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ON THE FACE-VECTOR OF A 5-VALENT CONVEX 3-POLYTOPE

MARIÁN TRENKLER

Introduction: Let M be a convex polytope in E_3. The symbol $p_k(M)$ denotes the *number of faces* of M having exactly k edges. The vector $(p_3(M), p_4(M), \ldots)$ is called the *face-vector* of M. The graph formed from vertices and edges of M is called the *graph of the polytope* M.

In the present paper we attempt to solve the problem of characterization of the face-vector of all convex polytopes M each vertex of which incides with 5 edges. This problem has not been solved yet completely.

The analogous problem for polytopes having each vertex of degree 3 or 4 has been treated in many papers in the last ten years. (See [1, 4, 5].)

We shall say that a finite sequence $(p_k \mid 3 \leqslant k)$ of non-negative integers is *5-realizable*, provided there exists a polytope M with a 5-valent regular graph such that $p_k(M) = p_k$ for all k; every such M is a *5-realization* of the sequence.

From Euler's formula the following necessary condition for the 5-realizability of a sequence $(p_k \mid 3 \leqslant k)$ follows:

$$p_3 = 20 + \sum_{4 \leq k} (3k - 10) p_k . \tag{1}$$

However, this condition is not sufficient. E.g. the sequence $(22, 1, 0, \ldots)$ satisfies (1) but is not 5-realizable.

Fisher [2] proved: *A sequence $(p_k \mid 3 \leqslant k)$ of non-negative integers satisfying* (1) *and $p_4 \geqslant 6$ is 5-realizable.*

In this paper we shall improve on this result.

In the proof we shall construct planar maps instead of convex polytopes. This is possible by the well-known Steinitz theorem [3, p. 235]: *A graph G is a graph of a convex 3-polytope if and only if G is planar and* 3*-connected.*

First, one lemma used in the construction will be proved. In a planar map N whose graph is connected let $p_k(N)$ $(v_k(N))$ denote the number of k-gons or k-valent vertices, respectively.

Lemma. *If there exists a planar map with a 2-connected graph without loops, multiple edges and vertices of second degree such that* $p_k(N) + v_k(N) = p'_k$ *for each* $k \geqslant 4$, *then the sequence* $(p'_k \mid 3 \leqslant k)$ *satisfying* (1) *is 5-realizable.*

Proof. We shall describe a transformation of the planar map N into a 5-valent planar map with a 3-connected graph and p'_k k-gons for each k. This transformation is called *transformation* μ and it consists of two steps.

In the first step each k-gon of N is replaced by a k-gon, each k-valent vertex is replaced by a k-gon and each edge by a quadrangle which in the second step will be divided into two triangles. (In Fig. 1 the map N is depicted by dashed lines.)

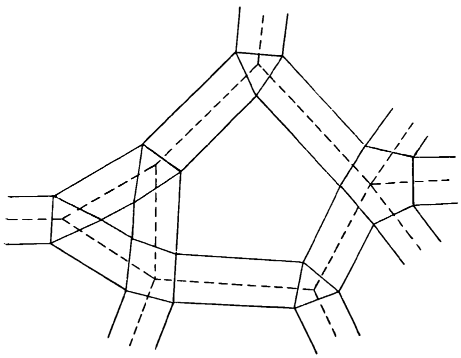

Fig. 1

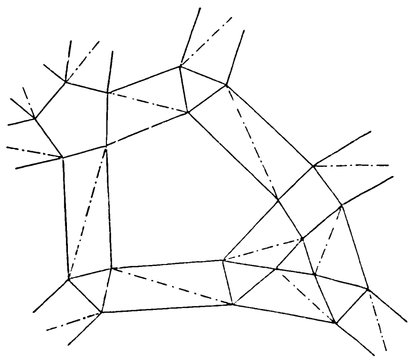

Fig. 2

We select such a quadrangle **Q** and by adding one new edge we divide **Q** into two triangles. Each of the remaining 4-valent vertices of **Q** belongs also to another quadrangle. By adding an edge beginning in this vertex we divide this quadrangle into two triangles as well. In this way $\sum_{4 \leq k} k \cdot p_k(N)$ couples of triangles are formed. (In Fig. 2 the added edges are shown as dot--and dashed lines.)

Thus we get a 5-valent planar map with a 3-connected graph and p'_k k-gons for all k.

Our results are contained in the following theorems.

Theorem 1. *A sequence* $(p_k \mid 3 \leqslant k)$ *of non-negative integers satisfying* (1) *and one of the following conditions*

α) $p_4 \geqslant 4$,

β) $p_4 = 3$ *and* $p_5 \geqslant 1$,

γ) $p_4 = 3$ *and* $p_i \geqslant 1$, $p_{i+1} \geqslant 1$ *or* $p_i \geqslant 1$, $p_{i+2} \geqslant 1$ *for* $i \geqslant 5$,

δ) $p_4 = 2$ *and* $p_5 \geqslant 2$,

ε) $p_4 = 2$ *and* $p_i \geqslant 2, \quad p_{i+1} \geqslant 2$ *or* $p_i \geqslant 2, \quad p_{i+2} \geqslant 2$ *for* $i \geqslant 5$,

is 5-realizable.

The proof consists of constructing, for every sequence $(p_k \mid 3 \leqslant k)$ satisfying the conditions of *Theorem* 1 its 5-realization M.

α) $p_4 \geqslant 4$

1. If $p_i \leqslant 1$ for all $i \geqslant 5$, $p_4 = 4$.

a) $\sum\limits_{5 \leq i} p_i = 0$

The map M can be obtained from the map depicted by full lines in Fig. 3 by performing the transformation μ.

b) $\sum\limits_{5 \leq i} p_i = 1$

Let $p_5 = 1$; in this case we obtain M from the map depicted in Fig. 3 by performing the transformation μ.

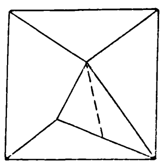

Fig. 3

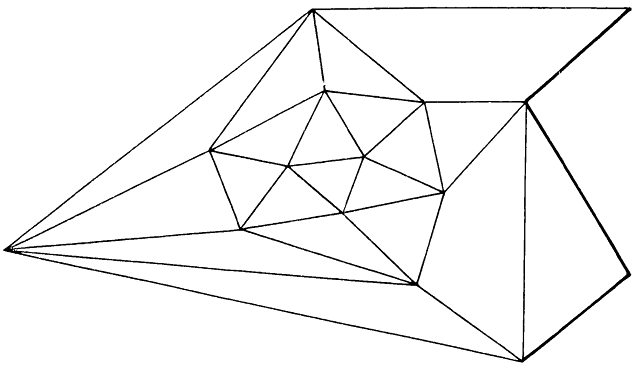

Fig. 4

Let $p_6 = 1$ or $p_7 = 1$. The map M is obtained by joining two submaps R^1 (Fig. 4) or the submap R^1 and R^2 (Fig. 5), respectively. The path which consists of three edges indicated in heavy lines is called the path ν. In the sequel, when speaking of joining two submaps, we mean that we identify the edges of the path ν in such a way that all vertices of the resulting map have degree 5.

Let $p_i = 1$ for $i = 6 + 2k$ or $i = 7 + 2k$. The map M will be the union of two submaps R^1 or the submaps R^1 and R^2 and k configurations $\boldsymbol{T}$ consisting of 6 triangles each. The *configuration* $\boldsymbol{T}$ is shown in Fig. 6.

c) $\sum\limits_{5 \leq i} p_i = 2$

From these conditions it follows that there exist numbers m, n such that $5 \leqslant m < n$ and $p_m = p_n = 1$. Three subcases must be considered.

c$_1$) $m \equiv 0 \pmod 3$

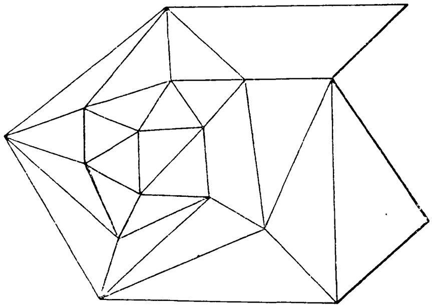

Fig. 5

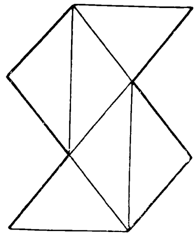

Fig. 6

The starting map of the construction is M_i, $i = m + n - 3$. The construction M_i is described in case a). It consists of one i-gon $A_1A_2 \dots A_{\frac{i}{2}}B_{\frac{i}{2}} \dots B_1$ or $A_1A_2 \dots A_{\frac{i-1}{2}}B_{\frac{i+1}{2}} \dots B_1$, four 4-gons and $18 + 3i$ triangles. The common edges of a i-gon and two 4-gons are indicated by A_1B_1 and $A_{\frac{i}{2}}B_{\frac{i}{2}}$ or $A_{\frac{i}{2}\,1}B_{\frac{i}{2}\,1}$, respectively.

In the starting map edges A_1A_2, A_4A_5, $\dots$, $A_{m-2}A_{m-1}$ are omitted and edges A_2A_4, A_5A_7, $\dots$, $A_{m-4}A_{m-2}$, $A_{m-1}A_1$ are added.

From an i-gon and $\frac{m}{3}$ triangles we obtain an m-gon and an n-gon and $\frac{m-3}{3}$ triangles $A_2A_3A_4$, $A_5A_6A_7$, $\dots$, $A_{m-4}A_{m-3}A_{m-2}$.

c₂) $m \equiv 1 \pmod 3$

In the starting map M_i, $i = m + n - 3$, the edges B_1A_1, A_4A_5, $\dots$, $A_{m-3}A_{m-2}$ (dashed lines in Fig. 7) are omitted and the edges A_1A_4, A_5A_7, $\dots$, $A_{m-5}A_{m-3}$, $A_{m-2}B_1$ (dot-and-dashed lines in Fig. 7) are added.

c₃) $m \equiv 2 \pmod 3$

If $p_5 = 1$, the starting map is M_n. A part of this map is shown in Fig. 8. We cut it along the path v (heavy lines in Fig. 8) and before the rejoining

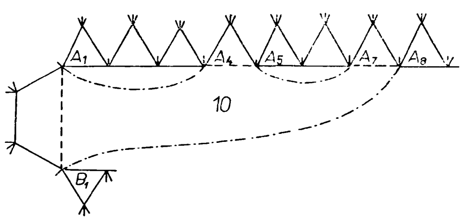

Fig. 7

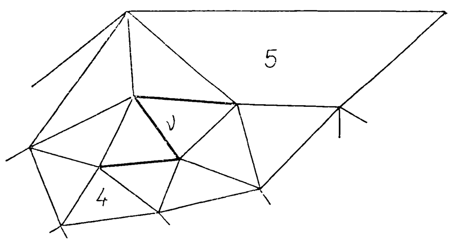

Fig. 8

we insert one configuration **T**. From a triangle and a 4-gon (in Fig. 8 indicated by the numbers 4 and 5) we obtain a 4-gon and a 5-gon.

Let $p_m = 1$ for $m > 5$. As before, we form a 4-gon and a 5-gon in the starting map M_i, $i = m + n - 5$. By omitting edges B_1A_1, A_3A_4, $\ldots$, $A_{m-5}A_{m-4}$ and adding edges A_1A_3, A_4A_6, $\ldots$, $A_{m-7}A_{m-5}$, B_1A_{m-4} the map M arises.

d) $\sum_{5 \leq i} p_i = 3$

From the conditions it follows that there exist number m, n, s such that $5 \leqslant m < n < s$ and $p_m = p_n = p_s = 1$.

The starting map is M_i, $i = m + n + s - K$, where

$K = 10$ if $m \equiv 2 (\mathrm{mod}\ 3)$ and $n \equiv 2 (\mathrm{mod}\ 3)$,

$K - \ 6$ if $m \not\equiv 2 (\mathrm{mod}\ 3)$ and $n \not\equiv 2 (\mathrm{mod}\ 3)$,

$K = \ 8$ in all other cases.

As in case c) one m-gon is formed. Only a small change takes place during the forming of the n-gon. The vertices of the omitted and added edges are indicated by B_k instead of A_j and A_h instead of B_l, where $k = \frac{i+2}{2}$ or $k \ \frac{i+3}{2}$ and $h = \frac{i}{2}$ or $h = \frac{i-1}{2}$, respectively.

e) $4 \leqslant \sum_{5 \leq i} p_i \leqslant 2 \left[\frac{t-3}{3}\right] + 3$, where t is such that $p_t = 1$ and $\sum_{t \leq i} p_i = 3$.

e$_1$) $p_i = 0$ for all $i \geqslant 9$

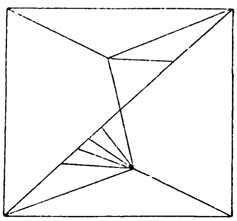

Fig. 9

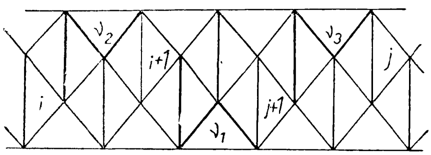

Fig. 10

The map M is obtained from the map in Fig. 9 by using the transformation μ.

e$_2$) $p_i > 0$ for $i \geqslant 9$

Let $t \neq 7$, $p_u = 1$, $\sum_{u \leq i} p_i = 2$. The set of indices $5 \leqslant j < t$ for which $p_j = 1$ is divided into two sets **A** and **B** so that the difference of these powers $|\boldsymbol{A}| - |\boldsymbol{B}| = 0$ or 1 and $0 \leqslant \sum_{j \in \boldsymbol{A}} j - \sum_{j \in \boldsymbol{B}} j < t$.

The starting map is M_i, $i = \sum_{5 \leq j} (j-3)p_j - K$, where

$K = 7$ if $t \equiv 2 (\text{mod } 3)$ and $u \equiv 2 (\text{mod } 3)$,

$K = 3$ if $t \not\equiv 2 (\text{mod } 3)$ and $u \not\equiv 2 (\text{mod } 3)$,

$K = 5$ in all other cases.

Analogously, a t-gon and a u-gon are formed similarly as in the case d). When forming a t-gon or a u-gon instead of $\mathbf{A}$ or $\mathbf{B}$ triangles j-gons are formed for all $j \in \mathbf{A}$ or $j \in \mathbf{B}$, respectively. Thus we obtain M.

Let $t = 7$ and $p_5 + p_6 = 1$. The construction is the same as before except that instead of a 4-gon a 5-gon or a 6-gon is formed. The necessary 4-gon may be obtained by forming a u-gon instead of one triangle.

Let $t = 7$ and $p_5 + p_6 = 2$. We construct a map with all prescribed k-gons, $k \geqslant 5$, except one 5-gon. This 5-gon is formed together with one 4-gon from a triangle and a 4-gon of a map R^1 a3 in c3) by inserting one configuration $\mathbf{T}$.

f) $2\left[\frac{t-3}{3}\right] + 3 < \sum_{5 \leq i} p_i$, where t is such that $p_t = 1$ and $\sum_{t \leq i} p_i = 3$.

In this case there exist at least $\frac{t}{6}$ couples of indices $(i, i+1)$ such that $p_i = p_{i+1} = 1$. We select the last possible number of such indices (the set of indices situated in selected couples is indicated by $\mathbf{X}$) so that a new defined sequence

$$p_3' = 20 + \sum_{4 \leq k} (3k - 10)p_k'$$

$$p_i' = p_i \qquad \text{for all } i \notin \mathbf{X}$$

$$p_j = 0 \qquad \text{for all } j \in \mathbf{X}$$

satisfies the conditions of the preceding cases. In the map consisting of p_i' i-gons for all $i \geqslant 3$ there exists a *configuration* $\mathbf{K}$ (shown in Fig. 10). Let $(i, i+1)$ and $(j, j+1)$ be two couples of the elements of $\mathbf{X}$. Choose four triangles in $\mathbf{K}$ (in Fig. 10 indicated by $i, i+1, j, j+1$) joined by paths v_i, $i = 1, 2, 3$. By cutting this map along v_1 and inserting one configuration $\mathbf{T}$ before the repeated joining two 4-gons will arise. By the next cutting of this map along v_2 or v_3 and by inserting $i-3$ or $j-3$ configurations $\mathbf{T}$ before the repeated joining four prescribed faces are formed. Therefore $i \geqslant 7$ or $j \geqslant 7$ and thus the configuration $\mathbf{K}$ is formed again. In this way all the prescribed faces will be constructed.

2. $p_i \leqslant 1$ for all $i \geqslant 5$, $p_4 = 5$

a) $p_i = 0$ for all $i \geqslant 6$

The map M can be obtained in all possible cases from maps shown in Fig. 11 by using the transformation μ. Two maps are drawn in full lines and the two others in full and dashed lines.

b) $p_i \neq 0$ for $i \geqslant 6$

Let $p_z = 1$ and $p_i = 0$ for all $i > z$. We define a new sequence

$p'_3 = p_3 - 6$

$p'_4 = 4$

$p'_{u-1} = p_{u-1} + 1$

$p'_u = 0$

$p'_i = p_i$ for all $i \neq 3, 4, u - 1, u$.

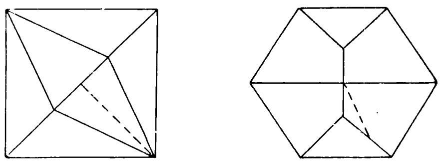

Fig. 11

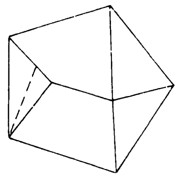

Fig. 12

Two cases must be distinguished: If $p'_{z\;1} = 1$, a sequence $(p'_k \mid 3 \leqslant k)$ satisfies the conditions of case 1. In the map consisting of p'_k k-gons for all k there exists one triangle in R^1 joined with the $(z - 1)$-gon by a path ν. Analogously as before, from these faces we obtain a z-gon and a 4-gon by inserting one configuration ***T***. If $p_z = 2$, the conditions of the case 1 are not fulfilled but the described construction is applicable with a slight variation.

3. $p_i \geqslant 2$ for $i \geqslant 5$

We define a new sequence

$$p'_3 = 20 + \sum_{4 \leq k} (3k - 10) p'_k$$

$$p'_4 = p_4 - 2 \left[\frac{p_4 - 4}{2} \right]$$

$$p'_i = p_i - 2 \left[\frac{p_i}{2} \right] \qquad \text{for all } i \neq 3, 4,$$

which satisfies condition of case 1 or 2. The starting map consists of p'_i i-gons for all i. In submap R^1 or R^2 a configuration ***W*** exists consisting of two triangles joined by a path ν. Two k-gons, $p_k \geqslant 2$, are formed from these triangles by cutting along a path ν and inserting a $k - 3$ configuration ***T*** and repeated joining. In this way all the necessary faces are formed, because the configuration ***W*** is reviewed after forming two k-gons.

β) $p_4 = 3$ and $p_5 \geqslant 1$

If $p_5 = 1$ or $p_5 = 2$ and $p_i = 0$ for all $i \geqslant 6$, we obtain the map M from a map drawn in Fig. 12 in full or full and dashed lines, respectively, by performing the transformation μ.

In all the other cases the map M is formed analogously as in α) with a small difference. The starting map M_i consists of one submap R^3 (Fig. 13) instead of a submap R^1 or R^2, respectively.

γ) $p_4 = 3$ and $p_i \geqslant 1$, $p_{i+1} \geqslant 1$ or
$p_i \geqslant 1$, $p_{i+2} \geqslant 1$ for $i \geqslant 5$.

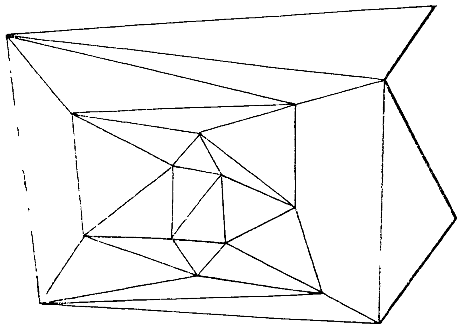

Fig. 13

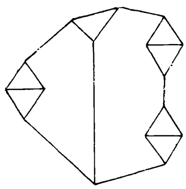

Fig. 14

First we construct a planar map with all required k-gons, $k \geqslant 5$, $p_4 + 1$ 4-gons except one i-gon and one $(i + 1)$-gon or $(i + 2)$-gon, respectively. This i-gon with a $(i + 1)$-gon or $(i + 2)$-gon is obtained from one 4-gon and one triangle of the submap R^i, $i = 1$ or 2, by inserting a $i - 3$ configuration $\boldsymbol{T}$, as before.

δ) $p_4 = 2$ and $p_5 \geqslant 2$

Let $\sum_{6 \leq i} p_i = 0$. In this case the construction is described in the proofs of the following theorems.

Let $\sum_{6 \leq i} p_i > 0$. If $i = 2k$, the starting map M_i is formed from two submaps R^3 and $k - 3$ configurations $\boldsymbol{T}$. If $i = 2k + 1$, three more triangles appear. The rest of the construction is the same as before.

ε) $p_4 = 2$ and $p_i \geqslant 2$, $p_{i+1} \geqslant 2$ or
$p_i \geqslant 2$, $p_{i+2} \geqslant 2$ for $i \geqslant 5$

In this case the construction is clear from the cases α), β), γ) and δ).

Theorem 2. *A sequence $(p_k \mid 3 \leqslant k)$ of non-negative integers is 5-realizable if it satisfies*

(i) *condition* (1),

(ii) $\sum_{4 \leq i} p_i = 3$

(iii) $p_{m_i} \neq 0$ *for* $m_i = k + \alpha_i$, $i = 1, 2, 3$, *where* $\sum_{1 \leq i} \alpha_i = 0 (\mathrm{mod}\ 2)$,

$1 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_1 + \alpha_2$ *and* $k = 2$ *or* 3 *or* 4

(iv) $p_j = 0$ *for all* $j \neq 3, m_1, m_2, m_3$.

Proof. The graph of the starting map has two vertices joined by three edges. Successively we form on the individual edges

$$\frac{\alpha_1 + \alpha_2 - \alpha_3}{2}, \quad \frac{\alpha_1 - \alpha_2 + \alpha_3}{2}, \quad \frac{-\alpha_1 + \alpha_2 + \alpha_3}{2}$$

couples of triangles with a common edge such that these couples have no common vertices. If $k = 3$, one original vertex of the starting map is replaced by a triangle and if $k = 4$, both original vertices are replaced by two triangles. (See Fig. 14, where $k = 3$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$. From this map we obtain by applying the transformation μ a map M with all the required k-gons.

Theorem 3. *A sequence* $(p_k \mid 3 \leqslant k)$ *satisfying* (1) *and decomposable into two sequences* $(p'_k \mid 3 \leqslant k)$ *and* $(c_k \mid 3 \leqslant k)$, *which satisfy the condition*

(i) $p_k = p'_k + 2c_k$ *for all* $k \geqslant 3$,

(ii) $p'_k = 0$ *for all* k *or*

$(p'_k \mid 3 \leqslant k)$ *satisfy the conditions of Theorem 2,*

(iii) c_k *are non-negative integer numbers.*

is 5-realizable.

Proof. a) Let $p'_k = 0$ for all k.

In this case the graph of the starting map is a complete graph of four vertices.

In the starting map there exists a triangle **F** and a 3-valent vertex **V** which is not the vertex of **F** but is joined with **F** by edge. If $p_k \neq 0$, $k \geqslant 4$, joining the vertex **V** by $k - 3$ new edges with $k - 3$ points on an edge of **F**, we get from the triangle **F** one k-gon and from **V** one k-valent vertex and $k - 3$ new triangles. Successively $\frac{1}{2} p_k$ couples of the k-gon and a k-valent vertex are constructed for all $k \geqslant 4$. From this map we obtain M by applying the transformation μ.

b) Let $p'_k \neq 0$ for $k \geqslant 4$.

The construction of M is obvious from case *a*, and the proof of *Theorem 2*.

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