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Additional Note to our Paper 'A Genesis for Combinatorial Identities'

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ADDITIONAL NOTE TO OUR PAPER „A GENESIS FOR COMBINATORIAL IDENTITIES"

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In the paper [1] we have described a certain method by means of which we can derive some combinatorial formulas. In this note we introduce another similar method.

Theorem. *Let n be a natural number, x an arbitrary complex number and $a_1, a_2, \ldots, a_n, a_{n+1}$ the given distinct complex numbers, with the condition $a_k = a_{k-n-1}$ for $k > n + 1$. Then the following relation holds*

$$\sum_{i=1}^{n+1} \frac{(x+a_i)(x+a_{i+1})\ldots(x+a_{i+n-1})}{(a_i - a_{i-1})(a_{i+1} - a_{i-1})\ldots(a_{i+n-1} - a_{i-1})} = 1 . \tag{1}$$

Proof. (1) is an algebraic equation of degree n in x. But it has $(n+1)$ roots

$$-a_1, -a_2, \ldots, -a_n, -a_{n+1} . \tag{2}$$

Therefore it is an identity.

In fact the factor $(x + a_k)$, $k = 1, 2, \ldots, n, (n + 1)$ occurs in all members on the left side of this equation except in member with $i = k + 1$. Thus for $x = -a_k$ only the member

$$\frac{(-a_k + a_{k+1})(-a_k + a_{k+2})\ldots(-a_k + a_{k+n})}{(a_{k+1} - a_k)(a_{k+2} - a_k)\ldots(a_{k+n} - a_k)} = 1 \tag{3}$$

is different from zero.

Example. Let $a_i = i$. In this case equation (1) gives

$$\frac{(x+1)(x+2)\ldots(x+n)}{(-n)[-(n-1)]\ldots(-2)(-1)} + \frac{(x+2)(x+3)\ldots(x+n+1)}{1 \,.\, 2 \,.\, \ldots \,.\, n} +$$

$$+ \frac{(x+3)(x+4)\ldots(x+n+1)}{1 \,.\, 2 \,.\, \ldots \,.\, (n-1)} \cdot \frac{x+1}{-1} + \frac{(x+4)(x+5)\ldots(x+n+1)}{1 \,.\, 2 \,.\, \ldots \,.\, (n-2)} \cdot$$

$$\cdot \frac{(x+1)(x+2)}{(-1)(-2)} + \ldots + \frac{x+n+1}{1} \cdot$$

$$\cdot \frac{(x+1)(x+2)\ldots(x+n-1)}{[-(n-1)][-(n-2)]\ldots(-2)(-1)} = 1$$

or

$$\sum_{k=0}^{n} (-1)^k \binom{x+n+1}{n-k}\binom{x+k}{k} = 1. \tag{4}$$

In virtue of identity

$$\binom{x+n+1}{n-k}\binom{x+k}{k} = \binom{x+n+1}{n+1}\binom{n}{k}\frac{n+1}{x+k+1} \tag{5}$$

we have therefrom

$$\sum_{k=0}^{n} (-1)^k \frac{1}{x+k+1}\binom{n}{k} = \left[\binom{x+n+1}{n+1}(n+1)\right]^{-1}. \tag{6}$$

This is a generalisation of the well-known relation

$$\sum_{k=0}^{n} (-1)^k \frac{1}{n+k+1}\binom{n}{k} = \frac{(n!)^2}{(2n+1)!}. \tag{7}$$

See [2].

Remark. Let us only remark that the identity (4) can be obtained in another with the aid of Cauchy's identity $\sum_{k=0}^{n}\binom{x}{k}\binom{y}{1-k} = \binom{x+y}{n}$

$$1 = \binom{n}{n} = \binom{x+n+1-(x+1)}{n} = \sum_{k=0}^{n}\binom{x+n+1}{n-k}\binom{-x-1}{k} = \tag{8}$$

$$= \sum_{k=0}^{n} (-1)^k \binom{x+n+1}{n-k}\binom{x+k}{k}.$$

REFERENCES

[1] Bartoš P., Kaucký J., *A genesis for combinatorial identities*, Mat.-fyz. časop. *16* (1966), 31—40.
[2] Turán P., *Problème* 51, Mat. lapok *7* (1956), 141.

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