Matematický časopis

Daniel Palumbíny; Štefan Znám

On Decompositions of Complete Graphs into Factors with Given Radii

Matematický časopis, Vol. 23 (1973), No. 4, 306--316

Persistent URL: http://dml.cz/dmlcz/126576

Matematický časopis 23 (1973), No. 4

ON DECOMPOSITIONS OF COMPLETE GRAPHS INTO FACTORS WITH GIVEN RADII

DANIEL PALUMBÍNY, Zvolen, ŠTEFAN ZNÁM, Bratislava

In paper [1] the decomposition of complete graphs into factors with given diameters is studied. A. Rosa proposed to study the decomposition of complete graphs into factors with given radii. Our article deals with this problem.

The mentioned problem is here completely solved for a decomposition into two factors and some partial results for a decomposition into three factors are given. Further, we consider the decomposition with equal radii.

Some of our results can also be used for solving the problems studied in [1].

General considerations

We shall consider undirected graphs without loops and multiple edges. Let G be such a graph and V_G its vertex set. The radius $r(G)$ of a graph G is defined as

$$r(G) = \inf_{x \in V_G} \sup_{y \in V_G} \varrho_G(x, y),$$

where $\varrho_G(x, y)$ denotes the distance between two vertices $x, y \in V_G$ in G. Hence $r(G)$ is ∞ if G is a disconnected graph or if $\sup_{y \in V_G} \varrho_G(x, y)$ is infinite for all x. Obviously $r(G) \leqq d(G)$ (the diameter of G) for any G. Suppose that G is finite and connected. Then the eccentricity $\varepsilon(x)$ of a vertex x in G is $\max \varrho_G(x, y)$ for all $y \in V_G$. Clearly $r(G) = \min_{x \in V_G} \varepsilon(x)$ and $d(G) = \max_{x \in V_G} \varepsilon(x)$. A vertex v is a center of G if $\varepsilon(v) = r(G)$. The remaining terms are used in the usual sense (see [2]). The complete graph with n vertices will be denoted by $\langle n \rangle$.

We shall study conditions for the existence of a decomposition of $\langle n \rangle$ into factors $F_1, F_2, \ldots, F_m$ with given radii $r_1, r_2, \ldots, r_m$, where $r_i = r(F_i)$ $(i = 1, 2, \ldots, m)$ are naturals or symbols ∞. Denote by $G(r_1, r_2, \ldots, r_m)$ the smallest natural n for which $\langle n \rangle$ is decomposable into m factors with radii $r_1, r_2, \ldots, r_m$; if such a natural does not exist then put $G(r_1, r_2, \ldots, r_m) = \infty$.

Theorem 1. *If $\langle n \rangle$ is decomposable into factors $F_1, F_2, \ldots, F_m$ with the radii*

$r_1, r_2, \ldots, r_m$, *then for any cardinal* $N > n$ *the graph* $\langle N \rangle$ *is decomposable in the same way.*

Proof. If $m = 1$, the assertion is trivial. Therefore let $m \geqq 2$. Denote $H = \langle N \rangle$ and let $U = \langle n \rangle$ be a complete subgraph of H. Denote $A = V_U$, $B = V_H - V_U$ and choose a vertex $v \in A$. Decompose U into factors U_1, $U_2, \ldots, U_m$ with radii $r_1, r_2, \ldots, r_m$. Decompose H into factors $H_1, H_2, \ldots, H_m$ as follows:

1. all the edges of U_i belong to H_i $(i = 1, 2, \ldots, m)$,
2. for $a \in A (a \neq v)$, $b \in B$ the edge $ab \in H_i$ if the edge $av \in U_i$,
3. the edges of the complete graph with vertex set $B \cup \{v\}$ belong to H_1.

Obviously, if $r(U_1) = 1$, then $r(H_1) = 1$, too. For $r_i > 1$ the statement that $r(H_i) = r_i$ can be proved in the same manner as the analogical assertion in Theorem 1 of [1].

From this theorem it follows that if $G(r_1, r_2, \ldots, r_m)$ is found, then the problem of the existence of a decomposition of $\langle N \rangle$ into m factors with radii $r_1, r_2, \ldots, r_m$ is solved for any cardinal number N.

Now we prove the following

Lemma 1. *Let r and n be positive integers, then for a graph G with n vertices and radius r we have*

$$(1) \qquad 2r \leqq n.$$

Proof. The case $r = 1$ is trivial. Therefore we can suppose $r \geqq 2$. Let v be an arbitrary center of G. Since $\varepsilon(v) = r$, there exists a vertex w in G such that $\varrho_G(v, w) = r$. Let $v v_1 v_2 \ldots v_{r-1} w$ be a shortest path from v to w. Denote by S the set of such vertices of G that no shortest path joining them to v is passing through v_1. It is easy to show that $\deg v \neq 1$, which implies $S \neq \emptyset$. Let $s = \max_{x \in S} \varrho_G(v, x)$. Clearly $s \geqq r - 1$. If the opposite were true, then $\varepsilon(v_1) \leqq r - 1$, which contradicts the fact that r is the radius of G. Thus there exists a path (beginning in v) of the length $r - 1$ in G not containing vertices in common with the path $v_1 v_2 \ldots v_{r-1} w$. Hence G contains at least $2r$ vertices.

In our considerations we shall need the following results (see [3]):

Theorem 2. *Let n and r be positive integers such that* $2r \leqq n$. *Then the maximal number of edges in a graph with n vertices and radius r is*

$$f(n, r) = \begin{cases} \dfrac{n(n-1)}{2}, & \textit{if } r = 1, \\[2mm] \left[\dfrac{n(n-2)}{2}\right], & \textit{if } r = 2, \\[2mm] \dfrac{n^2 - 4rn + 5n + 4r^2 - 6r}{2}, & \textit{if } r \geqq 3. \end{cases}$$

Corollary. *For* $2 \leqq r < \infty$ *we have* $f(2r, r) = 2r$.

Theorem 3. *Let* n *and* r *be positive integers such that* $4 \leqq 2r \leqq n$, *then the maximal degree of the vertices of a graph with* n *vertices and radius* r *is* $n - 2r + 2$.

Analogically as in [1] (see Theorem 2) it can be shown that if $\langle n\rangle$ is decomposable into m factors with natural radii, then

$$2m \leqq n. \tag{2}$$

Theorem 4. *Let naturals* $m, n, r_1, r_2, \dots, r_m$ *be given. If the complete graph* $\langle n\rangle$ *is decomposable into* m *factors with radii* $r_1, r_2, \dots, r_m$, *then*

$$n^2 - n - 2\sum_{i=1}^{m} f(n, r_i) \leqq 0, \tag{3}$$

$$2 \max r_i \leqq n. \tag{4}$$

Proof. Denote by h_i the number of edges in the factor F_i. Then obviously

$\binom{n}{2} = \sum_{i=1}^{m} h_i \leqq \sum_{i=1}^{m} f(n, r_i)$ and (3) follows. According to (1) we have (4).

Corollary. *For arbitrary naturals* $m, n, r_1, r_2, \dots, r_m$ *we have*

$$G(r_1, r_2, \dots, r_m) \geqq 2 \max (m, \max r_i).$$

Theorem 5. *For* $m \geqq 3$ *and* $r_2 = r_3 = \dots = r_m = \infty$ *we have*

$$G(r_1, r_2, \dots, r_m) = \begin{cases} 3, & \text{if } r_1 = \infty, \\ 2r_1, & \text{if } r_1 < \infty. \end{cases}$$

Proof. The proof of the first part is evident. If a graph contains a factor with natural radius r_1, then it has to have at least $2r_1$ vertices (see Lemma 1). Therefore it is sufficient to decompose the graph $\langle 2r_1\rangle$ into m factors with radii $r_1, \infty, \dots, \infty$. It can be done as follows. Denote the vertices of $\langle 2r_1\rangle$ by $v_1, v_2, \dots, v_{2r_1}$. The factor F_1 consists of the cycle $v_1v_2 \dots v_{2r_1}v_1$. The factor F_2 consists of all edges between $v_2, v_3, \dots, v_{2r_1}$ except of those contained in F_1. F_3 consists of the remaining edges and F_i for $i \geqq 4$ (if $m > 3$) are nullgraphs. It is easy to check that this decomposition fulfils the required conditions.

Theorem 6. *Let* $m \geqq 3$, $r_i \geqq 2$ $(i = 1, 2, \dots, m)$ *be naturals. Then*

$$G(r_1, r_2, \dots, r_m) \leqq 2(r_1 + r_2 + \dots + r_m) - 2m.$$

Proof. It is sufficient to find a decomposition of the graph $G = \langle 2(r_1 + r_2 + \dots + r_m) - 2m\rangle$ into factors $F_1, F_2, \dots, F_m$ with radii $r_1, r_2, \dots, r_m$. We shall use the construction from the proof of Theorem 4 of [1] with d_i

$= 2r_i - 1$. It is easy to show that for every factor F_i in this construction $v_{i,r_i 1}$ is a center of G and $r(F_i) = r_i$.

Corollary. *For every natural $m > 1$ the equality*

$$G(\underbrace{2, 2, \ldots, 2}_{m - times}) = 2m$$

holds.

Proof. For $m = 2$ see Theorem 9. For $m > 2$ our assertion follows from Theorem 6 and from Corollary of Theorem 4.

Theorem 7. *Let $3 \leqq r_1 \leqq r_2 \leqq r_3 \leqq r_4 < \infty$. Then we have*

$$G(r_1, r_2, r_3, r_4) \leqq 2(r_1 + r_2 + r_4) - 9.$$

Proof. We shall construct the four factors of the graph $\langle 2(r_1 + r_2 + r_4) - 9\rangle$ with the radii r_i ($i = 1, 2, 3, 4$). Denote the vertices of the graph $\langle 2(r_1 + r_2 + r_4) - 9\rangle$ by $u_1, u_2, \ldots, u_{2r_1-3}, v_1, v_2, \ldots, v_{2r_2-3}, w_1, w_2, \ldots, w_{2r_4-3}$.

I. The factor F_1 contains
- (a) the edges of the path $u_1u_2 \ldots u_{2r_1-3}$,
- (b) u_1w_1,
- (c) all the edges v_iw_j except of v_1w_1,
- (d) w_iw_j with $j - i \geqq 2$ except of:
  w_3w_1 and w_2w_i, $i = r_3 + 1, r_3 + 2, \ldots, 2r_4 - 3$ for $r_3 = 3$, (if they exist),
  w_1w_4 for $r_3 = 4$,
  the path $w_3w_1w_4w_{2r_4-3}w_5w_{2r_4-4}w_6 \ldots w_{2r_4-r_3+2}w_{r_3}$ and the edges w_2w_i, $i = r_3 + 1, r_3 + 2, \ldots, 2r_4 - r_3 + 1$ for $r_3 \geqq 5$.

II. The factor F_2 contains
- (a) the path $v_1v_2 \ldots v_{2r_2-3}$,
- (b) v_1u_1,
- (c) all u_iw_j except of u_1w_1, u_2w_2, u_3w_2, u_3w_3,
- (d) u_iu_j with $j - i \geqq 2$.

III. the factor F_3 contains
- (a) u_1v_2, u_2v_1, u_2v_2, u_3v_3,
- (b) u_3v_i and v_3u_i with $i > 3$ (if they exist),
- (c) u_2w_2, u_3w_2, u_3w_3,
- (d) w_3w_1,
- (e) w_2w_i, $i = r_3 + 1, r_3 + 2, \ldots, 2r_4 - 3$ for $r_3 = 3, 4$ (if they exist),
- (f) w_1w_4 if $r_3 = 4$,
- (g) the path $w_1w_4w_{2r_4-3}w_5w_{2r_4-4}w_6 \ldots w_{2r_4-r_3+2}w_{r_3}$ and w_2w_i, $i = r_3 + 1, r_3 + 2, \ldots, 2r_4 - r_3 + 1$ for $r_3 \geqq 5$.

IV. The factor F_4 contains

(a) the path $w_1 w_2 \dots w_{2r_4-3}$,
(b) $v_1 w_1$,
(c) $v_i v_j$ with $j - i \geqq 2$,
(d) $u_i v_j$ except of $u_1 v_1$, $u_1 v_2$, $u_2 v_1$, $u_2 v_2$, $u_3 v_3$ and $u_3 v_i$, $u_i v_3$ with $i > 3$ (if they exist).

It can be proved that the system of the factors F_i forms a decomposition of $\langle 2(r_1 + r_2 + r_4) - 9\rangle$ and that $r(F_i) = r_i$.

Remark 1. Analogical results can be stated (and proved by similar methods) in case of a decomposition into 5 and 6 factors with given radii.

Remark 2. It can be easily proved that for $r_i \geqq 4$ the factors F_i ($i = 1, 2, 3, 4$) in the preceding theorem have diameters $d_i = 2r_i - 1$. Denote by $F(d_1, d_2, d_3, d_4)$ the smallest natural N for which $\langle N\rangle$ can be decomposed into 4 factors with diameters d_1, d_2, d_3, d_4 (see [1]). Then we get

Theorem 8. *Let* $6 \leqq d_1 \leqq d_2 \leqq d_3 \leqq d_4 < \infty$, *then*

$$F(d_1, d_2, d_3, d_4) \leqq d_1 + d_2 + d_4 - 6.$$

Proof. If d_1, d_2, d_3, d_4 are odd, the proof follows from the considerations above. If some of them are even, it can be done by using a similar consideration.

This theorem can be developed for decomposition into 5 and 6 factors with given diameters, too.

The case $m = 2$

It is easy to prove the following

Lemma 2. *If* $r(G) = 1$, *then the complement* $\bar{G}$ *of* G *is a disconnected graph. If* G *is a disconnected graph, then* $r(\bar{G})$ *is* 1 *or* 2.

Lemma 3. *If* $r(G) \geqq 3$, *then* $r(\bar{G}) \leqq 2$.

Proof. According to Lemma 2 we may suppose that G is connected. We shall distinguish two cases.

(a) $d(G) \geqq 4$, then due to Lemma 3 of [1] we get $r(\bar{G}) \leqq d(\bar{G}) \leqq 2$.

(b) $(rG) = d(G) = 3$. Then for every vertex x there exists a vertex x′ with $\varrho_G(x, x') = 3$. We shall proceed indirectly: suppose there exist two vertices u, v for which $\varrho_{\bar{G}}(u, v) = 3$. (Then the edge uv belongs to G.) Let v' be a vertex for which $\varrho_G(v, v') = 3$. (Then the edge vv' belongs to $\bar{G}$.) Consider the edge uv'. If uv' belongs to G, then vuv' is a path of the length 2 in G between the vertices v and v', which is a contradiction. If uv' belongs to $\bar{G}$, then the path $uv'v$ is in $\bar{G}$ (the length is 2) — a cotradiction.

Theorem 9. Let $r_1 \leqq r_2$, then

$$G(r_1, r_2) = \begin{cases} 2 & \textit{if } r_1 = 1,\ r_2 = \infty, \\ 4 & \textit{if } r_1 = 2,\ r_2 = \infty, \\ 2r_2 & \textit{if } r_1 = 2,\ r_2 < \infty, \\ \infty & \textit{in the remaining cases.} \end{cases}$$

Proof. The proofs of the assertions $G(1, \infty) = 2$, $G(2, \infty) = 4$ and $G(2, 2) = 4$ are evident.

If $2 < r_2 < \infty$, then decompose $\langle 2r_2 \rangle$ into two factors as follows. The factor F_2 consists of a cycle containing all the vertices of $\langle 2r_2 \rangle$. F_1 contains all the remaining edges. Then obviously $r(F_1) = 2$ and $r(F_2) = r_2$.

Clearly $G(1, r) = \infty$ for any finite r. From Lemma 3 it follows that for $r_1 \geqq 3$, we have $r_2 \leqq 2$, hence $G(r_1, r_2) = \infty$ for $r_1, r_2 \geqq 3$.

The case $m = 3$

Theorem 10. *For* $3 \leqq r_1 \leqq r_2 \leqq r_3 < \infty$ *we have*

$$G(r_1, r_2, r_3) \leqq 2(r_1 + r_2 + r_3) - 11.$$

Proof. In the proof of the second part of Theorem 6 in [1] a decomposition of $\langle d_1 + d_2 + d_3 - 8 \rangle$ into factors F_1, F_2, F_3 of diameters d_1, d_2, d_3 is given. Put $d_i = 2r_i - 1$. It is easy to prove that the factor F_i of the mentioned decomposition has radius equal to r_i.

Theorem 11. *Let* $2 \leqq r_2 \leqq r_3 < \infty$, *then* $G(2, 2, 2) = 6$ *and* $G(2, r_2, r_3) = 2r_3$ *if* $r_3 \geqq 3$.

Proof. The first assertion follows from Corollary of Theorem 6. From Corollary of Theorem 4 we get $G(2, r_2, r_3) \geqq 2r_3$, hence it is sufficient to prove that $\langle 2r_3 \rangle$ can be decomposed into three factors with radii $2, r_2, r_3$ ($r_3 \geqq 3$). Denote the vertices of $\langle 2r_3 \rangle$ by $v_1, v_2, \ldots, v_{2r_3}$. We shall distinguish two cases.

(a) $r_2 = 2$. Let the factor F_3 consist of the path $v_1 v_2 \ldots v_{2r_3}$. Obviously $r(F_3) = r_3$. Let the edges $v_{2r_3}v_1, v_{2r_3}v_2, \ldots, v_{2r_3}v_{2r_3-3}$, $v_{2r_3-1}v_{2r_3-3}$, $v_{2r_3-2}v_{2r_3-4}$ belong to F_1 and the edges v_1v_3, v_1v_4, $v_1v_5, \ldots, v_1v_{2r_3-1}$, $v_{2r_3}v_{2r_3-2}$, $v_{2r_3-1}v_2$ belong to F_2; the remaining edges are distributed into the factors F_1 and F_2 in an arbitrary way. None of the vertices in F_i ($i = 1, 2$) is of degree $2r_3 - 1$ and hence $r(F_i) > 1$. It is easy to check that $v_{2r_3}(v_1)$ is a center of $F_1(F_2)$ and that $r(F_1) = r(F_2) = 2$.

(b) $r_2 \geqq 3$. $G(2, 3, 3) = 6 = 2r_3$ (see Fig. 1). Therefore we can suppose $r_3 \geqq 4$. Now we shall construct the factors F_i with radii $2, r_2, r_3$. The factor F_3 is equal to the path $v_{2r_3-1}v_{2r_3-3} \ldots v_9v_7v_3v_2v_4v_1v_5v_6v_8v_{10} \ldots v_{2r_3-2}v_{2r_3}$. Thus it has radius r_3. We must distinguish 4 cases:

(b1) If $r_2 = 3$, then F_2 contains the path $v_2v_1v_3v_6v_4v_5$.

(b2) If $r_2 = 4$, then F_2 contains the path $v_2v_1v_3v_6v_4v_5v_7v_8$.

(b3) If $r_2 \geqq 5$ and odd, then F_2 contains the path $v_{2r_2}v_{2r_2-1}v_{2r_2-4}v_{2r_2-5} \dots$ $\dots v_{10}v_9v_2v_1v_3v_6v_4v_5v_7v_8v_{11}v_{12} \dots v_{2r_2-3}v_{2r_2-2}$, where the vertices $v_7, v_8, \dots, v_{2r_2}$ were added to the path $v_2v_1v_3v_6v_4v_5$ in the evident way.

(b4) If $r_2 \geqq 6$ and even, then F_2 contains the path $v_{2r_2-2}v_{2r_2-3}v_{2r_2-6}v_{2r_2-7} \dots$ $\dots v_{10}v_9v_2v_1v_3v_6v_4v_5v_7v_8v_{11}v_{12} \dots v_{2r_2-1}v_{2r_2}$.

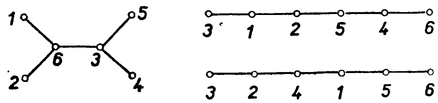

Fig. 1.

If $r_2 < r_3$, then F_2 contains besides the mentioned path also the edges $v_4v_{2r_2+1}, v_4v_{2r_2+2}, \dots, v_4v_{2r_3}$ (in all four cases). It can be shown that in all cases $r(F_2) = r_2$.

The factor F_2 (F_3) consists of $2r_3 - 1$ edges. Put all the remaining edges into the factor F_1. We have to prove that $r(F_1) = 2$. It can be shown that F_1 is a connected graph (it contains the path $v_1v_6v_2v_5v_3v_4$ and the edges v_1v_i for $i > 6$). F_1 contains

$$X = \binom{2r_3}{2} - 2(2r_3 - 1) = 2r_3^2 - 5r_3 + 2$$

edges. We now show that

$$X > f(2r_3, r) = 2r_3^2 - 4r_3r + 5r_3 + 2r^2 - 3r \tag{5}$$

for $3 \leqq r \leqq r_3$ and $r_3 \geqq 4$ (see Theorem 2). We have two cases:

(a) If $r = 3$, then $f(2r_3, 3) = 2r_3^2 - 7r_3 + 9$. Since $r_3 \geqq 4$, which implies $2r_3 > 7$, we have $2r_3^2 - 5r_3 + 2 > 2r_3^2 - 7r_3 + 9$ i. e. $X > f(2r_3, 3)$.

(b) If $r \geqq 4$, then $4r - 10 > 2r - 3$. Since $r_3 \geqq r > 0$, $4r - 10 > 0$ and $2r - 3 > 0$, we have $r_3(4r - 10) > r(2r - 3)$. The last inequality implies $2r_3^2 - 5r_3 + 2 > 2r_3^2 - 4r_3r + 5r_3 + 2r^2 - 3r$, i. e. $X > f(2r_3, r)$ for $4 \leqq r \leqq r_3$.

We have proved that (5) holds, hence $r(F_1) \leqq 2$. However $r(F_1) > 1$ because none of the vertices in F_1 is of degree $2r_3 - 1$. Thus $r(F_1) = 2$.

Theorem 12. *Let* $3 \leqq r_3 < \infty$. *Then*

$$G(3, 3, r_3) = 2r_3.$$

Proof. According to Corollary of Theorem 4 we have $G(3, 3, r_3) \geqq 2r_3$. It can be shown that $G(3, 3, 3) = 6$ (see Fig. 2) and $G(3, 3, 4) = 8$ (see Fig. 3).

Hence it is sufficient to find a decomposition of $\langle 2r_3 \rangle$ for $r_3 \geqq 5$ into three factors with radii 3, 3, r_3. Denote the vertices of $\langle 2r_3 \rangle$ by $u_1, u_2, \ldots, u_{r_3}, v_1, v_2, \ldots, v_{r_3}$. For $i > r_3$ we define u_i (v_i) in the following manner: $u_i(v_i) = u_s(v_s)$ with $s \equiv i (\operatorname{mod} r_3)$, $0 < s \leqq r_3$.

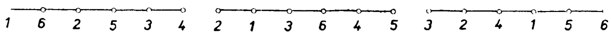

Fig. 2.

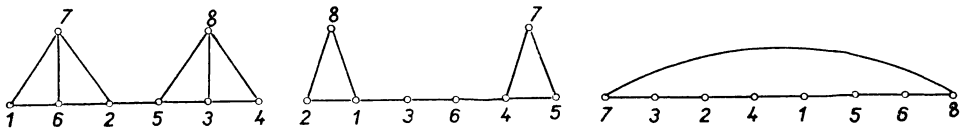

Fig. 3.

Let the factor F_1 contain the edges

(a) u_iu_j and v_iv_j for $j \not\equiv i + 1 (\operatorname{mod} r_3)$ and $j \not\equiv i - 1 (\operatorname{mod} r_3)$,

(b) all the edges $u_iv_{i+r_3-2}$.

Then $r(F_1) = 3$ $(\varrho_{F_1}(u_i, v_{i+r_3-1}) = 3$ and every vertex is a center of F_1).

The factor F_2 contains the edges

(a) $u_iv_{i+1}, u_iv_{i+2}, \ldots, u_iv_{i+r_3-3}$,

(b) u_iu_{i+1} and v_iv_{i+1}.

Obviously $r(F_2) = 3$ $(\varrho_{F_2}(u_i, v_{i+r_3-1}) = 3$ and every vertex is a center of F_2).

The factor F_3 contains the remaining edges u_iv_i and v_iu_{i+1} which form a cycle of the length $2r_3$.

Remark. Fig. 4 shows that $G(3, 4, 4) = 8$, but it can be easily proved that $G(3, r, r) > 2r$ for $4 < r < \infty$. To prove it (indirectly), we suppose that the graph $\langle 2r \rangle$ $(5 \leqq r < \infty)$ is decomposable into three factors with radii $3, r, r$. Then the factors F_2 and F_3 have at most $2r$ edges each (see Corollary of Theorem 2). Thus F_1 contains at least $Y = \binom{2r}{2} - 4r = 2r^2 - 5r$ edges. According to Theorem 2 we have $f(2r, 3) = 2r^2 - 7r + 9$. It is easy to check that $Y > f(2r, 3)$ for $r > 4$. Hence $r(F_1) \neq 3$, which is a contradiction.

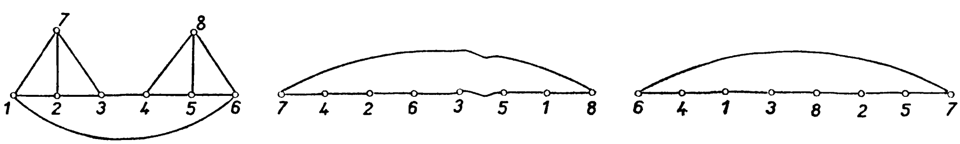

Fig. 4.

Theorem 13. *We have*

I. $G(2, r_2, \infty) = 2r_2$ for $2 \leqq r_2 < \infty$,

II. $\max\left\{2r_2, \frac{4}{3}(r_1 + r_2 - 2)\right\} \leqq G(r_1, r_2, \infty) \leqq 2(r_1 + r_2) - 6$ *for* $3 \leq$

$\leqq r_1 \leqq r_2 < \infty$.

Proof. I. The first assertion follows from Theorem 9 (take as F_3 the null-graph).

II. Suppose that for some n with

$$n < \frac{4}{3}(r_1 + r_2 - 2) \tag{6}$$

the graph $\langle n \rangle$ can be decomposed into three factors F_i with $r(F_1) = r_1$, $r(F_2) = r_2$, $r(F_3) = \infty$. The factor F_3 is disconnected, hence the vertices of n can be split into two disjoint sets A and B so that all the edges between A and B belong to F_1 or F_2. From (6) we get

$$2(2n - 2r_1 - 2r_2 + 4) < n .$$

Hence one of the sets — say A — contains at least $2n - 2r_1 - 2r_2 + 5$ elements. Let v be an arbitrary element of B. According to Theorem 3 the degree of v in F_1 is at most $n - 2r_1 + 2$ and in the factor F_2 at most $n - 2r_2 + 2$. This is a contradiction.

To complete the proof we must show that $G(r_1, r_2, \infty) \leqq 2r_1 + 2r_2 - 6$. It can be done by considerations analogical to those of the proof of Theorem 8 from [1] (see part I(b)). Namely, if we take $d_i = 2r_i - 1$ $(i = 1, 2)$, then we can see that the factors F_1 and F_2 of the graph $\langle d_1 + d_2 - 4 \rangle = \langle 2r_1 + 2r_2 - 6 \rangle$ have the radii r_1 and r_2. The factor F_3 is obviously disconnected. (There is no path from v_1 to any v_i with $i \geqq 2$ in F_3.)

Decomposition into 3 and 4 factors with equal radii

Denote $G(r, r, r) = g(r)$.

Theorem 14. *The following holds*

I. $g(\infty) = 3$, $g(1) = \infty$, $g(2) = g(3) = 6$,

II. $(3 + \sqrt{3})r - 9 < g(r) \leqq 6r - 11$ *for* $4 \leqq r < \infty$.

Proof. The first part follows from evident considerations. The estimation $g(r) \leqq 6r - 11$ holds due to Theorem 10. Now, if $\langle n \rangle$ is decomposable into three factors with equal radii r, then owing to Theorem 2 we get

$$3\,\frac{n^2 - 4rn + 5n + 4r^2 - 6r}{2} \geqq \binom{n}{2}.$$

After some modifications of the last inequality we get

$$s_r(n) = n^2 + (8 - 6r)n + (6r^2 - 9r) \geqq 0. \tag{7}$$

It can be easily checked that $s_r(2r) < 0$ and $s_r((3 + \sqrt{3})r - 9) < 0$ for all $r \geqq 4$. The function $s_r(n)$ is convex and hence from (7) we get $n > (3 + \sqrt{3})r - 9$. The theorem follows.

Now denote $G(r, r, r, r) = H(r)$.

Theorem 15. *For* $3 \leqq r < \infty$ *we have*

$$4r - 8 \leqq H(r) \leqq 6r - 9.$$

Proof. The estimation $H(r) \leqq 6r - 9$ follows from Theorem 7. Further we have to prove that $\langle 4r - 9 \rangle$ cannot be decomposed into 4 factors with equal radii r. For $r = 3$ and 4 this follows from (2); so we can suppose $r \geqq 5$. Suppose $\langle n \rangle$ is decomposable into 4 factors with radii r. Then according to Theorem 2 we have

$$4\,\frac{n^2 - 4rn + 5n + 4r^2 - 6r}{2} \geqq \binom{n}{2}. \tag{8}$$

After some modifications

$$t_r(n) = 3n^2 + (21 - 16r)n + (16r^2 - 24r) \geqq 0.$$

For $r \geq 5$ obviously $t_r(2r) < 0$ and it can be shown that (for $r \geqq 6$) $t_r(4r - 8) \leq 0$. $t_r(n)$ is a convex function of the variable n for any r and hence if n fulfils (8) where $r \geqq 6$, then $n \geqq 4r - 8$. As for $r = 5$ we have $t_5\left(\frac{35}{3}\right) = 0$ and $H(5) \geqq \frac{35}{3} = 11\frac{2}{3}$. Since $H(5)$ is an integer, we get $H(5) \geqq 12 = 4 . 5 - 8$.

Remark. From Corollary of Theorem 6 it follows that $H(2) = 8$.

REFERENCES

[1] BOSÁK, J., ROSA, A., ZNÁM, Š.: On decompositions of complete graphs into factors with given diameters, In: Theory of graphs, Budapest 1968, 37—56.
[2] ORE, O.: Theory of graphs. 1. ed. Providence 1962.
[3] ВИЗИНГ, В. Г.: О числе ребер в графе с данным радиусом, ДАН СССР. *173*, 1967, 1245—1246.

Received April 13, 1971

Katedra matematiky a deskriptívnej geometrie
Vysokej školy lesníckej a drevárskej
Zvolen

Katedra algebry a teórie čísel
Prírodovedeckej fakulty Univerzity Komenského
Bratislava