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I. INTRODUCTION. The complete k-chromatic graph $(k; n_1, n_2, \ldots, n_k)$, where $k \geqq 2$, is defined as an undirected graph whose vertices can be decomposed into k disjoint classes with $n_1, n_2, \ldots, n_k$ vertices such that any two vertices are joined by an edge if and only if they belong to different classes.

If we draw in the plane in the usual manner any nonplanar graph, that means we map vertices as points and edges as arcs which are homeomorphic with a straight segment, then necessarily certain two edges cross. The minimal number of crossings is called the crossing number of this graph. (See e. g. [1].)

The crossing number of graphs $(k; 1, 1, \ldots, 1)$ and $(2; n_1, n_2)$ is investigated in [1] to [8]. For the general complete k-chromatic graphs $(k; n_1, n_2, \ldots, n_k)$ the crossing number is studied in [9] to [12]. In [10] and [11] there is proved:

Theorem 1. Suppositions and denotations. *Let $(k; n_1, n_2, \ldots, n_k)$ be a complete k-chromatic graph and*

$$h(k; n_1, n_2, \ldots, n_k) =$$

$$= \sum_{i=1}^{k} \left[\frac{n_i}{2}\right]\left[\frac{n_i - 1}{2}\right]\left[\frac{N_i}{2}\right]\left[\frac{N_i - 1}{2}\right] - \sum_{\substack{i,j=1 \\ i>j}}^{k} \left[\frac{n_i}{2}\right]\left[\frac{n_i - 1}{2}\right]\left[\frac{n_j}{2}\right]\left[\frac{n_j - 1}{2}\right] +$$

$$+ \sum_{\substack{1 \leqq r < s < \\ < t < u \leqq k}} [a_r a_s (a_t a_u + b_t b_u) + a_u b_s (a_r b_t + b_r a_t) + b_r b_u (b_s b_t + a_s a_t)] ,$$

where

$$N_i = \sum_{\substack{j=1 \\ j \neq i}}^{k} n_j \qquad (i = 1, 2, \ldots, k)$$

and where a_i, b_i $(i = 1, 2, \ldots, k)$ *are non-negative integers such that*

$$a_i + b_i = n_i, \quad 0 \leqq (a_i - b_i)(-1)^{n_1+n_2+\ldots+n_i} \leqq 1.$$

Assertion. *The value* $h(k; n_1, n_2, \ldots, n_k)$ *is independent of the order of the numbers* $n_1, n_2, \ldots, n_k$ *and it is an upper bound for the crossing number* $p(k; n_1, n_2, \ldots, n_k)$ *of the graph* $(k; n_1, n_2, \ldots, n_k)$.

Now let us investigate this **problem:**

For given natural numbers N, k $(N \geqq k \geqq 2)$ the maximal and minimal crossing numbers

$$P(N, k) = \max_{R(N,k)} p(k; n_1, n_2, \ldots, n_k),$$

$$p(N, k) = \min_{R(N,k)} p(k; n_1, n_2, \ldots, n_k),$$

where $R(N, k)$ denote the set of all decompositions of the number N into k natural summands $n_1, n_2, \ldots, n_k$, should be estimated.

By means of statements from Theorem 1 we obtain in this article upper bounds for the numbers $P(N, k)$ and $p(N, k)$. The main results will be presented in Theorems 2 and 4.

II. UPPER BOUND FOR MAXIMAL CROSSING NUMBER $P(N, k)$. We shall investigate for all natural numbers N, k $(N \geqq k \geqq 2)$ the maximum

$$H(N, k) = \max_{R(N,k)} h(k; n_1, n_2, \ldots, n_k).$$

This function gives an upper estimate for the maximal crossing number $P(N, k)$, because obviously $P(N, k) \leqq H(N, k)$.

Let $(m_1, m_2, \ldots, m_k) \in R(N, k)$. For all different indices $i, j = 1, 2, \ldots, k$ we denote

$$S(m_i, m_j) = 8(N - m_{ij}) - \lambda(m_{ij}^2 - 4m_{ij} + 8) - 2m_{ij} \sum_{\substack{t=1 \\ t \neq i,j}}^{k} m_t \lambda(m_t),$$

where

$$m_{ij} = m_i + m_j, \quad \lambda = \frac{1}{2}[1 - (-1)^N], \ \lambda(m_t) = \frac{1}{2}[1 - (-1)^{N-m_t}].$$

Lemma 1. *Apart from the order of the summands, there are at least one, but at most three disjoint decompositions*

$$(m_1. m_2, \ldots, m_k) \in R(N, k),$$

which satisfy for all different indices $i, j = 1, 2, \ldots, k$ *the two conditions*:

(i) $$d(m_i, m_j) = |m_i - m_j| \leqq 2;$$

(ii) $$[d(m_i, m_j) - 1]\, S(m_i, m_j) \geqq 0, \quad \text{if } m_{ij} \equiv 0 \pmod 4.$$

Lemma 1 is a consequence of

Lemma 2. Suppositions and denotations: *Let N, k ($N \geqq k \geqq 2$) be given natural numbers and m, r such integers that*

$$N = km + r, \quad m \equiv 0 \pmod 2, \quad -k \leqq r < k.$$

We denote

$$a = \frac{1}{2}(k - r - b), \qquad c = \frac{1}{2}(k + r - b),$$

where the non-negative integer b is defined in Table 1.

Assertion: *The decomposition $(m_1, m_2, \ldots, m_k) \in R(N, k)$ satisfies the two conditions* (i) *and* (ii) *if and only if it has the form*

(1) $$\varrho(N, k) = (m_1, m_2, \ldots, m_k) = (m - 1)^a m^b (m + 1)^c \ (^1)$$

Proof. a) It is not difficult, but long enough, to show that in all cases we have

Table 1

N mod 2	w(*) mod m^2	$k + w/m^2$ mod 2	w, s, t(**)	b
1	0	1	—	$b = 2 + w/m^2$
1	0	0	$w \leqq 0$	$0 \leqq b \leqq 1,\ b \equiv k + 1 \pmod 2$
1	0	0	$w \geqq 0$	$b = k - \lvert r\rvert$
1	$\neq 0$	—	$w \leqq 0$	$0 \leqq b \leqq 1,\ b \equiv k + 1 \pmod 2$
1	$\neq 0$	—	$w \geqq 0$	$b = k - \lvert r\rvert$
0	—	—	$s \leqq k - \lvert r\rvert - 2,\ t \geqq 2$	$s \leqq b \leqq t,\ b \equiv k \pmod 2$
0	—	—	$s > k - \lvert r\rvert - 2$	$b = k - \lvert r\rvert$
0	—	—	$t < 2$	$0 \leqq b \leqq 1,\ b \equiv k \pmod 2$

(*) $w = 2N - 1 - (m + 1)^2$, (**) $s = \max\,[0; (Nm - 2N + 4m - 4m^2)/m^2]$, $t = \min\,[k - |r|; (Nm - 2N + 4m)/m^2]$

(1) The symbol $(m - 1)^a m^b (m + 1)^c$ denotes the decomposition in which there are a summands equal to $m - 1$, b summands equal to m and c summands equal to $m + 1$.

$$0 \leqq a, b, c \leqq k,$$
$$a + b + c = k,$$
$$a(m - 1) + bm + c(m + 1) = N.$$

Thus (1) is a decomposition of the number N into k natural summands.

b) Obviously the decomposition (1) satisfies condition (i). Conversely, if the decomposition $(m_1, m_2, \ldots, m_k) \in R(N, k)$ fulfills condition (i), then

$$\max [m_1, m_2, \ldots, m_k] - \min [m_1, m_2, \ldots, m_k] \leqq 2$$

and therefore it has the form (1), where a, b, c and $m \equiv 0 \pmod 2$ are suitable numbers.

Let

$$\min (a, c) \geqq 1, \quad b \geqq 2.$$

Then condition (ii) is fulfilled if and only if

$$\frac{1}{4} S(m, m) = (\lambda - 1)(Nm + m^2 + 2m - 2) + (w - 2m^2) + bm^2(-1)^N \leqq 0,$$

$$\frac{1}{4} S(m - 1, m + 1) = \frac{1}{4} S(m, m) - 4(\lambda - 1)m^2 \geqq 0.$$

Let

$$\min (a, c) < 1, \quad b \geqq 2.$$

Then condition (ii) is equivalent to

$$\frac{1}{4} S(m, m) \leqq 0.$$

Let

$$\min (a, c) \geqq 1, \quad b < 2.$$

In this case condition (ii) is equivalent to

$$\frac{1}{4} S(m - 1, m + 1) \geqq 0.$$

It is easy to verify that in all cases the number b, which is constructed in Table 1, satisfies even these inequalites.

c) It remains to prove: If the decomposition $(m_1, m_2, \ldots, m_k) \in R(N, k)$, for which conditions (i) and (ii) are fulfilled, has the form

$$(m_0 - 1)^{a_0} m_0^{b_0} (m_0 + 1)^{c_0},$$

where $m \neq m_0 \equiv 0 \pmod 2$, then

$$m_0 = m + 2, \quad b_0 = c_0 = a = b = 0, \quad a_0 = c = k \tag{2}$$

or

$$m_0 = m - 2, \quad a_0 = b_0 = b = c = 0, \quad c_0 = a = k. \tag{3}$$

Indeed if $m_0 < m$, then

$$N = a_0(m_0 - 1) + b_0 m_0 + c_0(m_0 + 1) \leqq k(m_0 + 1) \leqq$$
$$\leqq k(m - 1) \leqq a(m - 1) + bm + c(m + 1) = N$$

and hence we obtain (2). If $m_0 > m$, then

$$N = a_0(m_0 - 1) + b_0 m_0 + c_0(m_0 + 1) \geqq k(m_0 - 1) \geqq$$
$$\geqq k(m + 1) \geqq a(m - 1) + bm + c(m + 1) = N.$$

Hence we have (3).

Lemma 3. *Let*

$$(n_1, n_2, \ldots, n_k), \quad (n_1', n_2', \ldots, n_k') \in R(N, k)$$

be two decompositions, which are different only in two places i, j $(i, j = 1, 2, \ldots, k)$. *For these* i, j *let the numbers* n_i, n_j *fulfil the conditions* (i) *and* (ii). *Then we have*

$$D = h(k; n_1, n_2, \ldots, n_k) - h(k; n_1', n_2', \ldots, n_k') \geqq 0,$$

where the equality holds if and only if also the numbers n_i', n_j' *fulfil conditions* (i) *and* (ii).

Proof. We denote

$$\sum a_t a_u = AA, \qquad \sum b_t b_u = BB, \qquad \sum a_t b_u = AB,$$
$$\sum b_t a_u = BA, \qquad n_t = M,$$

where the indices t, u take on all values $1, 2, \ldots, k$ so that $t \neq i, j$, $u \neq i, j$, $t < u$ and where a_t, b_t are defined as in Theorem 1. Note that

$$AB - BA = \frac{1}{2} \sum_{\substack{t=1 \\ t \neq i,j}}^{k} n_t \lambda(n_t).$$

Hence we have

$$0 \leqq AB - BA \leqq \frac{1}{2} M.$$

Now we can find the difference D (see detailed enumerations in [10]):

I. Consider $|n_i - n_j| \leqq 1$. From this — according to condition (ii) — it follows: If $n_{ij} = n_i + n_j \equiv 0 \pmod 4$, then

$$S(n_i, n_j) \leqq 0 \tag{4}$$

1. Let

$$n_i = 2n + 1, \qquad n'_i = 2(n - x) + 1,$$

$$n_j = 2n, \qquad n'_j = 2(n + x),$$

where $-n < x \leqq n$, $x \neq 0$. Then

$$D =: D_1(x) = x^2[AB + (n - 1)M + n(n - \lambda)] +$$

$$+ x(x - 1)\left[BA + \left(n - \frac{1}{2}\right)(M + \lambda - 1) + n^2 - x^2\right].$$

2. Let

$$n_i = 2n - 1, \qquad n'_i = 2(n - x - 1) + 1,$$

$$n_j = 2n, \qquad n'_j = 2(n + x),$$

where $0 < |x| < n$. Then

$$D = D_2(x) = x^2[BA + (n - 1)(M + n - 1 - \lambda)] +$$

$$+ x(x + 1)\left[AB + \left(n - \frac{3}{2}\right)M + (n - 1)^2 + (1 - \lambda)\left(n - \frac{1}{2}\right) - x^2\right].$$

3. Let

$$n_i = 2n + 1, \qquad n'_i = 2(n + x + 1),$$

$$n_j = 2n + 1, \qquad n'_j = 2(n - x),$$

where $-n \leqq x < n$. Then

$$D = D_3(x) = \frac{1}{2}(AB + BA) - \frac{1}{2}(2n + 1)(AB - BA) + nM +$$

$$+ x(x + 1)[AB + BA + (2n - 1)M + 2n^2 - x(x + 1)] \geqq$$

$$\geqq \frac{1}{2}(AB + BA) + x(x + 1)[AB + BA + (2n - 1)M + 2n^2 - x(x + 1)].$$

4. Let

$$n_i = 2n + 1, \qquad n'_i = 2(n + x) + 1,$$

$$n_j = 2n + 1, \qquad n'_j = 2(n - x) + 1,$$

where $0 < |x| \leqq n$. Then

$$D = D_4(x) = x^2[AB + BA + (2n - 1)M + 2n^2 + \lambda - x^2].$$

5. Let

$$n_i = 2n, \qquad n'_i = 2(n + x),$$

$$n_j = 2n, \qquad n'_j = 2(n - x),$$

where $0 < |x| < n$. Then

$$D = D_5(x) = x^2[\lambda(AA + BB) + (1 - \lambda)(AB + BA) +$$
$$+ (n - 1)(2M + n - 1) + n^2 - x^2].$$

6. Let

$$n_i = 2n, \qquad n'_i = 2(n - x) + 1,$$

$$n_j = 2n, \qquad n'_j = 2(n + x - 1) + 1,$$

where $-n < x \leqq n$. Then

$$D = D_6(x) = 2(AB + BA) + n(AB - BA) + (8n - 9)M/8 +$$
$$+ 5\lambda\left[n(n - 1) + \frac{1}{2}\right] + 2[2n(n - 1) + \lambda - x(x - 1)] +$$
$$+ (x + 1)(x - 2)[AB + BA + 2(n - 1)(M + n) +$$
$$+ \lambda - x(x - 1)].$$

It is easy to see that for $j = 1, 2, \ldots, 5$ and all admissible x $D_j(x) > 0$. The inequality $D_6(x) > 0$ holds for all admissible $x \neq 0$. On the other hand $D_6(0) \geqq 0$ is fulfilled if and only if

$$\lambda(2n^2 - 2n + 1) + n \sum_{t=1}^{k} n_t \lambda(n_t) - M \geqq 0.$$

But this inequality is equivalent to $S(n_i, n_j) \leqq 0$. Furthermore the equality $D_6(0) = 0$ holds if and only if $S(n_i, n_j) = 0$.

II. Consider $|n_i - n_j| = 2$. Then $n_{ij} = n_i + n_j \equiv 0 \pmod 4$ and the inequality $S(n_i, n_j) \geqq 0$ must be true.

1. Let

$$n_i = 2n + 1, \qquad n'_i = 2(n + x),$$

$$n_j = 2n - 1, \qquad n'_j = 2(n - x),$$

where $|x| < n$. Then

$$D = D_7(x) = \begin{cases} D_5(x) - D_6(0) & \text{if } x \neq 0, \\ -D_6(0) & \text{if } x = 0. \end{cases}$$

2. Let

$$n_i = 2n + 1, \qquad n_i' = 2(n - x) + 1,$$

$$n_j = 2n - 1, \qquad n_j' = 2(n + x - 1) + 1,$$

where $-n < x \leqq n$, $x \neq 0, -1$. Then

$$D = D_8(x) = D_6(x) - D_6(0).$$

Also in these cases it is easy to see that for all admissible x is $D_7(x) \geqq 0$ and $D_8(x) > 0$ if and only if condition (ii) holds. Furthermore $D_7(0) = 0$ if and only if $S(n_i, n_j) = 0$.

From Lemmas 2, 3 we obtain immediately

Lemma 4. a) *If in a decomposition* $(m_1, m_2, \ldots, m_k) \in R(N, k)$ *there are such summands* m_i, m_j, *for which conditions* (i) *and* (ii) *are not fulfilled, then*

$$H(N, k) > h(k; m_1, m_2, \ldots, m_k).$$

b) *If* $(m_1, m_2, \ldots, m_k)$, $(m_1', m_2', \ldots, m_k') \in R(N, k)$ *are the decompositions satisfying conditions* (i), (ii), *then*

$$h(k; m_1, m_2, \ldots, m_k) = h(k; m_1', m_2', \ldots, m_k').$$

The following Theorem 2 is now obvious.

Theorem 2. *We have*

$$H(N, k) = h(k; m_1, m_2, \ldots, m_k)$$

if and only if $(m_1, m_2, \ldots, m_k)$ *is a decomposition*

$$\varrho(N, k) = (m - 1)^a m^b (m + 1)^c,$$

which is defined in Lemma 2.

Note 1. Theorem 2 gives an interesting result: The decomposition $\varrho(N, k)$, for which $h(k; n_1, n_2, \ldots, n_k)$, where $n_1 + n_2 + \ldots + n_k = N$, reaches its maximum $H(N, k)$, is asymmetrical. This probably also holds for the maximal crossing number $P(N, k)$. For example if $N = 6$, $k = 3$, we have $\varrho(6, 3) = (1, 2, 3)$ and

$$p(3; 1, 2, 3) = P(6, 3) = H(6, 3) = h(3; 1, 2, 3) = 1.$$

On the other hand for the symmetrical case we obtain

$$p(3;\, 2, 2, 2) = h(3;\, 2, 2, 2) = 0.$$

Note 2. As a consequence of Theorem 2 we have the following upper estimate:

$$P(N, k) \leqq H(N, k) \leqq h(k;\, m+1, m+1, \ldots, m+1).$$

If $b = k + r$ (we use the notation of Lemma 2), we obtain a better upper bound

$$P(N, k) \leqq H(N, k) \leqq h(k;\, m, m, \ldots, m).$$

Moreover if $b = k + r = 0$ (i. e. if $N = km - k$), then

$$P(N, k) \leqq H(N, k) \leqq h(k;\, m-1, m-1, \ldots, m-1).$$

In these cases we can obtain the number $h(k;\, n, n, \ldots, n)$ from Theorem 3, which is proved in [10]:

Theorem 3. *We have*

$$h(k;\, n, n, \ldots, n) = E(n)\left[nE(kn - n) - \binom{k}{2}E(n)\right] +$$

$$+ \frac{1}{2}F(k+3)F(k)E^2(n+1) + \frac{1}{4}E(k)E(k-2)[G^2(n) - 2E^2(n+1)] +$$

$$+ E(k+1)E(k-2)E(n+1)G(n),$$

where

$$E(p) = \left[\frac{p}{2}\right]\left[\frac{p-1}{2}\right], \quad F(p) = \left[\frac{p}{2}\right]\left[\frac{p-2}{2}\right], \quad G(p) = \left[\frac{p}{2}\right]^2 + \left[\frac{p+1}{2}\right]^2.$$

In the particular case for $n \equiv 0 \pmod 2$

$$h(k;\, n, n, \ldots, n) = E(n)\left[nE(kn - n) - \binom{k}{2}E(n)\right] + 3\binom{k}{4}n^4/8$$

holds.

III. UPPER BOUND FOR MINIMAL CROSSING NUMBER $p(N, k)$

We shall establish for the given natural numbers N, k ($N \geqq k \geqq 2$) the minimum

$$h(N, k) = \min_{R(N,k)} h(k;\, n_1, n_2, \ldots, n_k).$$

In this way we reach for the minimal crossing number $p(N, k)$ an upper bound, since certainly

$$p(N, k) \leqq h(N, k)$$

holds.

Theorem 4. *For all natural numbers N, k ($N \geqq k \geqq 2$) we have*

$$p(N, k) \leqq h(N, k) = h(k; 1, 1, \ldots, 1, N + 1 - k).$$

Proof. Let

$$(n_1, n_2, \ldots, n_k),\ (n_1', n_2', \ldots, n_k') \in R(N, k)$$

be decompositions, which differ at most in two places i, j. Let for these i, j

$$|n_i - n_j| \leqq 1.$$

We shall prove that the difference

$$D = h(k; n_1, n_2, \ldots, n_k) - h(k; n_1', n_2', \ldots, n_k')$$

reaches its maximum if and only if $n_i' = 1$ or $n_j' = 1$. But this follows from Table 2:

Table 2

$n_i + n_j$ (mod 4)	Maximum D
1	$\max D_1(x) = D_1(n)$
3	$\max D_2(x) = D_2(n - 1)$
2	$\max [\max D_3(x), \max D_4(x)] = D_4(n)$
0	$\max\ \max D_5(x), \max D_6(x)] = D_6(n)$

We can obtain the number $h(k; 1, 1, \ldots, 1, N + 1 - k)$ from Theorem 5, which has been proved in [10]:

Theorem 5. *We have*

$$h(k; 1, 1, \ldots, 1, n_k) = E(k - 1)\, E(n_k) + E(k - 2)] + a_k[T(k + 1) + 2T(k)] + \\ + b_k[T(k + 1) + T(k) + T(k - 1)],$$

where

$$E(p) = \left[\frac{p}{2}\right]\left[\frac{p - 1}{2}\right], \qquad T(p) = \binom{\left[\frac{p}{2}\right]}{3},$$

$$a_k + b_k = n_k, \quad 0 \leqq (-1)^k(a_k - b_k) \leqq 1.$$

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