Juraj Bosák

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ON RADICALS OF SEMIGROUPS

JURAJ BOSÁK, Bratislava

To a given semigroup S and its ideal J 6 further significant sets can be assigned: the radical of Schwarz $R_J(S)$, of Ševrin $L_J(S)$, of Clifford $R_J^*(S)$, of McCoy $M_J(S)$, of Jiang Luh $C_J(S)$ and the set $N_J(S)$ of all nilpotent elements (the definitions are given below). In this paper we prove (Theorem 2) that we always have (¹)

$$J \subseteq R_J(S) \subseteq M_J(S) \subseteq L_J(S) \subseteq R_J^*(S) \subseteq N_J(S) \subseteq C_J(S) \subseteq S;$$

however, there exists a semigroup U with an ideal I such that

$$I \subset R_I(U) \subset M_I(U) \subset L_I(U) \subset R_I^*(U) \subset N_I(U) \subset C_I(U) \subset U.$$

The essential part of the work is to elucidate the relation between the Ševrin and the McCoy radical (Theorem 1). The rest easily follows from known results (relations between individual radicals were studied already in [1], [3], [9], [10], [11], [19], [20] and [21]). The same results are valid if we consider only periodic semigroups. The case of commutative semigroups and that of finite semigroups are treated separately. (Corollaries 1 and 2.)

In terminology and notation we follow book [5].

First we state some definitions; all mentioned ideals are two-sided.

Suppose a semigroup S with an ideal J to be given. Denote by $N_J(S)$ the set of all elements of S nilpotent with respect to J, i. e. such that some power of them belongs to J. Now we define 5 radicals of S with respect to J.

By the *Schwarz* (or *nilpotent*) *radical* we understand (cf. [14]) the union $R_J(S)$ of all ideals of S nilpotent with respect to S (that is, such that some power of them is a subset of J).

By the *Ševrin* (*locally nilpotent*) *radical* we understand [16], [22] the union $L_J(S)$ of all ideals of S locally nilpotent with respect to J (i. e. of such ideals, every finitely generated subsemigroup of which is nilpotent with respect to J).

(¹) The symbol $\subseteq$ denotes the set-theoretic inclusion as well as the symbol $\subset$; the latter is used only in the case of proper subsets.

By the *Clifford* (*nil-*) *radical* we mean [4] the union $R_J^*(S)$ of all nil-ideals of S with respect to J (i. e. such that all its elements are nilpotent with respect to J).

By the *McCoy*(2) (*prime*) *radical* we mean [10] the intersection $M_J(S)$ of all prime ideals of S containing J (an ideal I of S is called prime if for any ideals A and B of S from the condition $AB \subseteq I$ it follows that either $A \subseteq I$ or $B \subseteq \subseteq I$). It is easy to prove (see [7], [10] and [19]) that the McCoy radical $M_J(S)$ consists exactly of such elements x that every m-system containig x has a non-empty intersection with J (by an m-system we mean a set $A \subseteq S$ with the property that to any a and b of A there exists $x \in S$ so that $axb \in A$).

By the *Jiang Luh* (*completely prime*) *radical*(3) we mean [10] the intersection $C_J(S)$ of all completely prime ideals of S including J (an ideal I of S is called completely prime (4) if for any $a, b \in S$ from the condition $ab \in I$ it follows that either $a \in I$ or $b \in I$).

In the case of a semigroup S with a zero element 0 if $J = \{0\}$, the index J in the symbols $N_J(S)$, $R_J(S)$, $L_J(S)$, $R_J^*(S)$, $M_J(S)$ and $C_J(S)$ will be omitted.

Let us remark that we shall not deal with radicals mentioned in [9] and [12] as well as with various types of radicals studied in a number of papers by Hoehnke (see, e. g. [6]) and Seidel [15]. We also do not consider radicals of topological semigroups [13].

In the proof of our Theorem 1 we shall use twice the following lemma by König (see [2], chapter 3, p. 17—18).

Lemma 1. *Let $\{A_1, A_2, \ldots, A_k, \ldots\}$ be a sequence of finite, non-empty sets which are pairwise disjoint, and let $\prec$ be a relation defined between the elements of two consecutive sets: if for all $x_k \in A_k$ ($k \geqq 2$) an element $x_{k-1} \in A_{k-1}$ exists such that $x_{k-1} \prec x_k$, a sequence $\{a_1, a_2, \ldots, a_k, \ldots\}$ exists with $a_k \in A_k$ for all k such that*:

$$a_1 \prec a_2 \prec a_3 \prec \ldots \prec a_k \prec \ldots$$

Theorem 1. *Let J be an ideal of a semigroup S. Then $M_J(S) \subseteq L_J(S)$. Further, there exists a semigroup T such that $M(T) \subset L(T)$.*

Proof. (i) To show that $M_J(S) \subseteq L_J(S)$ it is sufficient to prove that if $x \notin L_J(S)$, then $x \notin M_J(S)$. Suppose $x \notin L_J(S)$. Then the principal ideal $J(x) = x \cup Sx \cup xS \cup SxS$ is not locally nilpotent with respect to J, i. e. it has a subsemigroup $T \subseteq J(x)$ generated by a finite set $G = \{g_1, g_2, \ldots, g_n\}$ (in the notation of [5] we can write $\langle G \rangle = T$) such that T is not nilpotent, that is, $T^k \nsubseteq J$ for any natural number k. Evidently, we have:

(2) In [7] this radical is called the Baer-McCoy radical.

(3) In [9] this radical is called the Thierrin (compressed) radical.

(4) In [5] a somewhat different terminology is used.

$$T^k = G^k \cup G^{k+1} \cup G^{k+2} \cup \ldots \quad (k = 1, 2, 3, \ldots).$$

If $G^k \subseteq J$, then necessarily $G^{k'} \subseteq J$ for every $k' \geqq k$. Therefore from the condition $T^k \not\subseteq J$ it follows that $G^k \not\subseteq J$ $(k = 1, 2, 3, \ldots)$. Obviously each G^k contains only a finite number of elements. Denote by A_k $(k = 1, 2, 3, \ldots)$ the set of all (ordered) pairs of the form (k, y) where $y \in G^k$, $y \notin J$. Evidently $\{A_1, A_2, A_3, \ldots\}$ is a sequence of finite, non-empty and pairwise disjoint sets. Define a relation $\prec$ between the elements of two consecutive sets thus: $(k_1, y_1) \prec (k_2, y_2)$ if and only if $k_2 - k_1 = 1$ and $y_2 \in y_1 G$. Obviously for all $x_k \in A_k$ $(k \geqq 2)$ an element $x_{k-1} \in A_{k-1}$ exists such that $x_{k-1} \prec x_k$. According to Lemma 1 a sequence $\{a_1, a_2, \ldots, a_k, \ldots\}$ exists with $a_k \in A_k$ for all k such that $a_1 \prec a_2 \prec a_3 \prec \ldots \prec a_k \prec \ldots$ Further, if we put $a_k = (k, y_k^*)$, we have: $y_1^* \in G$, $y_k^* \notin J$, $y_{k+1}^* \in y_k^* G$ $(k = 1, 2, 3, \ldots)$.

(ii) From (i) it follows that there exists an infinite word of the form $s = g_{i_1} g_{i_2} g_{i_3} \ldots$ where $g_{i_k} \in G \subseteq J(x)$ $(k = 1, 2, 3, \ldots)$, no finite (connected) section of which belongs to J. Now, by means of s we shall construct an m-system containing x and no element of J so that $x \notin M_J(S)$ and the proof of the first assertion of Theorem 1 will be completed.

Denote by B the set of all finite non-empty (and connected) sections of s. Let us suppose that we have constructed a sequence H of non-empty finite sets $H_1, H_2, H_3, \ldots$ such that 1° $H_i \subseteq B$ $(i = 1, 2, 3, \ldots)$; 2° If $n > 1$, then every word of H_n can be written in the form $t = uvu$, where $u \in H_{n-1}$, $v \in B$; 3° $H_1 \subseteq G$.

Then the sequence $\{B_1, B_2, B_3, \ldots\}$, where B_j $(j = 1, 2, \ldots)$ consists of all pairs of the form (j, z), $z \in H_j$ will satisfy the suppositions of Lemma 1, if we define $(j_1, z_1) \prec (j_2, z_2)$ if and only if $j_2 - j_1 = 1$ and $z_2 = z_1 v z_1$, where $v \in B$. Therefore an infinite sequence of elements $b_1 \in B_1$, $b_2 \in B_2$, $\ldots$ exists such that $b_1 \prec b_2 \prec b_3 \prec \ldots$ Put $b_j = (j, z_j^*)$. Each of the words z_j^* consists of letters from G. As $G \subseteq J(x)$, every $g_i \in G$ can be written in the form $a_i x b_i$, where a_i and b_i are empty symbols or elements of S. If we replace in the words $z_1^*, z_2^*, z_3^*, \ldots$ every g_i by $a_i x b_i$, we obtain new words consisting of letters $x, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$. If we omit in each of these words the first and the last letter, we evidently get an m-system containing x and disjoint with J so that $x \notin M(S)$. (All this is valid under the assumption that a sequence H with the stated properties does exist.)

(iii) To accomplish the proof of the first part of Theorem 1 it suffices now to construct a sequence H of non-empty finite sets fulfilling the conditions 1°, 2° and 3° from (ii).

As before let n be the number of elements of G. Define a sequence $\{c_1, c_2, c_3, \ldots\}$ of positive integers thus: $c_1 = 1$, $c_{k+1} = 3c_k(n + 1)^{c_k}$ for $k = 1, 2, 3, \ldots$

Further, define a sequence $H = \{H_1, H_2, H_3, \ldots\}$ of sets as follows. $H_1 = G \setminus J$; H_{k+1} is the set of all words over G belonging to B whose length ($=$ number of letters) is not greater than c_{k+1} and which can be written in the form $t = uvu$, where $u \in H_k$, $v \in B$ ($k = 1, 2, 3, \ldots$). Obviously H is a sequence of finite sets satisfying the conditions 1°, 2° and 3°. It remains to prove only that all these sets are non-empty. By induction we shall even prove a stronger assertion ($k = 1, 2, 3, \ldots$):

(T_k) If the word s is decomposed into connected sections containing each c_k letters, then any of these sections contains as a subword at least one element of H_k.

The assertion (T_1) is evidently true. Suppose that for some k (T_k) takes place, that is, each of the sections with c_k letters contains as a subword an element of H_k. Let S_{k+1} be any section from the decomposition of s into sections with c_{k+1} letters. Evidently, s contains $3(n+1)c_k$ sections with c_k letters: any of these sections contains an element of H_k and the length of this element does not exceed c_k; thus the number of such elements cannot exceed $(n+1)^{c_k}$. Therefore at least one element $u \in H_k$ occurs in S_{k+1} at least three times as a subword of disjoint sections with c_k letters. Consequently in S_{k+1} there is a subword of the form $uvu, v \in B$. Thus the first assertion of Theorem 1 is proved.

(iv) Now we shall construct a semigroup T with a zero element containing an element $a \in T$ such that $a \in L(T)$, $a \notin M(T)$, so that $L(T) \neq M(T)$, and, taking into account the first, already proved part of Theorem 1, $M(T) \subset L(T)$.

Let T be the semigroup generated by a three-element set $\{0, a, b\}$ subject to the generating relations

$$a^2 = 0a = a0 = 0b = b0 = 0^2 = 0,$$
$$b^{s_1}ab^{s_2}ab^{s_3}a \ldots ab^{s_n} = 0$$

for every finite sequence $(s_1, s_2, \ldots, s_n)$ of natural numbers such that $s_1 + s_2 + \ldots + s_n < (n-2)^2$ where n runs through the set of all natural numbers.

(v) We first prove that $a \in L(T)$. It suffices to prove that $J(a) = a \cup Ta \cup aT \cup TaT$ is a locally nilpotent ideal. Let U be any subsemigroup of $J(a)$ generated by a finite set G. If $G \subseteq \{a, 0\}$, then evidently $U^2 = 0$, so that U is a nilpotent semigroup. If $G \nsubseteq \{a, 0\}$, then in G elements different from 0 and a must occur; such elements can be expressed uniquely as words over the alphabet $\{a, b\}$. Let t be the greatest exponent of b occuring in these expressions. We shall prove that $U^{2t+9} = 0$.

Let $x \in U^{2t+9}$, $x \neq 0$. Consequently $x = x_1x_2 \ldots x_{2t+9}$, where $x_i \in U \subseteq J(a)$ for $i = 1, 2, \ldots, 2t+9$. Each of the factors x_i, considered as a word over $\{a, b\}$, contains the element a. In addition, since $a^2 = 0$, at least $t+4$

of these factors $x_1, x_2, \ldots, x_{2t+9}$ contains — once or several times — the element b with some natural exponents. The element x can be written in the form

$$x = yb^{s_1}ab^{s_2}ab^{s_3}a \ldots ab^{s_{t+4}}z,$$

where y and z are the empty symbols or elements of S, and $s_i \leqq t$ for $i = 1, 2, \ldots, t + 4$.

Therefore $s_1 + s_2 + \ldots + s_{t+4} \leqq (t + 4)t < (t+2)^2$, so that $b^{s_1}ab^{s_2}a \ldots ab^{s_{t+4}} = 0$. Whence it follows that $x = y0z = 0$, which contradicts the assumption $x \neq 0$.

(vi) Now we prove that $a \notin M(T)$. Is is sufficient to construct an m-system M containing a and not containing 0. Put

$$M = \{a,\ ab^{x_1}a,\ ab^{x_1}ab^{x_2}a,\ ab^{x_1}ab^{x_2}ab^{x_3}a,\ \ldots\},$$

where $x_i = 10^c$, $i = 2^c d$, $d \equiv 1 \pmod 2$.

Words from M can be considered as certain sections of an infinite word

$$abab^{10}abab^{100}abab^{10}abab^{1000}abab^{10}abab^{100}abab^{10}abab^{10000}a \ldots,$$

formed successively from the word ab in such a way that in each step the whole word is repeated once more but the last exponent of b is multiplied by 10. Evidently M is an m-system containing a. If $0 \in M$, some word of M would have to contain a subword of the form $b^{s_1}ab^{s_2}a \ldots ab^{s_n}$, where

$$s_1 + s_2 + \ldots + s_n < (n - 2)^2, \tag{1}$$

which is possible only if $n \geqq 5$. It is easy to prove that among any 2^m consecutive members of the sequence $\{x_i\}$ at least one $\geqq 10^m$. It follows that

$$s_2 + s_3 + \ldots + s_{n-2} + s_{n-1} \geqq 10^{[\log_2 (n-2)]}.\text{[5]} \tag{2}$$

We need two auxiliary inequalities:

$$10^k > 4^{k+1} + 1 \quad (k = 2, 3, 4, \ldots). \tag{3}$$

$$10^{[\log_2 (n-2)]} > (n - 2)^2 \quad (n = 5, 6, 7, \ldots). \tag{4}$$

The former — (3) — can be easily proved by induction. The latter — (4) — is evident for $n = 5$; for $n \geqq 6$ (4) follows from (3) thus: Let $2^k \leqq n - 2 < 2^{k+1}$, where k is a natural number (obviously $k \geqq 2$). We have:

$$10^{[\log_2 (n-2)]} = 10^k > 4^{k+1} + 1 > (2^{k+1} - 1)^2 \geqq (n - 2)^2.$$

[5] The symbol [q] denotes the integral part of a number q.

Using (2) and (4) we obtain

$$s_1 + s_2 + \ldots + s_n > s_2 + \ldots + s_{n-1} \geqq 10^{[\log_2(n-2)]} > (n-2)^2,$$

but this contradicts (1). The theorem follows.

Theorem 2. *If J is an ideal of a semigroup S, then*

$$J \subseteq R_J(S) \subseteq M_J(S) \subseteq L_J(S) \subseteq R_J^*(S) \subseteq N_J(S) \subseteq C_J(S) \subseteq S.$$

But there exists a (periodic) semigroup U with a zero element 0 such that

$$0 \subset R(U) \subset M(U) \subset L(U) \subset R^*(U) \subset N(U) \subset C(U) \subset U.$$

Proof. The relation $M_J(S) \subseteq L_J(S)$ follows from Theorem 1. The relations $R_J(S) \subseteq M_J(S)$, $R_J^*(S) \subseteq N_J(S) \subseteq C_J(S)$ have been proved by Šulka [19] (and in the case $J = \{0\}$ already by Jiang Luh [10]). The relations $J \subseteq \subseteq R_J(S)$, $L_J(S) \subseteq R_J^*(S)$ and $C_J(S) \subseteq S$ are obvious. Thus the first part of the theorem is proved.

Now let us construct a semigroup U with the required properties. First we define the semigroups S_1, S_2, S_3 and S_4 as follows.

S_1 is defined by the multiplication table:

	0_1	z	a	b	c	d	e
0_1	0_1	0_1	0_1	0_1	0_1	0_1	0_1
z	0_1	0_1	0_1	0_1	0_1	0_1	z
a	0_1	0_1	a	b	0_1	0_1	a
b	0_1	0_1	0_1	0_1	a	b	b
c	0_1	0_1	c	d	0_1	0_1	c
d	0_1	0_1	0_1	0_1	c	d	d
e	0_1	z	a	b	c	d	e

It can be easily verified that the multiplication is associative.

S_2 is the semigroup generated by a set $\{0_2, a_1, a_2, a_3, \ldots\}$ subject to the generating relations

$$0_2 a_i = a_i 0_2 = a_i^2 = 0_2 \quad (i = 1, 2, 3, \ldots)$$

and

$$a_i x a_i = 0_2 \quad (i = 2, 3, 4, \ldots)$$

for every word x over the alphabet $\{0_2, a_1, a_2, a_3, \ldots\}$.

Put $S_3 = T \setminus \{b, b^2, b^3, \ldots\}$, where T is the semigroup constructed in Theorem 1. The zero element of S_3 will be denoted by 0_3.

Finally, let S_4 be the semigroup generated by $\{0_4, a, b\}$, subject to the generating relations $0_4 x = x 0_4 = x^3 = 0_4$ for every word x over the given alphabet.

Form now the direct product $U = S_1 \times S_2 \times S_3 \times S_4$, i. e. the semigroup of all ordered quadruples (s_1, s_2, s_3, s_4), $s_i \in S_i$ $(i = 1, 2, 3, 4)$ with a multiplication defined coordinate-wise. Evidently, all S_i are periodic semigroups with a zero element 0_i. It follows that U is a periodic semigroup with a zero element $0 = (0_1, 0_2, 0_3, 0_4)$.

Evidently, we have; $R(S_1) = R^*(S_1) = \{0_1, z\}$, $N(S_1) = \{0_1, z, b, c\}$, $C(S_1) = \{0_1, z, a, b, c, d\}$, so that $\{0_1\} \subset R(S_1) = R^*(S_1) \subset N(S_1) \subset C(S_1) \subset S_1$.

In the third part of [3] we proved that $R(S_2) \subset M(S_2)$.

From the proof of Theorem 1 it follows that $M(S_3) \subset L(S_3)$ (the deleting of the powers of b from T does not change the proof essentially).

Now we prove that $L(S_4) \subset R^*(S_4)$. (In the sixth part of [3] it has been proved only that $R(S_4) \neq R^*(S_4) = S_4$.) It suffices to prove that $a \notin L(S_4)$, i. e. that the principal ideal $J(a)$ is not locally nilpotent. But it is a consequence of the fact (see references quoted in [3] and [8]) that in the infinite word

$$abbabaabbaababbab\ldots$$

no (finite) subword occurs consecutively more than twice. It follows that the subsemigroup of $J(a)$ generated by ab and ba is not nilpotent so that $J(a)$ is not locally nilpotent.

We want the following relations:

$$\begin{aligned}
\{0\} &= \{0_1\} \times \{0_2\} \times \{0_3\} \times \{0_4\},\\
R(U) &= R(S_1) \times R(S_2) \times R(S_3) \times R(S_4),\\
M(U) &= M(S_1) \times M(S_2) \times M(S_3) \times M(S_4),\\
L(U) &= L(S_1) \times L(S_2) \times L(S_3) \times L(S_4),\\
R^*(U) &= R^*(S_1) \times R^*(S_2) \times R^*(S_3) \times R^*(S_4),\\
N(U) &= N(S_1) \times N(S_2) \times N(S_3) \times N(S_4),\\
C(U) &= C(S_1) \times C(S_2) \times C(S_3) \times C(S_4),\\
U &= S_1 \times S_2 \times S_3 \times S_4.
\end{aligned}$$

The first and the last of these relations are evident, the remaining ones follow from [1]. Since we already have proved that $\{0_1\} \subset R(S_1) = R^*(S_1) \subset N(S_1) \subset C(S_1) \subset S_1$, $R(S_2) \subset M(S_2)$, $M(S_3) \subset L(S_3)$, $L(S_4) \subset R^*(S_4)$ and considering that according to the first — already proved — part of Theorem 2 we have $\{0_i\} \subseteq R(S_i) \subseteq M(S_i) \subseteq L(S_i) \subseteq R^*(S_i) \subseteq N(S_i) \subseteq C(S_i) \subseteq S_i$ $(i = 1, 2, 3, 4)$, consequently $\{0\} \subset R(U) \subset M(U) \subset L(U) \subset R^*(U) \subset N(U) \subset C(U) \subset U$, q. e. d.

Corollary 1. *If J is an ideal of a commutative semigroup S, then $R_J(S) = M_J(S) = L_J(S) = R^*_J(S) = N_J(S) = C_J(S)$; but there exists a commutative semigroup V with a zero element 0 such that $\{0\} \subset R(V) \subset V$.*

Proof. Let V be the semigroup with three elements 0, z and 1 with a multiplication defined thus $1z = z1 = z$, $1^2 = 1$ and all remaining products equal to 0. Evidently, V has the required properties.

The relations $R_J(S) = M_J(S) = R_J^*(S) = N_J(S) = C_J(S)$ have been proved by Šulka [19]. The rest of the proof follows from Theorem 2.

Lemma 2. (6) *Let I be a nil-ideal of a semigroup S with respect to an ideal J of S. We have: If $x \in S$, $e \in I$, $x = xe$, then $x \in J$.*

Proof. Suppose $e^k \in J$. Then $x = xe = xe^2 = \ldots = xe^k \in J$.

Lemma 3. (7) *If I is a nil-ideal of a finite semigroup S with respect to an ideal J of S, then I is a nilpotent ideal of S with respect to J.*

Proof. Let n be the cardinality of S. We prove that $I^n \subseteq J$. Suppose $x \in I^n$, i. e. $x = a_1a_2\ldots a_n$, $a_i \in I$, $i = 1, 2, \ldots, n$. Put $b_i = a_1a_2\ldots a_i$. If $b_1, b_2, \ldots, b_n$ are mutually different, then they are all the elements of S, so that at least one of them belongs to J; but in this case x belongs to J, too. In the second case, i. e. if at least two elements of $b_1, b_2, \ldots, b_n$ are equal — e. g. let $b_i = b_j$, $i < j$ — we have: $b_i(a_{i+1}a_{i+2}\ldots a_j) = b_i$. Lemma 2 implies $b_i \in J$, so that also $b_n = x \in J$.

Corollary 2. *If J is an ideal of a finite semigroup S, then $R_J(S) = M_J(S) = L_J(S) = R_J^*(S)$; but there exists a finite semigroup W with a zero element 0 such that $\{0\} \subset R(W) \subset N(W) \subset C(W) \subset W$.*

Proof. From Lemma 3 it follows that every nil-ideal of a finite semigroup S with respect to J is nilpotent with respect to J. Therefore $R_J^*(S) \subseteq R_J(S)$. In view of Theorem 2 the first part of Corollary 2 is proved. For the proof of the second part if suffices to take as W the semigroup S_1 from the proof of Theorem 2.

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(6) Cf. Lemma 2.3 of [16] and Lemma 1 of [18].

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Matematický ústav
Slovenskej akadémie vied,
Bratislava