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Additional Note to our Paper 'A Genesis for Combinatorial Identities'

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**ADDITIONAL NOTE TO OUR PAPER
„A GENESIS FOR COMBINATORIAL IDENTITIES”**

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In the paper [1] we have described a certain method by means of which we can derive some combinatorial formulas. In this note we introduce another similar method.

Theorem. Let n be a natural number, x an arbitrary complex number and $a_1, a_2, \dots, a_n, a_{n+1}$ the given distinct complex numbers, with the condition $a_k = a_{k-n-1}$ for $k > n + 1$. Then the following relation holds

$$(1) \quad \sum_{i=1}^{n+1} \frac{(x + a_i)(x + a_{i+1}) \dots (x + a_{i+n-1})}{(a_i - a_{i-1})(a_{i+1} - a_{i-1}) \dots (a_{i+n-1} - a_{i-1})} = 1.$$

Proof. (1) is an algebraic equation of degree n in x . But it has $(n + 1)$ roots

$$(2) \quad -a_1, -a_2, \dots, -a_n, -a_{n+1}.$$

Therefore it is an identity.

In fact the factor $(x + a_k)$, $k = 1, 2, \dots, n$, $(n + 1)$ occurs in all members on the left side of this equation except in member with $i = k + 1$. Thus for $x = -a_k$ only the member

$$(3) \quad \frac{(-a_k + a_{k+1})(-a_k + a_{k+2}) \dots (-a_k + a_{k+n})}{(a_{k+1} - a_k)(a_{k+2} - a_k) \dots (a_{k+n} - a_k)} = 1$$

is different from zero.

Example. Let $a_i = i$. In this case equation (1) gives

$$\begin{aligned} & \frac{(x + 1)(x + 2) \dots (x + n)}{(-n)[- (n - 1)] \dots (-2)(-1)} + \frac{(x + 2)(x + 3) \dots (x + n + 1)}{1 \cdot 2 \cdot \dots \cdot n} \\ & + \frac{(x + 3)(x + 4) \dots (x + n + 1)}{1 \cdot 2 \cdot \dots \cdot (n - 1)} + \dots + \frac{(x + 1)}{-1} + \frac{(x + 4)(x + 5) \dots (x + n - 1)}{1 \cdot 2 \cdot \dots \cdot (n - 2)}. \end{aligned}$$

$$\frac{(x+1)(x+2)}{(-1)(-2)} + \dots + \frac{x+n+1}{1} \\ = \frac{(x+1)(x+2)\dots(x+n-1)}{[-(n-1)][-(n-2)]\dots(-2)(-1)} = 1$$

or

$$(4) \quad \sum_{k=0}^n (-1)^k \binom{x+n+1}{n-k} \binom{x+k}{k} = 1.$$

In virtue of identity

$$(5) \quad \binom{x+n+1}{n-k} \binom{x+k}{k} = \binom{x+n+1}{n+1} \binom{n}{k} \frac{n+1}{x+k+1}$$

we have therefrom

$$(6) \quad \sum_{k=0}^n (-1)^k \frac{1}{x+k+1} \binom{n}{k} = \left[\binom{x+n+1}{n+1} (n+1) \right]^{-1}.$$

This is a generalisation of the well-known relation

$$(7) \quad \sum_{k=0}^n (-1)^k \frac{1}{n+k+1} \binom{n}{k} = \frac{(n!)^2}{(2n+1)!}.$$

See [2].

Remark. Let us only remark that the identity (4) can be obtained in

another with the aid of Cauchy's identity $\sum_{k=0}^n \binom{x}{k} \binom{y}{1-k} = \binom{x+y}{n}$

$$(8) \quad 1 = \binom{n}{n} = \binom{x+n+1-(x+1)}{n} = \sum_{k=0}^n \binom{x+n+1}{n-k} \binom{-x-1}{k} \\ = \sum_{k=0}^n (-1)^k \binom{x+n+1}{n-k} \binom{x+k}{k}.$$

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