the ideal I is locally nilpotent with respect to J_1, then it is evidently also locally nilpotent with respect to J_2. But every element of $L(J_1)$ is contained in some locally nilpotent ideal I with respect to J_1 and therefore it is contained in $L(J_2)$.

Lemma 3. $L(J_1 \cap J_2) = L(J_1) \cap L(J_2)$.

Proof. a) from $J_1 \cap J_2 \subseteq J_1$ and $J_1 \cap J_2 \subseteq J_2$ according to Lemma 2 we obtain $L(J_1 \cap J_2) \subseteq L(J_1)$ and $L(J_1 \cap J_2) \subseteq L(J_2)$. Hence $L(J_1 \cap J_2) \subseteq$ $\subseteq L(J_1) \cap L(J_2)$.

b) If $x \in L(J_1) \cap L(J_2)$, then $x \in L(J_1)$ and $x \in L(J_2)$. Thus x is contained in some locally nilpotent ideal I_1 with respect to J_1 and in some locally nilpotent ideal I_2 with respect to J_2, therefore $x \in I_1 \cap I_2$ and this is by Lemma 1 a locally nilpotent ideal with respect to $J_1 \cap J_2$. Hence $x \in L(J_1 \cap J_2)$ and $L(J_1) \cap L(J_2) \subseteq L(J_1 \cap J_2)$.

From a) and b) our statement follows.

Lemma 4. $L(J_1) \cup L(J_2) \subseteq L(J_1 \cup J_2)$.

Proof. From $J_1 \subseteq J_1 \cup J_2$, $J_2 \subseteq J_1 \cup J_2$ and by Lemma 2 we have $L(J_1) \subseteq L(J_1 \cup J_2)$, $L(J_2) \subseteq L(J_1 \cup J_2)$ and this implies Lemma 4.

Remark. In Lemma 4 the equality need not hold. This can be shown on the following example (cf. [3], p. 213).

Example. Let S be the free semigroup generated by elements a and b. Let (a) and (b^2) be the principal two-sided ideals generated by a and b^2, respectively. Since $(b^2) \subseteq (a) \cup (b^2)$, the ideal (b) is localy nilpotent with respect to $(a) \cup (b^2)$ and hence $b \in L((a) \cup (b^2))$. But no power of $b \in (b)$ is contained in (a) and no power of $ba \in (b)$ is contained in (b^2), therefore (b) is a locally nilpotent ideal neither with respect to (a) nor with respect to (b^2). Thus b is not contained in $L((a)) \cup L((b^2))$. We proved that $b \in L((a) \cup (b^2))$ but b is not in $L((a)) \cup L((b^2))$. Hence $L((a) \cup (b^2)) \neq L((a)) \cup L((b^2))$.

From the foregoing lemmas we have:

Theorem. *The mapping which assigns to each two-sided ideal J of the semigroup S the Ševrin radical $L(J)$ is a $\cap$-endomorphism of the lattice of all ideals of S.*

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