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ON AN ABSTRACT FORMULATION OF REGULARITY

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There is a general concept of regularity presented in book [2] and in paper [5] including many known cases in topological spaces (differences between concepts presented in [2] and [5] are only formal). On the other hand it was shown in [4], [6] and [7] that many problems of measure theory can be formulated only by means of systems of sets „of small measure". Hence in such a theory we have no measure, we have only a sequence $\{\mathcal{N}_n\}_{n=0}^{\infty}$ of systems of measurable sets satisfying some axioms.

The purpose of the present article is to construct a common generalization of both theories, since the regularity in [4] was studied only in a very special case.

Let X, $\mathbf{C}$, $\mathbf{U}$, $\mathbf{S}$ and $\{\mathcal{N}_n\}_{n=0}^{\infty}$ satisfy the following assumptions: X is a nonempty set of elements, $\mathbf{C}$, $\mathbf{U}$, $\mathbf{S}$ are systems of subets of X with the following properties:

V_1 $\emptyset \in \mathbf{C}$, $\emptyset \in \mathbf{U}$

V_2 If $U_n \in \mathbf{U}$ for $n = 1, 2, \ldots$, then also $\bigcup_{n=1}^{\infty} U_n \in \mathbf{U}$.

V_3 If $C_1, C_2 \in \mathbf{C}$, then $C_1 \cup C_2 \in \mathbf{C}$.

V_4 $U - C \in \mathbf{U}$, $C - U \in \mathbf{C}$ for any $U \in \mathbf{U}$, $C \in \mathbf{C}$.

V_5 To any $C \in \mathbf{C}$ there are $U \in \mathbf{U}$, $D \in \mathbf{C}$ such that $C \subset U \subset D$.

V_6 $\mathbf{U} \subset \mathbf{S}(\mathbf{C}) = \mathbf{S}$, where $\mathbf{S}(\mathbf{C})$ is the σ-ring generated by $\mathbf{C}$.

$\{\mathcal{N}_n\}_{n=0}^{\infty}$ is a sequence of subsystems of the system $\mathbf{S}$ with the following properties:

(i) $\emptyset \in \mathcal{N}_n$ for $n = 0, 1, 2, \ldots$; $E, F \in \mathcal{N}_0 \Rightarrow E \cup F \in \mathcal{N}_0$.

(ii) To any positive integer n there exists a sequence $\{k_i\}_{i=1}^{\infty}$ of positive integers such that $\bigcup_{i=1}^{\infty} E_i \in \mathcal{N}_n$, whenever $E_i \in \mathcal{N}_{k_i}$ $(i = 1, 2, \ldots)$.

(iii) If $\{E_i\}_{i=1}^{\infty}$ is a sequence of sets of $\mathbf{S}$, $E_{i+1} \subset E_i$ $(i = 1, 2, \ldots)$ $\bigcap_{i=1}^{\infty} E_i = \emptyset$

and $E_t \in \mathscr{N}_0$ for some positive integer t, then to any positive integer n there is a positive integer m such that $E_m \in \mathscr{N}_n$.

(iv) If $E \in \mathscr{N}_n$, $F \subset E$, $F \in \mathbf{S}$, then $F \in \mathscr{N}_n$ $(n = 0, 1, 2, \ldots)$.

(v) $C \in \mathscr{N}_0$ for every $C \in \mathbf{C}$.

Note 1. From the axioms $V_1 - V_6$ the following properties of $\mathbf{C}$ and $\mathbf{U}$ follow:

1. If $C_i \in \mathbf{C}$ $(i = 1, 2, \ldots)$, then $\bigcap_{i=1}^{\infty} C_i \in \mathbf{C}$ (see [5] Lemma 1).

2. If $U_1, U_2 \in \mathbf{U}$, then $U_1 \cap U_2 \in \mathbf{U}$. Endeed, if $U_1, U_2 \in \mathbf{U}$, then according to V_6 we have $U_1 \cap U_2 \subset \bigcup_{n=1}^{\infty} C_n$, where $C_n \in \mathbf{C}$ $(n = 1, 2, \ldots)$. According to V_5 there are sets $V_n \in \mathbf{U}$, $D_n \in \mathbf{C}$ such that $C_n \subset V_n \subset D_n$. According to V_4 we have $U_2 \cap V_n = V_n - (D_n - U_2) \in \mathbf{U}$ and also $U_1 \cap U_2 \cap V_n = (V_n \cap U_2) - (D_n - U_1) \in \mathbf{U}$. Hence $U_1 \cap U_2 = \bigcup_{n=1}^{\infty} (U_1 \cap U_2 \cap V_n) \in \mathbf{U}$ according to V_2.

3. To any $E \in \mathbf{S}$ there is a set $U \in \mathbf{U}$ such that $E \subset U$ (see [5] Lemma 3).

4. To any $E \in \mathbf{S}$ there are $A_i \in \mathbf{S}$, $C_i \in \mathbf{C}$ $(i = 1, 2, \ldots)$ such that $A_i \subset A_{i+1}$, $A_i \subset C_i$ $(i = 1, 2, \ldots)$, $E = \bigcup_{i=1}^{\infty} A_i$ (see [5] Lemma 2).

We shall use also the following consequence of (i) and (ii):

(vi) To any positive integer n there are positive integers k, m such that $M \in \mathscr{N}_m$, $K \in \mathscr{N}_k$ implies $M \cup K \in \mathscr{N}_n$.

Definition 1. *Put*

$\mathbf{R}_1 = \{E \in \mathbf{S}$: *to any positive integer* n *there is a set* $U \in \mathbf{U}$ *such that* $E \subset U$, $U - E \in \mathscr{N}_n\}$,

$\mathbf{R}_2 = \{E \in \mathbf{S}$: *to any positive integer* n *there is a set* $C \in \mathbf{C}$ *such that* $C \subset E$, $E - C \in \mathscr{N}_n\}$,

$\mathbf{R}_3 = \{E \in \mathbf{S}$: there are sets $C_k \in \mathbf{C}$, $C_k \subset E$ $(k = 1, 2, \ldots)$ such that $\bigcup_{k=1}^{\infty} C_k \notin \mathscr{N}_0\}$.

Definition 2. *Put* $\mathbf{P}_1 = \mathbf{R}_1 \cup (\mathbf{S} - \mathscr{N}_0)$, $\mathbf{P}_2 = \mathbf{R}_2 \cup \mathbf{R}_3$, $\mathbf{P} = \mathbf{P}_1 \cap \mathbf{P}_2$.

Note 2. Evidently $E \in \mathbf{R}_3 \Rightarrow E \notin \mathscr{N}_0$. Hence $E \in \mathbf{P}_2$, $E \in \mathscr{N}_0$ implies $E \in \mathbf{R}_2$. According to (i) we have $\mathbf{C} \subset \mathbf{P}_2$, $\mathbf{U} \subset \mathbf{P}_1$. At the end of the article examples are given of systems $\{\mathscr{N}_n\}_{n=0}^{\infty}$ as well as $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}$.

Lemma 1. *If* $E_i \in \mathbf{P}_2$, $E_i \subset E_{i+1}$ $(i = 1, 2, \ldots)$ *then* $\bigcup_{i=1}^{\infty} E_i \in \mathbf{P}_2$.

Proof. 1. Assume that $E_i \in \mathbf{R}_2 (i = 1, 2, \ldots)$. Let n be any positive integer and let k, m be the positive integers fulfilling the property (vi). To k there exists a sequence $\{k_i\}_{i=1}^{\infty}$ of positive integers with the property (ii). To k_i and the set E_i there is a set $C_i \in \mathbf{C}$ such that $C_i \subset E_i$, $E_i - C_i \in \mathscr{N}_{k_i}$ $(i = 1, 2, \ldots)$.

According to (ii) we have $\bigcup_{i=1}^{\infty} (E_i - C_i) \in \mathcal{N}_k$. Further $\left(\bigcup_{i=1}^{\infty} E_i\right) - \left(\bigcup_{i=1}^{\infty} C_i\right) \subset \bigcup_{i=1}^{\infty} (E_i - C_i)$ and hence $\left(\bigcup_{i=1}^{\infty} E_i\right) - \left(\bigcup_{i=1}^{\infty} C_i\right) \in \mathcal{N}_k$ according to (iv). If $\bigcup_{i=1}^{\infty} C_i \notin \notin \mathcal{N}_0$, then $\bigcup_{i=1}^{\infty} E_i \in \mathbf{R}_3 \subset \mathbf{P}_2$. If $\bigcup_{i=1}^{\infty} C_i \in \mathcal{N}_0$, then according to (iii) to the sequence $\left\{\left(\bigcup_{i=1}^{\infty} C_i\right) - \left(\bigcup_{i=1}^{k} C_i\right)\right\}_{k=1}^{\infty}$ and to the positive integer m there is a positive integer t such that $\left(\bigcup_{i=1}^{\infty} C_i\right) - \left(\bigcup_{i=1}^{t} C_i\right) \notin \mathcal{N}_m$. Hence according to (vi) we have $\left(\bigcup_{i=1}^{\infty} E_i\right) - \left(\bigcup_{i=1}^{t} C_i\right) = \left[\left(\bigcup_{i=1}^{\infty} E_i\right) - \left(\bigcup_{i=1}^{\infty} C_i\right)\right] \cup \left[\left(\bigcup_{i=1}^{\infty} C_i\right) - \left(\bigcup_{i=1}^{t} C_i\right)\right] \in \mathcal{N}_n$, i. e. $\bigcup_{i=1}^{\infty} E_i \in \mathbf{R}_2 \subset \mathbf{P}_2$.

2. If $E_j \notin \mathbf{R}_2$ for some positive integer j, then $E_j \in \mathbf{R}_3$ and there are sets $C_k \in \mathbf{C}$, $C_k \subset E_j$ $(k = 1, 2, \ldots)$ such that $\bigcup_{k=1}^{\infty} C_k \notin \mathcal{N}_0$. Then evidently $\bigcup_{i=1}^{\infty} E_i \in \mathbf{R}_3$ i. e. $\bigcup_{i=1}^{\infty} E_i \in \mathbf{P}_2$.

Lemma 2. *Let $C \in \mathbf{C}$. Then the system $\mathbf{N} = \{B \in \mathbf{P} : B \subset C\}$ is monotone.*

Proof. According to (v) and (iv) we have $\mathbf{N} \subset \mathcal{N}_0 \cap \mathbf{P} = \mathbf{R}_1 \cap \mathbf{R}_2$. Let $\{A_i\}_{i=1}^{\infty}$, $\{B_i\}_{i=1}^{\infty}$ be sequences of sets of $\mathbf{N}$ and let $A_i \subset A_{i+1}$, $B_i \supset B_{i+1}$ $(i = 1, 2, \ldots)$. Then evidently $\bigcup_{i=1}^{\infty} A_i \subset C$, $\bigcap_{i=1}^{\infty} B_i \subset C$. Let n be any positive integer and let $\{k_i\}_{k=1}^{\infty}$ be a sequence of positive integers with the property (ii). To any i the exists sets $U_i \in \mathbf{U}$, $C_i \in \mathbf{C}$ such that

(1) $\quad U_i \supset A_i,\ B_i \supset C_i,\ U_i - A_i \in \mathcal{N}_{k_i},\ B_i - C_i \in \mathcal{N}_{k_i}.$

We have $\bigcup_{i=1}^{\infty} U_i \in \mathbf{U}$ according to V_2 and $\bigcap_{i=1}^{\infty} C_i \in \mathbf{C}$ according to 1 of note 1. From (1) and (ii) it follows

(2) $\quad \bigcup_{i=1}^{\infty} U_i \supset \bigcup_{i=1}^{\infty} A_i,\ \left(\bigcup_{i=1}^{\infty} U_i\right) - \left(\bigcup_{i=1}^{\infty} A_i\right) \subset \bigcup_{i=1}^{\infty} (U_i - A_i) \in \mathcal{N}_n$ and hence $\bigcup_{i=1}^{\infty} A_i \in \mathbf{P}_1$,

(3) $\quad \bigcap_{i=1}^{\infty} B_i \supset \bigcap_{i=1}^{\infty} C_i,\ \left(\bigcap_{i=1}^{\infty} B_i\right) - \left(\bigcap_{i=1}^{\infty} C_i\right) \subset \bigcup_{i=1}^{\infty} (B_i - C_i) \in \mathcal{N}_n$ and hence $\bigcap_{i=1}^{\infty} B_i \in \mathbf{P}_2$.

To an arbitrarily chosen positive integer n there are positive integers k, m with the property (vi). According to (iii) to the number m and to the sequence $\left\{B_j - \left(\bigcap_{i=1}^{\infty} B_i\right)\right\}_{j=1}^{\infty}$ there exists a positive integer such that $B_t - \left(\bigcap_{i=1}^{\infty} B_i\right) \in \mathcal{N}_m$. To the number k there exists a set $U \in \mathbf{U}$ such that $U \supset B_t$, $U - B_t \in \mathcal{N}_k$. Hence according to (vi) we have

$$U \supset \bigcap_{i=1}^{\infty} B_i,\ U - \left(\bigcap_{i=1}^{\infty} B_i\right) \subset (U - B_t) \cup \left[B_t - \left(\bigcap_{i=1}^{\infty} B_i\right)\right] \in \mathcal{N}_n$$

and therefore $\bigcap_{i=1}^{\infty} B_i \in \mathbf{P}_1$; $\bigcup_{i=1}^{\infty} A_i \in \mathbf{P}_2$ according to Lemma 1.

Lemma 3. *$\mathbf{P} \cap \mathcal{N}_0$ is a ring.*

Proof. Let n be any positive integer. Choose m, k according to (vi). To the number k and a set $E \in \mathbf{P} \cap \mathcal{N}_0$ there are according to the definition of $\mathbf{P}$ sets $C \in \mathbf{C}$, $U \in \mathbf{U}$ such that $C \subset E \subset U$, $E - C \in \mathcal{N}_k$, $U - E \in \mathcal{N}_k$. Similarly to a set $F \in \mathbf{P} \cap \mathcal{N}_0$ there are sets $D \in \mathbf{C}$, $V \in \mathbf{U}$ such that $D \subset F \subset V$, $F - D \in \in \mathcal{N}_m$, $V - F \in \mathcal{N}_m$.

Hence we have $U \cup V \supset E \cup F \supset C \cup D$, $(U \cup V) - (E \cup F) \subset (U - E) \cup \cup (V - F) \in \mathcal{N}_n$, $(E \cup F) - (C \cup D) \subset (E - C) \cup (F - D) \in \mathcal{N}_n$. Moreover $U \cup V \in \mathbf{U}$ according to V_2 and $C \cup D \in \mathbf{C}$ according to V_3. Hence $E \cup F \in \mathbf{P}$ according to (iv) and the definition of $\mathbf{P}$.

Further $U - D \in \mathbf{U}$, $C - V \in \mathbf{C}$ according to V_4. Evidently $U - D \supset \supset E - F \supset C - V$. Further $(U - D) - (E - F) \subset (U - E) \cup (F - D) \in \in \mathcal{N}_n$, $(E - F) - (C - V) \subset (E - C) \cup (V - F) \in \mathcal{N}_n$. Hence $E - F \in \mathbf{P}$, according to (iv).

Definition 3. *Put*
$\mathbf{V} = \{U \in \mathbf{U}$: *there is* $C \in \mathbf{C}$ *such that* $U \subset C\}$.

Theorem 1. *$\mathbf{C} \subset \mathbf{P}_1$ if and only if $\mathbf{V} \subset \mathbf{P}_2$.*

Proof 1. Let $\mathbf{C} \subset \mathbf{P}_1$. Let $U \in \mathbf{V}$ be an arbitrary set. Then there is a set $C \in \mathbf{C}$ such that $U \subset C$. We have $C - U \in \mathbf{C}$ according to V_4 and $C - U \in \mathbf{P}_1$ according to the assumption. Let n be any positive integer. Then there is $V \in \mathbf{U}$, $V \supset C - U$, $V - (C - U) \in \mathcal{N}_n$. We have $C - V \in \mathbf{C}$ according to V_4 and $C - V \subset U$, $U - (C - V) = U \cap V \subset V - (C - U)$. Hence $U - (C - V) \in \mathcal{N}_n$ and $\mathbf{U} \subset \mathbf{P}_2$ according to the definition of $\mathbf{P}_2$.

2. Let $\mathbf{V} \subset \mathbf{P}_2$. Let $C \in \mathbf{C}$ be an arbitrary set. According to V_5 there are $U \in \mathbf{U}$, $D \in \mathbf{C}$ such that $C \subset U \subset D$. $U - C \in \mathbf{U}$ according to V_4. Evidently $U - C \subset D$ and hence $U - C \in \mathbf{P}_2$. To any positive integer n there is $C_1 \in \mathbf{C}$ such that $C_1 \subset U - C$, $(U - C) - C_1 \in \mathcal{N}_n$. Further $C = U - (U - C) \subset \subset U - C_1$, $(U - C_1) - C = (U - C) - C_1 \in \mathcal{N}_n$. We have $\mathbf{C} \subset \mathbf{P}_1$ since $U - C_1 \in \mathbf{U}$ according to V_4.

Theorem 2. *Let* X, $\mathbf{C}$, $\mathbf{U}$, $\mathbf{S}$ *satisfy the conditions* V_1—V_6 *and* $\{N_n\}_{n=0}^{\infty}$ *satisfy the condition* (i)—(v). *Then* $\mathbf{P} = \mathbf{S}$ *if and only if one of the following conditions is satisfied*:

A $\mathbf{C} \subset \mathbf{P}_1$,

B $\mathbf{V} \subset \mathbf{P}_2$.

Proof. The necessity of the conditions A, B is obvious. With respect to Theorem 1 is suffices to prove that A is sufficient. Hence let $\mathbf{C} \subset \mathbf{P}_1$.

Let $A \in \mathbf{S}$ and there exist $C \in \mathbf{C}$ such that $A \subset C$. Put $\mathbf{N} = \{B \in \mathbf{P}: B \subset C\}$. Then evidently $\mathbf{C} \subset \mathbf{P}$ hence $\mathbf{C} \cap C \subset \mathbf{N}$ (where $\mathbf{C} \cap C$ is the system of all all $E \in \mathbf{S}$ such that $E = D \cap C$ for some $D \in \mathbf{C}$). $\mathbf{N}$ is a σ-ring according to Lemmas 2 and 3. Hence $\mathbf{N} \supset \mathbf{S}(\mathbf{C} \cap C) = \mathbf{S}(\mathbf{C}) \cap C = \mathbf{S} \cap C$. Therefore $A \in \mathbf{N}$ since $A = A \cap C \in \mathbf{S} \cap C$.

Let $E \in \mathbf{S}$ be any set. According to 4 of Note 1 there are sets $A_i \in \mathbf{S}$, $C_i \in \mathbf{C}$, $A_i \subset C_i$, $A_i \subset A_{i+1}$ $(i = 1, 2, \ldots)$ such that $E = \bigcup_{i=1}^{\infty} A_i$. Hence $A_i \in \mathbf{P}$ $(i = 1, 2, \ldots)$ and $E \in \mathbf{P}_2$ according to Lemma 1. If $E \notin \mathcal{N}_0$, then $E \in \mathbf{P}_1$. If $E \in \mathcal{N}_0$, then $E = \bigcup_{i=1}^{\infty} A_i \in \mathbf{P}_1$ can be proved similarly as in the Lemma 2.

Theorem 3. *Let* X, $\mathbf{C}$, $\mathbf{U}$, $\mathbf{S}$ *satisfy the conditions* V_1—V_6 *and*

V_7: *To any* $C \in \mathbf{C}$ *there are* $U_k \in \mathbf{U}$ $(k = 1, 2, \ldots)$ *such that* $C = \bigcap_{k=1}^{\infty} U_k$. *Let* $\{N_u\}_{n=0}^{\infty}$ *satisfy the conditions* (i)—(v). *Then* $\mathbf{P} = \mathbf{S}$.

Proof. Let $C \in \mathbf{C}$ be an arbitary set. There are $U_k \in \mathbf{U}$ $(k = 1, 2, \ldots)$ such that $C = \bigcap_{k=1}^{\infty} U_k$. According to V_5 there are sets $V \in \mathbf{U}$, $D \in \mathbf{C}$ such that $C \subset V \subset D$. Hence $C = \bigcap_{k=1}^{\infty} (U_k \cap V) = \bigcap_{i=1}^{\infty} V_i$, where $V_i = \bigcap_{k=1}^{i} (U_k \cap V)$ $(i = 1, 2, \ldots)$. $V_i \in \mathbf{U}$ $(i = 1, 2, \ldots)$ according to 2 of Note 1. Further $V_i \subset V_{i+1}$ and $V_i \in \mathcal{N}_0$ $(i = 1, 2, \ldots)$ according to (iv). Hence $C \in \mathbf{P}_1$ according to the second part of the proof of Lemma 2. Now apply Theorem 2.

Theorem 4. *Let* X, $\mathbf{C}$, $\mathbf{U}$, $\mathbf{S}$ *satisfy the conditions* V_1—V_6 *and*

V_8: *To any* $U \in \mathbf{V}$ *there are* $C_k \in \mathbf{C}$ $(k = 1, 2, \ldots)$ *such that* $U = \bigcup_{k=1}^{\infty} C_k$. *Let* $\{N_n\}_{n=0}^{\infty}$ *satisfy the conditions* (i)—(v). *Then* $\mathbf{P} = \mathbf{S}$.

Proof. Let $U \in \mathbf{V}$ and $U = \bigcup_{k=1}^{\infty} C_k$, $C_k \in \mathbf{C}$ $(k = 1, 2, \ldots)$. Evidently $U \in \mathcal{N}_0$. Put $D_i = \bigcup_{k=1}^{i} C_k$. Then $D_i \subset D_{i+1}$ and $D_i \in \mathbf{C}$ $(i = 1, 2, \ldots)$ according to V_3. Since $\mathbf{C} \subset \mathbf{P}_2$ and $U = \bigcup_{i=1}^{\infty} D_i$ we have $U \in \mathbf{P}_2$ according to

Lemma 2. Hence $\mathbf{P} = \mathbf{S}$ according to Theorem 2.

Finally, let us mention some applications.

Examples of spaces X and systems C, U, S satisfying the conditions V_1—V_6 are, e. g. in [5], examples 1—5.

Hence let X, $\mathbf{C}$, $\mathbf{U}$, $\mathbf{S}$ satisfy the conditions V_1—V_6. Let μ be a measure defined on $\mathbf{S}$ and finite on $\mathbf{C}$. If $\mathcal{N}_n = \{E \in \mathbf{S}\colon \mu(E) < 1/n\}$ for $n = 2, \ldots$ and $\mathcal{N}_0 = \{E \in \mathbf{S}\colon \mu(E) < \infty\}$ then the sequence $\{N_n\}_{n=0}^{\infty}$ satisfies the conditions (i)—(v). The condition $E \in \mathbf{P}_1$ is equivalent to the condition $\mu(E) = \inf\{\mu(U)\colon E \subset U \in \mathbf{U}\}$ and the condition $E \in \mathbf{P}_2$ is equivalent to the condition $\mu(E) = \sup\{\mu(C)\colon E \supset C \in \mathbf{C}\}$.

Hence Theorem 8 of paper [5] is a consequence of Theorem 2. Similarly Theorem 4 p. 198 of [2] is a consequence of Theorem 2. Namely it can be shown that she system I—VII of axioms of Theorem 4 of [2] is equivalent to the system V_1—V_6 (cf. Note 1).

The well-known theorem on regularity of Borel measure (theorem F of [1] p. 228) is a consequence of these theorems, as well as the assertion included in examples 3 and 4 of [5].

The theorem on the regularity of the Baire measure (theorem C of [1] p. 228) and Theorems 10 and 11 of [5] are consequences of Theorem 3.

Also Theorem 2 of paper [4] is a consequence of Theorem 2. If we put $\mathcal{N}_0 = \mathbf{S}$, then $\mathbf{P}_1 = \mathbf{R}_1$, $\mathbf{P}_2 = \mathbf{R}_2$. If $X = \langle 0, 1\rangle$ and $\mathbf{S}$ is the system of all Borel subsets of $\langle 0, 1\rangle$, if $\mathbf{C}$ is the system of all closed and $\mathbf{U}$ the system of all open subsets of $\langle 0, 1\rangle$, we get a special case of Theorem 2.

Let $\mathbf{m}$ be a vector-valued measure defined on $\mathbf{S}$ with values in a normed space, $|\mathbf{m}|$ be the variation of $\mathbf{m}$ (see [3]). Let $|\mathbf{m}|(C) < \infty$ for every $C \in \mathbf{C}$. Then the sequence $\{\mathcal{N}_n\}_{n=0}^{\infty}$ defined by the equalities $\mathcal{N}_0 = \{E \in \mathbf{S}\colon |\mathbf{m}|(E) < \infty\}$ and $\mathcal{N}_n = \{E \in \mathbf{S}\colon |\mathbf{m}|(E) < 1/n\}$ for $n = 1, 2, \ldots$ satisfies the conditions (i)—(v). If $\mathcal{N}_0 = \mathbf{S}$, then the equality $\mathbf{P} = \mathbf{S}$ implies the ($\mathbf{C}$, $\mathbf{U}$)-regularity of $\mathbf{m}$ on $\mathbf{S}$, i. e. the following condition: If $A \in \mathbf{S}$ then to any $\varepsilon > 0$ there are $C \in \mathbf{C}$, $U \in \mathbf{U}$ such that $C \subset A \subset U$ and $|\mathbf{m}(B)| < \varepsilon$ for any $B \subset U - C$. Indeed, if $\mathcal{N}_0 = \mathbf{S}$, then $P_1 = \mathbf{R}_1$, $\mathbf{P}_2 = \mathbf{R}_2$. A set A is regular is and only if to any $\varepsilon > 0$ there are sets $C \in \mathbf{C}$ and $U \in \mathbf{U}$ such that $C \subset A \subset U$ and $|\mathbf{m}(B_1)| < \varepsilon$, $(\mathbf{m}(B_2)| < \varepsilon$ for any $B_1 \subset A - C$ and any $B_2 \subset U - A$. Our assertion follows from the inequatlity $|\mathbf{m}(B)| \leqq |\mathbf{m}|(B)$ for any $B \in \mathbf{S}$.

If $\mathcal{N}_0 \neq \mathbf{S}$ then $\mathbf{P} = \mathbf{S}$ implies $\mathbf{S} \cap \mathcal{N}_0 = \mathbf{P} \cap \mathcal{N}_0 = \mathbf{R}_1 \cap \mathbf{R}_2$. Hence if $A \in \mathbf{S} \cap \mathbf{R}_2$ and the regularity of A can be shown similarly as in the previous case. As a corrolary of Theorems 3 and 4 we get the following theorem.

Theorem 5. *Let* X, $\mathbf{C}$, $\mathbf{U}$, $\mathbf{S}$ *satisfy the assumptions* V_1—V_6 *and* V_7, *resp.* V_8. *Let* $\mathbf{m}$ *be a vector-valued measure defined on* $\mathbf{S}$ *with values in a normed space such* *that* $|\mathbf{m}|(C) < \infty$ *for* $C \in \mathbf{C}$. *Then* $\mathbf{m}$ *is* $(\mathbf{C}, \mathbf{U})$*-regular on* $\mathbf{S} \cap \mathscr{N}_0 = \{E \in \mathbf{S}: |\mathbf{m}|(E) < \infty\}$, *i. e. for every* $A \in \mathbf{S} \cap \mathscr{N}_0$ *the following holds*:

To any $\varepsilon > 0$ *there are* $C \in \mathbf{C}$, $U \in \mathbf{U}$ *such that* $C \subset A \subset U$ *and for every* $B \in \mathbf{S}$, $B \subset U - C$ *we have* $|\mathbf{m}(B)| < \varepsilon$.

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