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CHAINS OF DECOMPOSITIONS AND n-ARY RELATIONS

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In some theorems of universal algebra (e. g. in the Schreier and the Jordan—Hölder theorems) the notion of a chain of congruences is used. The aim of this paper is to show how a chain of congruences of an algebra can be described by using an n-ary relation on the same algebra.

First we shall show the description of the finite chain of equivalences on a set by means of an n-ary relation.

Throughout the paper the following symbols are used: The letters x, y, z with and without indexes always stand for the elements of a set M. Decompositions and the corresponding equivalences are identified in the well-known way. The letter i denotes an element of the standard set $\{1, \ldots, n - 1\}$. If a is an arbitrary symbol, then a^i denotes the same as a.

Definition 1. *Let R be an n-ary relation on a set M.*

R is n-reflexive if $(x, \ldots, x)R$ holds for every x and $(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)R$ implies $(x_1, \ldots, x_{i-1}, x_{i+1}, x_{i+1}, \ldots, x_n)R$.

R is n-symmetric if $(x_1, \ldots, x_{i-1}, x, y, \ldots, y)R$ implies $(x_1, \ldots, x_{i-1}, y, x, \ldots, x)R$.

R is n-transitive if $(x_1, \ldots, x_{i-1}, x, y, \ldots, y)R$, $(x_1, \ldots, x_{i-1}, y, z, \ldots, z)R$ imply $(x_1, \ldots, x_{i-1}, x, z, \ldots, z)R$ and $(x_1, x_2, \ldots, x_2)R$, $(x_2, x_2, x_3, \ldots, x_3)R, \ldots$, $(x_{n-1}, \ldots, x_{n-1}, x_n)R$ imply $(x_1, \ldots, x_n)R$.

Definition 2. *A decomposition of degree n on a set M is a sequence of decompositions $R_1, \ldots, R_{n-1}$ on the set M with $R_1 \supset R_2 \supset \ldots \supset R_{n-1}$.*

Theorem 1. *Let $R_1, \ldots, R_{n-1}$ be a decomposition of degree n on a set M. Then the relation R defined by $(x_1, \ldots, x_n)R \Leftrightarrow x_iR_ix_{i+1}$ for each i is n-reflexive, n-symmetric and n-transitive. Conversely, let S be an n-reflexive, n-symmetric and n-transitive relation on a set M. Then $S_1, \ldots, S_{n-1}$ with $xS_iy \Leftrightarrow (x^1, \ldots, x^i, y, \ldots, y)\, S$ is a decomposition of degree n on a set M. Moreover, if $F(R_1, \ldots, R_{n-1})$ denotes the corresponding n-ary relation and $G(S)$ denotes the corresponding decomposition of degree n, then $G(F(R_1, \ldots, R_{n-1})) = R_1, \ldots, R_{n-1}$ and $F(G(S)) = S$.*

Proof. Let $R_1, \dots, R_{n-1}$ be a decomposition of degree n on a set M. Let R be defined as above. Since R_i is a decomposition, xR_ix holds for all i, x. The relation R can be readily verified to be n-reflexive. Suppose $(x_1, \dots, x_{i-1}, x, y, \dots, y)R$, this means $x_jR_jx_{j+1}$ for $j = 1, \dots, i-2$, $x_{i-1}R_{i-1}x$, xR_iy. Since $R_i \subset R_{i-1}$, $xR_{i-1}y$ holds. This and $x_{i-1}R_{i-1}x$ give $x_{i-1}R_{i-1}y$ by the transitivity of R_{i-1}. The symmetry of R_i gives yR_ix. Both shown for each i prove R to be n-symmetric. Next suppose $(x_1, \dots, x_{i-1}, x, y, \dots, y)R$, $(x_1, \dots, x_{i-1}, y, z, \dots, z)R$. Hence xR_iy, yR_iz, which implies xR_iz. This holds for each i. The second part of the definition of n-transitivity can be readily verified.

Conversely, let S be an n-ary relation on a set M satisfying the assumptions of the theorem and the S_i relations constructed as in the theorem. All the S_i are evidently reflexive. Suppose xS_iy, that is $(x^1, \dots, x^i, y, \dots, y)S$, by the n-symmetry $(x^1, \dots, x^{i-1}, y, x, \dots, x)S$ holds and the n-reflexivity follows $(y^1, \dots, y^i, x, \dots, x)S$. This means yS_ix, hence all relations S_i are symmetric. Suppose xS_iy, yS_iz, that is $(x^1, \dots, x^i, y, \dots, y)S$, $(y^1, \dots, y^i, z, \dots, z)S$. Since S_i have been shown symmetric, $(y^1, \dots, y^i, x, \dots, x)S$ holds. By the n-symmetry we get $(y^1, \dots, y^{i-1}, x, y, \dots, y)S$ and by the n-transivity $(y^1, \dots, y^{i-1}, x, z, \dots, z)S$. From the n-reflexivity we get $(x^1, \dots, x^i, z, \dots, z)S$, that means xS_iz. This shows S_i to be transitive. Now we shall prove $S_i \supset S_{i+1}$ for each i. Suppose $xS_{i+1}y$, hence $(x^1, \dots, x^i, x, y, \dots, y)S$, from this by the n-symmetry there follows that $(x^1, \dots, x^i, y, x, \dots, x)S$, by the n-transitivity $(x^1, \dots, x^i, y, \dots, y)S$, that is xS_iy. Hence $S_1, \dots, S_n$ is a decomposition of degree n on the set M.

Let $R_1, \dots, R_{n-1}$ be a decomposition of degree n and let $G(F(R_1, \dots, R_{n-1})) = S_1, \dots, S_{n-1}$. If xS_iy, then $(x^1, \dots, x^i, y, \dots, y)F(R_1, \dots, R_n)$ and so xR_iy for each i. If xR_iy, then $(x^1, \dots, x^i, y, \dots, y)F(R_1, \dots, R_{n-1})$ and xS_iy. Let S be an n-reflexive, n-symmetric and n-transitive relation and let $F(G(S)) - R$. If $(x_1, \dots, x_n)R$ then $x_iG_i(S)x_{i+1}$ for each i where $G(S) = G_1(S), \dots, G_{n-1}(S)$. It follows that $(x_i^1, x_i^2, \dots, x_i^i, x_{i+1}, \dots, x_{i+1})S$ for each i and $(x_1, \dots, x_n)S$. Similarly we get that if $(x_1, \dots, x_n)S$, then $(x_1, \dots, x_n)R$. This completes the proof of the theorem.

Now we shall describe the chain of the congruences of the algebra by means of the n-ary relation. Let M be an algebra.

Definition 3. *The n-ary relation R on the algebra M is said to be compatible with an m-ary operation f if $(x_1, \dots, x_{i-1}, y_1, x_{i+1}, \dots, x_n)R$, $(x_1, \dots, x_{i-1}, y_2, x_{i+1}, \dots, x_n)R$, $\dots$, $(x_1, \dots, x_{i-1}, y_m, x_{i+1}, \dots, x_n)R$ imply $(x_1, \dots, x_{i-1}, f(y_1, \dots, y_m), x_{i+1}, \dots, x_n)R$ for all i.*

Theorem 2. *Let $R_1, \dots, R_{n-1}$ be a non-ascending chain of congruences on the algebra M. Let R be the n-ary relation defined as $(x_1, \dots, x_n)R \Leftrightarrow x_iR_ix_{i+1}$ for all i. Then R is n-reflexive, n-symmetric, n-transitive and compatible with* *all operations. Conversely, let S be an n-reflexive, n-symmetric and n-transitive relation on M, compatible with all operations. Then $G(S)$ is a non-ascending chain of congruences on M.*

Proof. To prove the first part of the theorem it is sufficient to show that R is compatible with all operations. The rest follows by Theorem 1. Let f be an m-ary operation on M and let $(x_1, \ldots, x_{i-1}, y_1, x_{i+1}, \ldots, x_n)R$, $(x_1, \ldots, x_{i-1}, y_2, x_{i+1}, \ldots, x_n)R$, $\ldots$, $(x_1, \ldots, x_{i-1}, y_m, x_{i+1}, \ldots, x_n)R$. Then $x_{i-1}R_{i-1}y_1$, $x_{i-1}R_{i-1}y_2, \ldots, x_{i-1}R_{i-1}y_m$ hold. Since R_{i-1} is a congruence, $x_{i-1}R_{i-1}f(y_1, \ldots, y_m)$ holds. Similar arguments prove $f(y_1, \ldots, y_m)R_i x_{i+1}$. Clearly $x_jR_jx_{j+1}$ for all $j \neq i-1$, i thus $(x_1, \ldots, x_{i-1}, f(y_1, \ldots, y_m), x_{i+1}, \ldots, x_n)R$. It shows the compatibility of R with all operations. To prove the second part, let S be a relation as it is assumed in the theorem. Let $xG_i(S)y_1, \ldots, xG_i(S)y_m$, that means $(x^1, \ldots, x^i, y_1, \ldots, y_1)S, \ldots, (x^1, \ldots, x^i, y_m, \ldots, y_m)S$. The n-symmetry follows $(x^1, \ldots, x^{i-1}, y_1, x, \ldots, x)S, \ldots, (x^1, \ldots, x^{i-1}, y_m, x, \ldots, x)S$. The compatibility of S gives $(x^1, \ldots, x^{i-1}, f(y_1, \ldots, y_m), x, \ldots, x)S$. Using the n-symmetry we get $(x^1, \ldots, x^i, f(y_1, \ldots, y_m), \ldots, f(y_1, \ldots, y_m))S$, that is $xG_i(S)f(y_1, \ldots, y_m)$. The theorem is proved.

The Schreier and the Jordan—Hölder theorems use the notion of a refinement of a chain of congruences. We shall give the definition of this notion in terms of n-ary relations.

Definition 4. *Let R be a decomposition of degree n on an algebra M. If all $G_i(R)$ are congruences, R is said to be a congruence of degree n.*

Definition 5. *Let $1 \leqslant n_1 < n_2 < \ldots < n_{k-1} \leqslant n$. A congruence R of degree n on an algebra M is called the $n_1, \ldots, n_{k-1}$ — refinement of a congruence S of degree k on M if $(x_1, \ldots, x_n)R$ implies $(x_{n_1}, x_{n_2}, \ldots, x_{n_{k-1}}, x_{n_{k-1}+1})S$ and $(x_1, \ldots, x_k)S$ implies $(x_1^1, \ldots, x_1^{n_1}, x_2^{n_1+1}, \ldots, x_2^{n_2}, \ldots, x_k^{n_{k-1}+1}, \ldots, x_k^n)R$.*

Theorem 3. *A congruence R of degree n on an algebra M is the $n_1, \ldots, n_{k-1}$-refinement of a congruence S of degree k on M if and only if $G_1(S) = G_{n_1}(R)$, $G_2(S) = G_{n_2}(R), \ldots, G_{k-1}(S) = G_{n_{k-1}}(R)$.*

Proof. We shall write R_i instead of $G_i(R)$ and S_i instead of $G_i(S)$. To prove the necessity, let xS_jy, that is $(x^1, \ldots, x^j, y, \ldots, y)S$. Because R is the $n_1, \ldots, n_{k-1}$-refinement of S, $(x^1, \ldots, x^{n_j}, y, \ldots, y)R$ holds, thus $xR_{n_j}y$. Let $xR_{n_j}y$, that is $(x^1, \ldots, x^{n_j}, y, \ldots, y)R$. Because R is the $n_1, \ldots, n_{k-1}$-refinement of S, $(x^{n_1}, x^{n_2}, \ldots, x^{n_j}, y, \ldots, y)S$ holds, thus xS_jy. To prove the sufficiency, let $S_1 = R_{n_1}$, $S_2 = R_{n_2}, \ldots, S_{k-1} = R_{n_{k-1}}$. If $(x_1, \ldots, x_n)R$ holds, it means that $x_1R_1x_2, x_2R_2x_3, \ldots, x_{n-1}R_{n-1}x_n$. From $R_1 \supset R_2 \supset \ldots \supset R_{n-1}$ it follows $x_{n_1}R_{n_1}x_{n_2}$, $x_{n_2}R_{n_2}x_{n_3}$, $\ldots$, $x_{n_{k-1}}R_{n_{k-1}}x_{n_{k-1}+1}$, that gives $x_{n_1}S_1x_{n_2}$, $x_{n_2}S_2x_{n_3}$, $\ldots$, $x_{n_{k-2}}S_{k-2}x_{n_{k-1}}$, $x_{n_{k-1}}S_{k-1}x_{n_{k-1}+1}$, and so $(x_{n_1}, x_{n_2}, \ldots, x_{n_{k-1}}, x_{n_{k-1}+1})S$. Conversely, $(x_1, \ldots, x_k)S$ means $x_1S_1x_2, \ldots, x_{k-1}S_{k-1}x_k$. From the assumption it follows $x_1R_{n_1}x_2, \dots, x_{k-1}R_{n_{k-1}}x_k$. This by the reflexivity gives $x_1R_1x_1, \dots,$ $x_1R_{n_1-1}x_1$, $x_1R_{n_1}x_2$, $x_2R_{n_1+1}x_2$, $\dots$, $x_2R_{n_2-1}x_2$, $x_2R_{n_2}x_3$, $\dots$, $x_{k-1}R_{n_{k-1}}x_k$, $x_kR_{n_{k-1}+1}x_k, \dots, x_kR_{n-1}x_k$, which implies $(x_1^1, \dots, x_1^{n_1}, x_2^{n_1+1}, \dots, x_2^{n_2}, \dots, x_k^{n_{k-1}+1}, \dots, x_k^n)R$. The theorem is proved.

Definition 6. *Let R be a congruence of degree n on an algebra M and let $e \in M$. Then we denote*

$e_i(R) = \{x \mid$ *there are elements* $x_{i+1}, x_{i+2}, \dots, x_{n-1}$ *such that*
$(x^1, \dots, x^i, x_{i+1}, x_{i+2}, \dots, x_{n-1}, e)R$ *for all* $i\}$.

Finally we formulate the Schreier and the Jordan—Hölder theorems.

The Schreier theorem. *Let M be any algebra with a one-element subalgebra $\{e\}$ and permutable congruences. Let R and S be congruences on M of degrees n and m, respectively such that $G_1(R) = G_1(S) = I$, $G_{n-1}(R) = G_{m-1}(S) = O$. Then there exist congruences R' and S' on M of degree $(n-1)(m-1)+1$ such that R' is the $1, m, 2m-1, 3m-2, \dots, (n-2)m-n+3$-refinement of R, S' is the $1, n, 2n-1, 3n-2, \dots, (m-2)n-m+3$-refinement of S and $e_j(R)/R_{j+1}$ are pairwise isomorphic with $e_k(S)/S_{k+1}$ for $j, k = 1, 2, \dots, (n-1)(m-1)$.*

The Jordan—Hölder theorem. *Let M be any algebra with a one-element subalgebra $\{e\}$ and permutable congruences. Let R, S be unrefinable congruences on M of degrees n and m, respectively, such that $G_1(R) = G_1(S) = I$, $G_{n-1}(R) = G_{m-1}(S) = O$. Then $m = n$ and $e_j(R)/R_{j+1}$ are pairwise isomorphic with $e_k(S)/S_{k+1}$ for $j, k = 1, 2, \dots, n-2$.*

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