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The Decomposition of a General Graph according to a Given Abelian Group

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Proof. First take the case when either $r = 0$, or $r = 1$. In this case $m \leqq \leq \frac{1}{2}n(n-1)$. According to [5] the complete graph with n edges is an R_H-graph. Thus construct the decomposition $\mathscr{R}'$ of the complete graph $\langle n \rangle$ according to the group H. Let $m' = \frac{1}{2}n(n-1) - m$. The number m' is evidently divisible by the number $o(H)$. Choose an edge h_1 of the graph $\langle n \rangle$ and from the graph $\langle n \rangle$ omit all edges $\alpha(h_1)$ for $\alpha \in M$. The number of such edges is evidently exactly $o(H)$. In the reverse case there would have to be $\alpha(h_1) = \beta(h_1)$ for some $\alpha \in H$ and $\beta \in H$, $\alpha \neq \beta$; therefore if the edge h_1 were contained in the graph $G'(\gamma)$ of the decomposition $\mathscr{R}'$, then the edge $\alpha(h_1) = \beta(h_1)$ would have to be contained at the same time in the graphs $G'(\alpha\gamma)$ and $G'(\beta\gamma)$, which must be edge-disjoint, because $\alpha \neq \beta$ implies $\alpha\gamma \neq \beta\gamma$. In the graph thus obtained, choose an edge h_2 and proceed in the same way. We shall do this on the whole $m'/o(H)$ times, whereby we omit exactly m' edges. The resulting graph G is the wanted graph; if we denote $G(\alpha) = G \cap G'(\alpha)$ for any $\alpha \in M$, we obtain the decomposition $\mathscr{R}$ of the graph G according to the group H.

Now let $r \geqq 2$ and let $n' = n - r$. If $m \leqq \frac{1}{2}n'(n'-1)$, we shall construct in the above dessribed way an R_H-graph with n' vertices and m edges and add r isolated vertices to it. Therefore let $m > \frac{1}{2}n'(n'-1)$ and let $m'' = = m - \frac{1}{2}n'(n'-1)$. The number m'' is evidently also divisible by the number $o(H)$. Construct the complete graph $\langle n' \rangle$ with n' vertices (which is an R_H-graph, see [5]) and add the vertices $u_1, \ldots, u_r$ to it. Take an arbitrary vertex v of the graph $\langle n' \rangle$ and join some vertex u_i $(1 \leqq i \leqq r)$ with all vertices $\alpha(v)$, where $\alpha \in H$. In this way $o(H)$ edges are created. If we do this $m''/o(H)$ times, we obtain the wanted graph, in which the vertices $u_1, \ldots, u_r$ are fixed. (No two of these vertices may be joined by an edge, thus we put the condition $m \leqq \leqq \frac{1}{2}[n(n-1) - r(r-1)]$, because $\frac{1}{2}r(r-1)$ is the number of all unordered pairs of these vertices.)

Now we shall consider an Abelian group of an even order. According to [1] any finite Abelian group is a direct product of cyclic groups, whose orders are powers of prime numbers (they are so-called primary cyclic groups). Therefore let us have a finite Abelian group H and let the mentioned primary cyclic groups be $H_1, \ldots, H_l$. If the order of the group H is even, the order of at least one of the groups $H_1, \ldots, H_k$ must be a power of two. Assume without the loss of generality that all such groups are $H_1, \ldots, H_l$, where $1 \leqq l \leqq k$. If a_i is a generator of the group H_i, $(1 \leqq i \leqq l)$ and its order is 2^q, where q is a positive integer, the element $a_i^{2^{q-1}}$ is evidently the unique element of the group H_i whose order is two. Each of the groups $H_1, \ldots, H_l$ contains therefore exactly one element of the order 2. If $b = \prod_{i=1}^{k} b_i \in H$, where $b_i \in H_i$ for $i = 1, \ldots, k$, then the order of the element b is the least common multiple of the orders of the elements b_i in H_i. The element b has the order 2 if and only if for all $i = 1, \ldots, k$ the element b_i either is a unit element of the group H_i, or has the order 2 in H_i, while all b_i cannot be at the same time unit elements. If $l < j \leqq k$, then according to the described facts the group H_j is of an odd order, therefore it does not contain any element of the order 2. Therefore any element of the group H of the order 2 can be expressed as $b = \prod_{i=1}^{l} b_i$, where for $1 \leqq i \leqq l$ the element b_i is either a unit element of the group H_i, or has the order 2 in H_i, while at least for one i the element b_i has the order 2. The number of such elements is exactly $2^l - 1$.

After this introductory consideration we shall prove the following theorem.

Theorem 3. *Let an Abelian group H of an even order $o(H)$ and two positive integers m and n be given. In expressing the group M as the direct product of primary cyclic groups let exactly l of these groups be of even orders. Let r be the remainder of the division of the number n by the number $o(H)$ and let $m \equiv 0 \pmod{o(H)}$, $m \leqq \frac{1}{2}[n(n - 2^l) - r(r - 1)]$. Then there exists an R_H-graph with n vertices and m edges.*

Proof. First take the case when $r = 0$ and therefore $m \leqq \frac{1}{2}n(n - 2^l)$. The number n is divisible by the number $o(H)$, thus take the vertex set U with n elements and decompose it into subsets (pairwise disjoint) $U_1, \ldots, U_p$, where $p = n/o(H)$; each of these sets contains $o(H)$ elements. For $1 \leqq i \leqq p$ assign to each element α of the group H a vertex $u_i(\alpha)$ of the set U_i. Now join by edges all pairs of vertices of U, except the pairs $\{u_i(\alpha), u_i(\alpha\beta)\}$, where $1 \leqq i \leqq p, \alpha \in H$, β is an element of the order 2 in H. The graph thus obtained will be denoted by G_0. Evidently for $\alpha_1 \neq \alpha_2$ we have $\alpha_1\beta \neq \alpha_2\beta$ and at the same time $(\alpha\beta)\beta = \alpha\beta^2 = \alpha\varepsilon = \alpha$. Therefore there exist exactly $o(H)/2$ pairwise different pairs $\{\alpha, \alpha\beta\}$ at the given β. As the group H contains $2^l - 1$ elements of the order 2, there exist exactly $(2^{l-1} - \frac{1}{2})o(H)$ different pairs $\{\alpha, \alpha\beta\}$, where $\alpha \in H$, β is an element of the group H of the order 2. The number of the pairs $\{u_i(\alpha), u_i(\alpha\beta)\}$ of the vertices of U_i assigned to those elements is therefore also $(2^{l-1} - \frac{1}{2})o(H)$, the total number of such pairs of elements in the whole the set U is $p(2^{l-1} - \frac{1}{2})o(H) = (2^{l-1} - \frac{1}{2})n$. The graph G_0 contains therefore $\frac{1}{2}n(n - 1) - (2^{l-1} - \frac{1}{2})n = \frac{1}{2}n(n - 2^l)$ edges. To prove that G_0 is an R_H-graph it is sufficient to prove that for an arbitrary edge h of the graph G_0 the inequality $\alpha \neq \beta$ implies $\alpha(h) \neq \beta(h)$. If the edge h joins the vertices $u_i(\gamma)$, $u_j(\delta)$, where $i \neq j$, we can have $\alpha(h) = \beta(h)$ only of $u_i(\alpha\gamma) = u_i(\beta\gamma)$, $u_j(\alpha\delta) = u_j(\beta\delta)$ and this (as the assigning of vertices of U_i or U_j to the elements of H is a one-to-one) holds only if $\alpha\gamma = \beta\gamma$, $\alpha\delta = \beta\delta$, which implies $\alpha = \beta$. If the edge h joins the vertices $u_i(\gamma)$, $u_i(\delta)$, then $\alpha(h) = \beta(h)$ holds, if either $u_i(\alpha\gamma) = u_i(\beta\gamma)$, $u_i(\alpha\delta) = u_i(\beta\delta)$ (which is a case quite analogous to the preceding), or $u_i(\alpha\gamma) = u_i(\beta\delta)$, $u_i(\alpha\delta) = u_i(\beta\gamma)$. In the latter case $\alpha\gamma = \beta\delta$,

$\alpha\delta = \beta\gamma$, which implies $\alpha\gamma(\alpha\delta)^{-1} = \beta\delta(\beta\gamma)^{-1}$, which means $\gamma\delta^{-1} = \gamma^{-1}\delta$. The element $\gamma^{-1}\delta$ is equal to its inverse element and its order is 2. The element δ is created by multiplying the element γ by the element $\gamma^{-1}\delta$, which is of the order 2, and therefore the pair $\{u_i(\gamma), u_i(\delta)\}$ is not joined by an edge in G_0, which is a contradiction. Therefore the graph G_0 is an R_H-graph. The following procedure of the proof is analogous to the proof of Theorem 2 with the difference that instead of $\langle n \rangle$ we take G_0.

Now we shall consider R_H-graphs with the minimal number of vertices and edges. If we omit trivial R_H-graphs not containing any edges, this means that $m = o(H)$ and therefore each of the graphs of the corresponding decomposition contains exactly one edge. Among such graphs we shall look for those which have the minimal number of vertices.

Theorem 4. *Let H be a primary cyclic group of the order $o(H) = p^r$, where p, r are positive integers, $p > 1$. Then every R_H-graph with $o(H)$ edges with the minimal number of vertices consists of p^s connected components, where $0 \leqq s < r$ and each of these components is a circuit with p^{r-s} vertices.*

Proof. It is well-known that all subgroups of a primary cyclic group form a chain (a totally ordered set) with respect to the inclusion. Let u be a vertex of the graph G which is not isolated, let H' be the group of mappings of H, in which the vertex u is fixed. Let the vertex u be joined by an edge h with a vertex v. The vertex v cannot be fixed in any non-identical mapping of H', because in this case there would be $\alpha(h) = h$, which is impossible. But if v is fixed in some non-identical mapping β of H, it is fixed in all mappings of the cyclic subgroup H'' of the group H generated by the element β. According to the above mentioned either $H'' \subset H'$, or $H' \subset H''$. In the first case the vertices u and v are fixed in all mappings of the group H'', therefore also in the mapping β, which is not identical, therefore $\beta(h) = h$, which is impossible. In the second case the vertices u and v are fixed in all mappings of H'; thus this group cannot contain non-identical mappings and $H' = \{\varepsilon\}$, where ε is the unit element of the group H. Therefore the vertex u is fixed only in the identical mapping and there exist $o(H)$ pairwise different vertices u, $\eta(u)$, $\eta^2(u)$, $\ldots$, $\eta^{p^r-1}(u)$. If the graph G contains the edge $h = u\eta^k(u)$, where $0 < k \leqq p^r - 1$, k is a positive integer and the element η^k has not the order 2 in the group H, it contains also all edges $\eta^i(h)$ for $i = 1, \ldots, p^r - 1$, that is the edges $\eta^i(u)\eta^{i+k}(u)$. Let G' be the graph generated by exactly all these edges. Let s be the greatest positive integer such that the number k is divisible by the number p^s. Then kp^{r-s} is divisible by the number $p^s p^{r-s} = p^r$ and therefore $\eta^{kp^{r-s}}(u) = u$. The edges $u\eta^k(u)$, $\eta^k(u)\eta^{2k}(u)$, $\ldots$, $\eta^{k(p^{r-s}-1)}(u)u$ form a circuit with p^{r-s} edges and therefore also with p^{r-s} vertices. The same consideration as for u can be applied to any other vertex;

thus each vertex of G is contained in a circuit of the graph G' with p^{r-s} vertices. It is easy to prove that any two such circuits are edge-disjoint. As the number of edges of the graph G is p^r, the number of such circuits must be p^s. It is easy to prove that the graph G' is an R_H-graph, thus it is identical with the graph G, about which we have assumed that it is minimal. The case $p = 2$, $s = r - 1$ is excluded, because the edge $u\eta^k(u)$ would be fixed in the mapping η^k.

Theorem 5. *Let H be an Abelian group which can be expressed as the direct product of primary cyclic groups $H_1, \ldots, H_k$, where $k \geqq 2$. Let $o(H), o(H_1), \ldots, o(H_k)$ be the orders of the groups $H, H_1, \ldots, H_k$, respectively. Let I be such a subset of the set $\{1, 2, \ldots, k\}$ that $|\prod_{i \in I} o(H_i) - \sqrt{o(H)}|$ is minimal and let $\prod_{i \in I} o(H_i) = t$. Then an R_H-graph G with $o(H)$ edges with the minimal number of vertices is the complete bipartite graph consisting of the set A containing t vertices and of the set B containing $o(H)/t$ vertices, while each vertex of A is joined in G with each vertex of B and each edge of the graph G joins a vertex of A with a vertex of B.*

Proof. Let a vertex u of the graph G be given and let H' be the group of mappings of H in which the vertex u is fixed. If the vertex u is joined by an edge with the vertex v, then the vertex v cannot be fixed in any non-identical mapping of H'; if H'' is the group of mappings of H in which the vertex v is fixed, then the groups H' and H'' have only the unit element in common. The product of H' and H'' is therefore a direct product. If G is to have the minimal number of vertices, H'' (or H' respectively) must be a maximal subgroup of the group H such that it has only the unit element in common with H' (or H'' respectively). Thus H is the direct product of H' and H''. Let H' be the direct product of primary cyclic groups $H'_1, \ldots, H'_l$ and let H'' be the direct product of primary cyclic groups $H'_{l+1}, \ldots, H'_m$. Then H is the direct product of the primary cyclic groups $H'_1, \ldots, H'_m$. Therefore k m and $H_i \cong H_{p(i)}$, where p is some permutation of the set $\{1, \ldots, k\}$. Thus H' is evidently isomorphic with the direct product of the groups H_i for $i \in I$, where I is some subset of the set $\{1, 2, \ldots, k\}$ and H'' is the direct product of the groups H_j for $j \in \{1, 2, \ldots, k\} - I$ (if $I = \emptyset$, or $\{1, 2, \ldots, k\} - I - \emptyset$, then instead of the direct product over this set we take only the unit element). Then the group H is the direct product of the groups H' and H''. The graph G contains the vertices $\alpha(u)$ for $\alpha \in H''$ (pairwise different) and the vertices $\beta(v)$ for $\beta \in H'$ (also pairwise different). As the vertices u, v are joined by the edge h, also the vertices $\alpha(u)$, $\beta(v)$ for each $\alpha \in H''$, $\beta \in H'$ are joined. We have namely $\alpha\beta(h) = \alpha\beta(uv) = \alpha\beta(u)\alpha\beta(v) = \alpha(u)\beta(v)$, because $\beta(u) = u$, $\alpha(v) = v$. The number of these edges is $\prod_{i \in I} o(H_i) \prod_{j \in \{1,2,\ldots,k\} - I} o(H_j) = \prod_{l=1}^{k} o(H_l) = o(H)$. The number of the vertices of the graph G is $\prod_{i\in I} o(H_i) + \prod_{j\in\{1,2,\dots,k\}-I} o(H_j)$; this number has to be the minimal possible one. We know that the function $f(x) = x + n/x$ attains its minimum at $x = \sqrt{n}$. Thus the number $\prod_{i\in I} o(H_i)$ must differ as little as possible from $\sqrt{o(H)}$. If we consider different subsets J of the set $\{1, 2, \dots, k\}$, the numbers $\prod_{i\in J} o(H_i)$ are always positive integers. If $\sqrt{o(H)} \leqq \prod_{i\in J_1} o(H_i) < \prod_{i\in J_2} o(H_j)$ or $\prod_{i\in J_2} o(H_i) < \prod_{i\in J_1} o(H_i) \leqq \sqrt{o(H)}$, evidently $\prod_{i\in J_2} o(H_i) + \prod_{j\in\{1,2,\dots,k\}-J_2} o(H_i) \geqq \prod_{i\in J_1} o(H_i) + \prod_{j\in\{1,2,\dots,k\}-J_1} o(H_j)$. Thus, as $\prod_{i\in I} o(H_i) + \prod_{j\in\{1,2,\dots,k\}-I} o(H_j)$ must be minimal, we have either $\prod_{i\in I} o(H_i) \leqq o(H)$ and for no $J \subset \{1, 2, \dots, k\}, J \neq I$ we have $\prod_{i\in I} o(H_i) < \prod_{i\in J} o(H_i)$ or $\prod_{i\in I} o(H_i) \geqq o(H)$ and for no $J \subset \{1, 2, \dots, k\}$, $J \neq I$, we have $\prod_{i\in I} o(H_i) > \prod_{i\in J} o(H_i)$. But as $\prod_{j\in\{1,2,\dots,k\}-J} o(H_j) = o(H)[\prod_{i\in J} o(H_i)]^{-1}$ for each $J \subset \{1, 2, \dots, k\}$, we see that if I satisfies the first (or the second respectively) condition, the set $\{1, 2, \dots, k\} - I$ satisfies the second (or the first respectively) condition. Thus we need not distinguish, whether I satisfies the first or the second condition and may state that I is such a subset of $\{1, 2, \dots, k\}$ that $|\prod_{i\in I} o(H_i) - \sqrt{o(H)}|$ is minimal (in this case I evinently satisfies either the first, or the second condition).

Now let us have the edge $\alpha(u)\beta(v)$ of G, where $\alpha \in H''$, $\beta \in H'$ and two mappings $\varphi \in H$, $\psi \in H$, while $\varphi = \varphi'\varphi''$, $\psi = \psi'\psi''$, where $\varphi' \in H'$, $\varphi'' \in H''$, $\psi' \in H''$. Let the images of the edge $\alpha(u)\beta(v)$ in the mappings φ, ψ be identical. This means that there must be $\varphi\alpha(u) = \psi\alpha(u)$, $\varphi\beta(v) = \psi\beta(v)$, because a vertex of A (or of B, respectively) can be transformed by a mapping of H only into a vertex of A (or of B, respectively). Evidently $\varphi\alpha(u) = \varphi''\alpha(u)$, $\psi\alpha(u) = \psi''\alpha(u)$, while $\varphi''\alpha \in H''$, $\psi''\alpha \in H''$. The images of the vertex u in mappings of H'' are pairwise different, therefore there must be $\varphi''\alpha = \psi''\alpha$, which implies φ'' ψ Analogously $\varphi\beta(v) = \psi\beta(v)$ implies $\varphi' = \psi'$, therefore also $\varphi = \varphi'\varphi'' = \psi'\psi'' = \psi$. Thus it is proved that the images of the edge $\alpha(u)\beta(v)$ in different mappings of H are different; therefore any of such edges form together with all vertices of the graph G one graph of the decomposition of the graph G according to the group H.

Now we shall consider one generalization of the concept of the bipartite graph, namely so-called simplex-like graphs [2]. A simplex-like graph is by definition the graph whose vertex set can be decomposed into pairwise disjoint sets $U_1, \dots, U_k$, while exactly all such pairs of vertices are joined by edges that the vertices of the pair belong to different sets U_i ($i = 1, \dots, k$). They are the critical graphs with respect to the chromatic number.

Theorem 6. *Let H be a finite Abelian group which can be expressed as the direct product of the groups $H_1, \dots, H_k$ (not necessarily non-decomposable). Let $o(H_1), \dots, o(H_k)$ be the orders of the groups H, $H_1, \dots, H_k$ respectively. Let $P(x_1, \dots, x_k)$ (or $Q(x_1, \dots, x_k)$, respectively) be an elementary symmetric polynomial* [3] *of the degree $n - 1$ (or $n - 2$, respectively) of the undetermined $x_1, \dots, x_k$. Then there exists a simplex-like R_H-graph G, whose number of vertices is $P(o(H_1), \dots, o(H_k))$ and whose number of edges is $o(H)Q(o(H_1), \dots, o(H_k))$.*

Proof. We shall construct the graph G. Its vertex set is the union of pairwise disjoint sets $U_1, \dots, U_k$, while the set U_i contains $o(H)/o(H_i)$ vertices for each $i = 1, \dots, k$. The vertices of the set U_i $(i = 1, \dots, k)$ are denoted by $u_i(\alpha)$ for all α of the direct product $\bar{H}_i$ of the groups $H_1, \dots, H_{i-1}, H_{i+1}, \dots, H_k$, therefore of the group isomorphic to the factor-group H/H_i. Let $\beta \in H$ and let $\beta = \prod_{i=1}^{k} \beta_i$, where $\beta_i \in H_i$ for each $i = 1, \dots, k$. Then $\beta(u_i(\alpha)) = u_i(\bar{\beta}_i\alpha)$, where $\bar{\beta}_i = \beta\beta_i^{-1} \in H_i$. We see that the vertex $u_i(\alpha)$ is fixed exactly in all mappings of H_i. The proof that G is an R_H-graph is analogous to the proof of Theorem 5. We shall also compute easily the number of vertices and edges of the graph G. The number of vertices is $\sum_{i=1}^{k} o(H)/o(H_i) = P(o(H_1), \dots, o(H_k))$. The number of edges going from U_i into U_j is the product of the numbers of vertices of the sets U_i and U_j, i. e. $[o(H)]^2/o(H_i)o(H_j)$. The number of edges of the whole graph G is therefore $\sum_{j=1}^{k} \{\sum_{i=1}^{k} [o(H)]^2/o(H_i)o(H_j) - [o(H)]^2/[o(H_j)]^2\} =$

$$= o(H) \sum_{j=1}^{k} \{ \sum_{i=1}^{k} o(H)/o(H_i)o(H_j) - o(H)/[o(H_j)]^2\} = o(H)Q(o(H_1), \dots, o(H_k)).$$

Theorem 7. *Let there be given a vertex set U of the cardinality $4k$ or $4k + 1$, where k is a positive integer, and on it a permutation p^* such that the number of vertices of each of its cycles is divisible by four, with the exception of at most one cycle formed by a fixed vertex. Further let an Abelian group H, whose order $o(H)$ is odd and is a divisor of the number k, be given. Then there exists a self-complementary R_H-graph G, whose vertex set is U and the permutation p^* is induced by the isomorphic mapping g of the graph G onto its complement G.*

The proof is similar to the proof of Theorem 4 of [4]. Let the permutation p^* have no fixed vertex and let $\mathscr{C}_1, \dots, \mathscr{C}_q$ be its cycles. The cycle $\mathscr{C}_i$ $(i = 1, \dots, q)$ contains the vertices $u_1^{(i)}, \dots, u_{r_i}^{(i)}$. The numbers r_i are divisible by four for any $i = 1, \dots, q$. Let n be the number of vertices of the set U, evidently $n = \sum_{i=1}^{q} r_i$. For each vertex introduce another notation so that the vertex $u_j^{(i)}$ will be denoted by v_k, where for $j \leqq r_i/2$ we have $k = j + \frac{1}{2} \sum_{z<i} r_z$,

for $j > r_i/2$ we have $k = j + \frac{1}{2}(n - r_i + \sum_{z>i} r_z)$. It can be proved that if $v_k = u_j^{(i)}, v_{k'} = u_{j'}^{(i)}$, then $k \equiv j \pmod 4$ and therefore $k - k' \equiv j - j' \pmod 4$. Now we construct the graph G in the following way. By an edge we join any two vertices with odd subscripts at v. Further, for any even y we join the vertex v_y with all vertices whose subscripts at v are congruent with $y + 1$ modulo 4. We can easily verify that the graph thus constructed by an isomorphic mapping inducing the permutation p^*can be transformed onto its complement. Now let us decompose the set U into the sets $U_0, U_1, \ldots, U_{4k/o(H)-1}$ so that to the set U_i $(i = 0, 1, \ldots, 4k/o(H) - 1)$ exactly all such vertices v_l belong, for which $l \equiv i \pmod{4k/o(H)}$ holds. As $o(H)$ is a divisor of k, such sets are at least four and each of them contains exactly $o(H)$ elements. In a one-to-one manner assign to each element $\alpha \in H$ a vertex $w_0(\alpha)$ of U_0. Now if $v_l \in U_0$ and $v_l = w_0(\alpha)$, then for any $j = 1, \ldots, 4k/o(H) - 1$ denote $v_{l+j} = w_j(\alpha)$; this vertex evidently is contained in U_j. Now for any $\alpha \in H$, $\beta \in H$ and any $j = 0, 1, \ldots, 4k/o(H) - 1$ define $\beta(u_j(\alpha)) = u_j(\beta\alpha)$. The number $4k/o(H)$ is divisible by four, therefore the image of any vertex v_l in any mapping of H is a vertex whose subscript at v is congruent with l modulo 4. Thus the image of an arbitrary pair of vertices is joined by an edge if and only if the original pair is joined. We can also easily prove that for any edge h the equality $\gamma(h) = \delta(h)$ implies $\gamma = \delta$. Now if p^*has a fixed vertex x, we shall make the described construction for the set $U - \{x\}$ and for the restriction of the permutation p^* on $U - \{x\}$ .Then we join x with all vertices which have odd subscripts at v and define $\alpha(x) = x$ for all $\alpha \in H$.

§ 2.

In this paragraph we shall study directed graphs. The definition of the decomposition according to a given Abelian group is quite analogous. We have the same situation in the concept of a directed R_H-graph. By the term complete directed graph we shall mean the graph which with any two vertices u, v, contains both the edges $\overrightarrow{uv}$ and $\overrightarrow{vu}$.

We shall express one lemma and some theorems analogous to the lemmas and theorems of the first paragraph.

Lemma 3. *Let G be a directed R_H-graph. Let H'be a subgroup of the order greater than one of the group H, let u, v be two vertices of the graph G which are fixed in all mappings of H'. Then the vertices u, v are not joined by an edge in G.*

Proof. In the same way as in Lemma 1 we could prove that in the reverse case the mentioned edge would be fixed in all mappings, therefore also in non-identical mappings, of H' and would have to belong to more than one graph of the decomposition R, which is impossible.

The lemma analogous to Lemma 2 for directed graphs does not hold. If α is an involutory element of the group H and u a vertex of the directed R_H-graph G, then there may exist the edge $\overrightarrow{u\alpha(u)}$, because its image in the mapping α is the edge $\overrightarrow{\alpha(u)u}$, which is evidently different from $\overrightarrow{u\alpha(u)}$. Therefore the edge $\overrightarrow{u\alpha(u)}$ is not fixed in the mapping α.

Theorem 8. *Let G be a directed R_H-graph. Let m be the number of edges of the graph G, let $o(H)$ be the order of the group H. Then the number m is an integral multiple of the number $o(H)$.*

The proof is the same as the proof of Theorem 1.

Theorem 9. *Let an Abelian group H of the order $o(H)$ and further two positive integers m and n be given. Let $m \equiv 0 \pmod{o(H)}$, $m \leqq n(n-1) - r(r-1)$, where r is the remainder of the division of the number n by the number $o(H)$. Then there exists a directed R_H-graph with n vertices and m edges.*

The proof is analogous to the proof of Theorem 2. In the assumptions of the theorem there is the inequality $m \leqq n(n-1) - r(r-1)$ instead of the inequality $m \leqq \frac{1}{2}[n(n-1) - r(r-1)]$ from Theorem 2, because the number of edges of the complete directed graph with n vertices is $n(n-1)$, while the number of edges of the complete undirected graph with n vertices is $\frac{1}{2}n(n-1)$; analogously for the complete graph with r vertices. Further, the assumption that the order of the group H is odd is omitted. In the proof of Theorem 2 one starts from the complete undirected graph which according to [5] cannot be decomposed according to an Abelian group of an even order (for the sake of involutory elements — see Lemma 2). But in Theorem 9 we start from the complete directed graph which according to [6] can be decomposed according to an arbitrary Abelian group whose order is a divisor either of the number of vertices of that graph, or of this number minus one.

Theorem 10. *Let H be a primary cyclic group of the order p^r, where p, r are positive integers, $p > 1$. Then every directed R_H-graph with $o(H)$ edges with the minimal number of vertices consists of p^s connected components, $0 \leqq s < r$, and each of these components is a cycle with p^{r-s} vertices.*

Theorem 11. *Let H be an Abelian group which can be expressed as the direct product of cyclic primary groups $H_1, \ldots, H_k$, where $k \geqq 2$. Let $o(H), o(H_1), \ldots, o(H_k)$ be the orders of the groups $H, H_1, \ldots, H_k$ respectively. Let I be such a subset of the set $\{1, 2, \ldots, k\}$ that $|\prod_{i \in I} o(H_i) - \sqrt{o(H)}|$ is minimal and let $\prod_{i \in I} o(H_i) = t$. Then a directed R_H-graph G with $o(H)$ edges with the minimal number of vertices is the bipartite graph consisting of the set A with t vertices and of the set B with $o(H)/t$ vertices, while either each edge of this graph has its beginning vertex in A and its end vertex in B, or each edge of this graph has its beginning vertex in B and its end vertex in A, and the mentioned graph contains all such edges.*

These theorems are quite analogous to Theorems 4 and 5, also their proofs are completely analogous.

If we define a directed simplex-like graph as the graph whose vertex set can be decomposed into subsets $U_1, \ldots, U_k$ pairwise disjoint, in which from the vertex u into the vertex v a directed edge goes if and only if the vertices u and v belong to different U_i, we can express a theorem:

Theorem 12. *Let H be a finite Abelian group which can be expressed as the direct product of the groups $H_1, \ldots, H_k$ (not necessarily non-decomposable). Let $o(H), o(H_1), \ldots, o(H)$ be the orders of the groups $H, H_1, \ldots, H_k$, respectively. Let $P(x_1, \ldots, x_k)$ (or $Q(x_1, \ldots, x_k)$, respectively) be the elementary symmetric polynomial of the degree $n - 1$ (or $n - 2$ respectively) with the undetermined $x_1, \ldots, x_k$. Then there exists a directed simplex-like R_H-graph G whose number of vertices is $P(o(H_1), \ldots, o(H_k)$ and the number of edges $2l(H)Q(o(H_1), \ldots, o(H_k))$.*

The proof is analogous to the proof of Theorem 6. The number of edges is the doubled number of edges from Theorem 6, because each pair of vertices (which is joined) is joined by two directed edges.

Finally we shall express a theorem analogous to Theorem 7.

Theorem 13. *Let there be given a set U with n vertices and on it a permutation p^* such that the number of vertices of each of its cycles, with the exception of at most one cycle formed by a fixed vertex, is even. Further let an Abelian group be given, whose order $o(H)$ is a divisor of the number $\frac{1}{2}n$ in the case if n is even and of the number $\frac{1}{2}(n - 1)$ in the case if n is odd. Then there exists a directed self-complementary R_H-graph G whose vertex set is U and the permutation p^* is induced by an isomorphic mapping of the graph G onto its complement G.*

Proof. First assume that the permutation p^* has no fixed vertex and therefore it contains only cycles with even numbers of elements. Let $\mathscr{C}_1, \ldots, \mathscr{C}_q$ be these cycles. The cycle $\mathscr{C}_i$ $(i = 1, \ldots, q)$ contains the vertices $u_1^{(i)}, \ldots, u_{r_i}^{(i)}$. The numbers r_i are even for any $i = 1, \ldots, q$. Evidently $n = \sum_{i=1}^{q} r_i$. For each vertex we introduce, as in the proof of Theorem 7, another notation, so that the vertex $u_j^{(i)}$ will be denoted by v_k, where for $j \leqq r_i/2$ there is $k = j + \frac{1}{2} \sum_{z<i} r_z$, for $j > r_i/2$ there is $k = j + \frac{1}{2}(n - r_i + \sum_{z>i} r_z)$. It can be proved that if $v_k \quad u_j^{(i)}$, then $k \equiv j \pmod 2$. Now we construct a directed graph G so that we join by both edges (i. e. from the first vertex into the second and from the second vertex into the first) any two vertices whose subscripts at v are odd; further, from any vertex with an even subscript at v we put directed edges into all vertices with odd subscripts at v. We can again easily verify that the graph thus constructed can be transformed by an isomorphic mapping inducing the permutation p^* into its complement. Now decompose the set U