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ON THE GENERALIZED BANACH INDICATRIX

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In paper [7] the notion of the Banach indicatrix is generalized in the following natural way: Let X, Y be two sets, let f be a mapping from X to Y. If the set $f^{-1}(\{y\}) = \{x \in X; f(x) = y\}$ is finite, then $\tau_f(y)$ denotes the number of its elements, if $f^{-1}(\{y\})$ is infinite, then we put $\tau_f(y) = +\infty$. The so defined function $\tau_f(\tau_f : Y \to \{0, 1, \ldots, +\infty\})$ is called a (generalized) Banach indicatrix of the function f.

It follows from the results of paper [7] that, if I_0 is an interval (it may be $I_0 = (-\infty, +\infty) = E_1$) and $f : I_0 \to E_1$ is a Darboux function, then τ_f is a Borel measurable function in the second class (and so τ_f is a Lebesgue measurable function, too).

We shall give some further classes of functions $f : I_0 \to E_1$, for which $\tau_f(\tau_f : E_1 \to \{0, 1, \ldots, +\infty\})$ is Lebesgue measurable.

Theorem 1. *Let $f : I_0 \to E_1$ be a monotone function. Then τ_f is a function in the second Baire class.*

Proof. Let, e. g., f be a nondecreasing function on I_0 and $Y_0 = f(I_0)$. If x_0 is a discontinuity point of the function f and x_0 is an interior point (left-hand and right-hand endpoint, respectively) of the interval I_0, then for $y \neq f(x_0)$, $y \in (f(x_0 - 0), f(x_0 + 0))$ $(y \in (f(x_0), f(x_0 + 0))$ and $y \in (f(x_0 - 0), f(x_0))$, respectively) we have $\tau_f(y) = 0$. Further for each $y \in Y_0$ precisely one of the following possibilities holds:

1) There exists the only x such that $y = f(x)$;

2) The set $\{x \in I_0; f(x) = y\}$ is an interval.

Obviously the set of all such y for which 2) occurs is countable. Thus for all $y \in Y_0$ but the points of a countable set we have $\tau_f(y) = 1$ and Y_0 arises from E_1 by ommitting a countable set of intervals. From this the assertion of Theorem follows at once.

The following theorem gives a criterion for the continuity of monotone functions.

Theorem 2. *The monotone function $f : \langle a, b \rangle \to E_1$ is continuous if and only if*

$$\overset{b}{\underset{a}{V}}(f) = \int_{-\infty}^{\infty} \tau_f(y)\,\mathrm{d}y \qquad (*)$$

($\overset{b}{\underset{a}{V}}(f)$ *denotes the variation of the function* f).

Proof. 1) S. Banach proved that if f is a continuous function, then (*) holds (cf. [5], p. 246—248; [6], p. 374—375).

2) Let, e. g., f be discontinuous at a point $x_0 \in (a, b)$ and f be nondecreasing on $\langle a, b\rangle$. Then

$$Y_0 = f(\langle a, b\rangle) \subset \{ < f(a), f(b) > - (f(x_0 - 0), f(x_0 + 0))\} \cup \{f(x_0)\} = M.$$

Further $\tau_f(y) = 1$ almost everywhere on the set Y_0 (see the proof of Theorem 1) and $\tau_f(y) = 0$ for $y \notin Y_0$. Hence

$$\int_{-\infty}^{\infty} \tau_f(y)\,\mathrm{d}y = \int_{Y_0} \mathrm{d}y \leqq \int_M \mathrm{d}y = f(b) - f(a) - (f(x_0 + 0) - f(x_0 - 0)) <$$
$$< f(b) - f(a) = \overset{b}{\underset{a}{V}}(f).$$

In connection with Theorem 1 we shall prove the measurability of the function τ_f for functions f of a certain more extensive class which contains the class of monotone functions.

Theorem 3. *Let* $f: I_0 \to E_1$ *be a Baire function. Then* τ_f *is Lebesgue measurable.*

Proof. Let, e. g., $I_0 = \langle a, b\rangle$. We put

$$D_1^n = \left\langle a, a + \frac{b-a}{2^n}\right\rangle, \quad D_{i+1}^n = \left(a + i\,\frac{b-a}{2^n},\ a + (i+1)\frac{b-a}{2^n}\right\rangle$$
$$(i = 1, 2, \ldots, 2^n).$$

Further let $E_i^n = f(D_i^n)$ $(i = 1, 2, \ldots, 2^n)$. The sets E_i^n are analytic since D_i^n are Borel sets and f is a Baire function (cf. [3], p. 458, § 38. III. Th. 5). Hence E_i^n are Lebesgue measurable. Put

$$\chi_{n,i}(y) = \begin{cases} 1 & \text{if } y \in E_i^n, \\ 0 & \text{if } y \notin E_i^n \quad (i = 1, 2, \ldots, 2^n) \end{cases}$$

and $L_n(y) = \sum_{i=1}^{2^n} \chi_{n,i}(y)$ $(n = 1, 2, \ldots)$. Then the functions L_n are Lebesgue measurable, the sequence $\{L_n(y)\}_{n=1}^{\infty}$ is nondecreasing and $\tau_f(y) = \lim_{n\to\infty} L_n(y)$. Hence τ_f is Lebesgue measurable.

The assertion of Theorem 3 cannot be improved even if the function f is supposed to have only countable many points of discontinuity. This shows the following

Theorem 3'. *There exists a function $f: I_0 \to E_1$ with the following propertise:*
i) *the set of discontinuity points of the function f is countable,*
ii) *τ_f is not a Baire function.*

Proof. Let E be an analytic set which is not a Borel set. It is well-known (cf. [8], p. 78—80) that there exists a function f such that $f(I_0) = E$ and the set of all discontinuity points of the function f is countable. According to Theorem 3 τ_f is Lebesgue measurable. Now we have $\{y; \tau_f(y) > 0\} = E$. Hence τ_f is not a Baire function.

In connection with the result of paper [7] quoted at the beginning of this paper the question arises whether to each ordinal number α, $0 \leqq \alpha < \Omega$ (Ω denotes the first uncountable ordinal number) there exists such a function $f: E_1 \to E_1$ that τ_f belongs precisely to the Baire class α. The following theorem gives a positive answer to this question.

Denote by $\mathbf{B}_\xi$ $(0 \leqq \xi < \Omega)$ the class of all functions $g: E_1 \to E_1^*$ $(E_1^* = \langle -\infty, +\infty \rangle)$ belonging to the Baire class ξ. Put $\mathbf{C}_0 = \mathbf{B}_0$, $\mathbf{C}_\xi = \mathbf{B}_\xi - \bigcup_{\eta<\xi} \mathbf{B}_\eta$ $(1 \leqq \xi < \Omega)$.

Theorem 4. a) *There exists for each γ, $0 \leqq \gamma < \Omega$ a Borel measurable function $f: E_1 \to E_1$ in the class γ such that $\tau_f \in \mathbf{C}_\gamma$.*

b) *There exists a function $f: E_1 \to E_1$ such that τ_f is a Lebesgue but not a Borel measurable function.*

We prove first the following auxiliary result.

Lemma 1. *Let $\mathbf{A}_\alpha$ and $\mathbf{M}_\alpha$, respectively, denote the system of all Borel subsets of E_1 belonging to the additive and multiplicative, respectively, class α, $0 \leqq \alpha < \Omega$. Then for each γ, $1 \leqq \gamma < \Omega$ there exists a set $E \in \mathbf{A}_p \cap \mathbf{M}_p$ such that $E \notin \bigcup_{\beta<\gamma} (\mathbf{A}_\beta \cup \mathbf{M}_\beta)$.*

Proof of Lemma. It is well known that for each α there exists a set $H \subset E_1$ such that $H \notin \mathbf{A}_\alpha$, $H \in \mathbf{M}_\alpha - \bigcup_{\beta<\alpha} (\mathbf{A}_\beta \cup \mathbf{M}_\beta)$ (cf. [6], p. 196). If we put $H' = E_1 - H$, then $H' \notin \mathbf{M}_\alpha$, $H' \in \mathbf{A}_\alpha - \bigcup_{\beta<\alpha} (\mathbf{A}_\beta \cup \mathbf{M}_\beta)$. There exists an interval (a, b) $(a < b)$ such that

$$H \cap (a, b) \in \mathbf{M}_\alpha - \bigcup_{\beta<\alpha} (\mathbf{A}_\beta \cup \mathbf{M}_\beta),\ (a, b) - H \in \mathbf{A}_\alpha - \bigcup_{\beta<\alpha} (\mathbf{A}_\beta \cup \mathbf{M}_\beta).$$

Without loss of generality it can be supposed that $a = 0$, $b = \frac{1}{2}$. Put $H_{\alpha+1} = [H \cap (0, \frac{1}{2})] \cup \{[(0, 1) - H] + \frac{1}{2}\}$ ($M + \frac{1}{2}$ denotes the set which arises through the translation of the set M by $\frac{1}{2}$). Then $H_{\alpha+1} \in \mathbf{A}_{\alpha+1} \cap \mathbf{M}_{\alpha+1}$, $H_{\alpha+1} \notin \bigcup_{\beta<\alpha+1} (\mathbf{A}_\beta \cup \mathbf{M}_\beta)$. The same is true also for $H'_{\alpha+1}$. Let γ be an ordinal number, $1 \leqq \gamma < \Omega$. There are two possibilities: 1) γ is an isolated number,

2) γ is a limit number. In case 1) we put $\gamma - 1 = \alpha$, thus $\gamma = \alpha + 1$. From the foregoing the existence of such a set $F_p(= H_{\alpha+1})$ follows that $F_\gamma \subset (0, 1)$, $F_\gamma \in \mathbf{A}_\gamma \cap \mathbf{M}_\gamma$, $F_\gamma \notin \bigcup_{\beta<\gamma} (\mathbf{A}_\beta \cup \mathbf{M}_\beta)$. Without loss of generality we can take $(n - 1, n)$ $(n \geqq 2)$ instead of $(0, 1)$.

In case 2) denote by Γ the set of all isolated ordinal numbers $\alpha < \gamma$. Then Γ is a countable set, $\Gamma = \{\alpha_1, \alpha_2, \ldots\}$. It follows from the foregoing that there exists an F_{α_n} $(n = 1, 2, \ldots)$ such that

$$F_{\alpha_n} \subset (n - 1, n), F_{\alpha_n} \in \mathbf{A}_{\alpha_n} \cap \mathbf{M}_{\alpha_n}, \; F_{\alpha_n} \notin \bigcup_{\beta<\alpha_n} (\mathbf{A}_\beta \cup \mathbf{M}_\beta).$$

Put $F_\gamma = \bigcup_{n=1}^{\infty} F_{\alpha_n}$. Then $F_\gamma \in \mathbf{A}_\gamma$, $F_\gamma \subset (0, +\infty)$. Further $F_\gamma = \bigcap_{n=1}^{\infty} \{[F_\gamma \cap \cap (0, n)] \cup (n, +\infty)\}$. From the definition of F_γ we obtain $F_\gamma \cap (0, n) \in \mathbf{M}_\tau$, where $\tau = \max(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and obviously $\mathbf{M}_\tau \subset \mathbf{M}_\gamma$. Hence $F_\gamma \in \mathbf{M}_\gamma$, $F_\gamma \in \mathbf{A}_\gamma \cap \mathbf{M}_\gamma$. But $F_\gamma \notin \bigcup_{\beta<\gamma} (\mathbf{A}_\beta \cup \mathbf{M}_\beta)$ since in the reverse case we have $F_\gamma \in \mathbf{A}_\beta \cup \mathbf{M}_\beta$ for a suitable $\beta < \gamma$. Then there exists an $\alpha_n, \beta < \alpha_n < \gamma$ such that $F_{\alpha_n} = F_\gamma \cap (n - 1, n) \in \mathbf{A}_\beta \cup \mathbf{M}_\beta$ and this contradicts the properties of the set F_{α_n}.

Putting in both cases 1), 2) $E = F_\gamma$, we see that E has the required properties.

Proof of Theorem 4. a) Put $f(x) = x$ for $x \in E_1$. Then f is continuous and $\tau_f(y) = 1$ for each $y \in E_1$. Hence $\tau_f \in \mathbf{C}_0$.

Let $1 \leqq \gamma < \Omega$. Choose $E \subset E_1$ such that $E \in \mathbf{A}_\gamma \cap \mathbf{M}_\gamma, E \notin \bigcup_{\beta<\gamma} (\mathbf{A}_\beta \cup \mathbf{M}_\beta)$. Such a set exists on account of Lemma 1. Let $t_0 \in E$. Put $f(x) = x$ for $x \in E$ and $f(x) = t_0$ for $x \in E_1 - E$. Then f is a Borel measurable function in the class γ. Further we have $\tau_f(y) = 1$ for $y \in E - \{t_0\}$, $\tau_f(t_0) = +\infty$ and $\tau_f(y) = 0$ for $y \in E_1 - E$. We show that for each set $G \subset E_1^*$ open in E_1^* the set $\tau_f^{-1}(G)$ is a Borel set in the class γ. It suffices to take for G the sets of the following form:

1) $G = (b, +\infty\rangle$, 2) $G = \langle-\infty, a)$,

3) $G = (b, +\infty)$, 4) $G = (-\infty, a)$.

$$1) \; \{y; \tau_f(y) > b\} = \begin{cases} \{t_0\} & \text{for} \quad b \geqq 1, \\ E & \text{for} \quad 0 \leqq b < 1, \\ E_1 & \text{for} \quad b < 0; \end{cases}$$

$$2) \; \{y; \tau_f(y) < a\} = \begin{cases} \emptyset & \text{for} \quad a \leqq 0, \\ E_1 - E & \text{for} \quad 0 < a \leqq 1, \\ E_1 - \{t_0\} & \text{for} \quad 1 < a; \end{cases}$$

3) $\{y; b < \tau_f(y) < +\infty\} = \begin{cases} E_1 - \{t_0\} & \text{for } b < 0, \\ E - \{t_0\} & \text{for } 0 \leqq b < 1, \\ \emptyset & \text{for } 1 \leqq b; \end{cases}$

4) we proceed in the same way as in case 2).

From these facts it is obvious that $\tau_f \in \mathbf{B}_\gamma$ and since $\{y; \tau_f(y) > 0\} = E$, we have $\tau_f \in \mathbf{C}_\gamma$.

b) The assertion follows at once from Theorems 3 and 3′.

We put in the sequel $\mathbf{T} = Z^{E_1}$ (= the system of all functions $g : E_1 \to Z$, $Z = \{0, 1, 2, \ldots, +\infty\}$). In connection with Theorem 4 a) we prove the following result.

Theorem 5. *The functions $g(y) \equiv 1$ and $g(y) \equiv m$ $(m \geqq 3)$ are the only continuous functions from $\mathbf{T}$ which are Banach indicatrices of some continuous functions $f : E_1 \to E_1$.*

Remark 1. It will be shown that the function $g(y) \equiv 2$ is a Banach indicatrix of a function $f : E_1 \to E_1$ (see Theorem 6 b)) which on account of the foregoing theorem cannot be continuous on E_1.

Proof of Theorem 5. Put $f_1(x) = x$ and $f_2(x) = x \sin x$ for $x \in E_1$. Obviously $\tau_{f_1}(y) \equiv 1$ and $\tau_{f_2}(y) \equiv +\infty$.

Further it is well-known (cf. [4]) that for each natural number $m \geqq 3$ there exists a continuous function $f : E_1 \to E_1$ such that for each $y \in E_1$ the set $\{x \in E_1; f(x) = y\}$ consists of precisely m points. It is also well-known (cf. [1], [2], [4]) that there exists no continuous function $f : E_1 \to E_1$ for which the set $\{x \in E_1; f(x) = y\}$ would consist of precisely two points for each $y \in E_1$. This completes the proof.

Denote by $\mathbf{S}(\mathbf{S}_0)$ the class of all Banach indicatrices of real functions defined on arbitrary non-void sets (defined on E_1). Then $\mathbf{S}$, $\mathbf{S}_0$ are subsets of the set $\mathbf{T}$. We shall investigate the structure of the set $\mathbf{T}$ from the point of view of sets $\mathbf{S}$, $\mathbf{S}_0$.

Theorem 6 a) *Let g_0 denote the function which is identically equal to zero on E_1. Then $\mathbf{S} = \mathbf{T} - \{g_0\}$.*

b) *Let $\mathbf{T}_0$ denote the system of all such functions $g \in \mathbf{T}$ for which the set $A_g = \{y \in E_1; g(y) > 0\}$ is countable and for each $y \in A_g$ we have $g(y) < +\infty$. Then $\mathbf{S}_0 = \mathbf{T} - \mathbf{T}_0$.*

Proof. a) Obviously g_0 cannot be a Banach indicatrix of any function $f : X \to E_1$ with $X \neq \emptyset$. Let $g \in \mathbf{T} - \{g_0\}$. Put $C_k = g^{-1}(\{k\})$ $(k = 1, 2, \ldots, +\infty)$, $D_k = \{1, 2, \ldots, k\} \times C_k$ $(k = 1, 2, \ldots)$, $D_\infty = \{1, 2, \ldots, n, \ldots\} \times C_\infty$. Let $X = \bigcup_{k=1}^{\infty} D_k \cup D_\infty$. Since $g \neq g_0$ at least one of the sets C_k and consequently at least one of the sets $D_k (1 \leqq k \leqq +\infty)$ is non-void. Hence $X \neq \emptyset$.

Let us define the function f on X in the following way: If $x \in$

$\in D_k$ ($1 \leqq k \leqq +\infty$) then $x = (l, y)$ for some natural l and $y \in C_k$, and we put $f(x) = y$. Then $f: X \to E_1$ and obviously $\tau_f = g$.

b) Obviously any function $g \in \mathbf{T}_0$ cannot be a Banach indicatrix of any function $f: E_1 \to E_1$. Hence $\mathbf{S}_0 \subset \mathbf{T} - \mathbf{T}_0$.

Let $g \in \mathbf{T} - \mathbf{T}_0$. Then we have the following possibilities:

1) The set $A_g = \{y; g(y) > 0\}$ is uncountable;

2) The set A_g is countable but for some $y \in A_g$ we have $g(y) = +\infty$.

Case 1) can be decomposed into the following two cases:

11) For each $y \in A_g$ we have $1 \leqq g(y) < +\infty$;

12) There exists some $y \in A_g$ such that $g(y) = +\infty$.

In case 11) let $\overline{\overline{A}}_g$ denote the cardinal number of the set A_g and Ω^* be the least ordinal number of the cardinality $\overline{\overline{A}}_g$. Let

$$y_0, y_1, \ldots, y_\xi, \ldots \; (\xi < \Omega^*) \tag{1}$$

denote the one-to-one transfinite sequence of all elements of the set A_g and

$$x_0, x_1, \ldots, x_\eta, \ldots \; (\eta < \Omega) \tag{2}$$

denote the one-to-one transfinite sequence of all elements of the set E_1. Define the function $f: E_1 \to E_1$ by transfinite induction in the following way:

1) Put $f(x_0) = f(x_1) = \ldots = f(x_{g(y_0)-1}) = y_0$.

2) If for each y_ξ, $\xi < \gamma$ from (1) the numbers $g(y_\xi)$ in (2) were found in which the function f is equal to y_ξ, then let β denote the least ordinal number such that the function f was yet not defined in x_β. Then we put

$$f(x_\beta) = f(x_{\beta+1}) = \ldots = f(x_{\beta+g(y_\gamma)-1}) = y_\gamma .$$

Thus we obtain the function $f: E_1 \to E_1$ for which $\tau_f = g$.

In case 12) let (1) have the previous meaning and for a δ, $0 \leqq \delta < \Omega^*$ let $g(y_\delta) = +\infty$. Let

$$F_0, F_1, \ldots, F_\xi, \ldots \; (\xi < \Omega^*)$$

be a sequence of such infinite pair-wise disjoint sets that $\bigcup_{0 \leqq \xi < \Omega^*} F_\xi = E_1$. Define $f: E_1 \to E_1$ in the following way: If $g(y_\xi) < +\infty$ ($0 \leqq \xi < \Omega^*$), then we take from the set F_ξ the points $x_1, x_2, \ldots, x_{g(y_\xi)}$ ($x_i \neq x_j$ for $i \neq j$) and put $f(x_j) = y_\xi$ ($j = 1, 2, \ldots, g(y_\xi)$). For $x \in F_\xi$, $x \neq x_j$ ($j = 1, 2, \ldots, g(y_\xi)$) we put $f(x) = y_\delta$. If $g(y_\xi) = +\infty$, then we put $f(x) = y_\xi$ for each $x \in F_\xi$. Thus we get the function $f: E_1 \to E_1$ and obviously $\tau_f = g$.

In case 2) the existence of a function $f: E_1 \to E_1$ with $\tau_f = g$ can be proved in an analogous way as in case 12). This ends the proof.

Let $\mathbf{T}^*$ denote the set of all functions $g: \langle 1, +\infty) \to Z$. Let $\mathbf{S}^*$, $\mathbf{S}_0^*$, $\mathbf{T}_0^*$

have an analogous meaning to the sets $\mathbf{S}, \mathbf{S}_0, \mathbf{T}_0$ in Theorem 6 (i. e. $\mathbf{S}^*(\mathbf{S}_0^*)$ denotes the set of all $g \in \mathbf{T}^*$ for which there exists an $f : X \to \langle 1, +\infty), X \neq \emptyset$ $(f : E_1 \to \langle 1, +\infty))$ such that $g = \tau_f \mid \langle 1, +\infty)$; $\mathbf{T}_0^*$ denotes the set of all $g \in \mathbf{T}^*$ for which the set $A_g = \{y \in \langle 1, +\infty); g(y) > 0\}$ is countable and for each $y \in A_g$ we have $g(y) < +\infty$). It is easy to see from the proof of Theorem 6 that $\mathbf{S}^* = \mathbf{T}^* - \{g_0^*\}$, where g_0^* denotes the function which is identically equal to zero on $\langle 1, +\infty)$ and $\mathbf{S}_0^* = \mathbf{T}^* - \mathbf{T}_0^*$.

We can illustrate the mutual relation between the sets $\mathbf{S}_0^*$ and $\mathbf{T}_0^*$ also from the topological point of view.

If $g, h \in \mathbf{T}^*$ and $g = h$, then we put $\varrho(g, h) = 0$. In the reverse case we put $\varrho(g, h) = \dfrac{1}{\inf\{x; g(x) \neq h(x)\}}$. It is easy to see that ϱ is a metric (cf. [6], p. 67) and the space $\mathbf{T}^*$ with this metric is a complete metric space.

Theorem 7. *The set* $\mathbf{T}_0^*$ *is non-dense in* $\mathbf{T}^*$.

Corollary. *The set* $\mathbf{S}_0^*$ *is residual in* $\mathbf{T}^*$.

Proof. Let $g \in \mathbf{T}^*$, $0 < \varepsilon < 1$, $0 < \varepsilon' < \varepsilon$. Define $g_1(x) = g(x)$ for $x \in \in \langle 1, 1/\varepsilon'\rangle$ and $g_1(x) = 1$ for $x > 1/\varepsilon'$. Then we have $\varrho(g, g_1) \leqq \varepsilon' < \varepsilon$. It is easy to check that for $0 < \delta_1 < \varepsilon'$ we have $S(g_1, \delta_1) \subset S(g, \varepsilon)$ $(S(h, \delta) = \{f \in \mathbf{T}^*; \varrho(h, f) < \delta\})$. Let $f \in S(g_1, \delta_1)$. Then $f(x) = g_1(x)$ for $1 \leqq x \leqq 1/\delta_1$ and from this and from the definition of the function g_1 we get that $f(x) = 1$ for $1/\varepsilon < x < 1/\delta_1$. Hence $f \notin \mathbf{T}_0^*$. The proof is complete.

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